THE CATEGORY OF ORDERED SPACES

by

Steven Purisch

Research Report 71-34

Based on a series of talks given in the
Categorical Topology Seminar
1970-71 at Carnegie-Melion University

July, 1971

In Section 3 the ordered absolutes of ordered spaces are studied, and it is shown that they are the projectives for an appropriate class of maps in the category of ordered spaces and order preserving maps.

See Herrlich [H^] for the definitions and properties in categorical topology*

Incorporated in this paper is most of the theorems from a paper by V. Fedorchuk. His theorems are identified by his name in parenthesis following the word "theorem". His proofs have been modified when I believed it would simplify matters or would better serve the purposes of this paper.

SECTION 1. The Category Lots.

1.1. Define LOTS to be the category of ordered spaces and order preserving (continuous) maps. The monomorphisms are the one-to-one maps and the isomorphisms are the one-to-one onto maps.

1.2. PROPOSITION. The epimorphisms are the maps with dense range.

Proof. Since every ordered space is Hausdorff, obviously a map with dense range is an eipmorphism.

Conversely let f:X-»Y be an epimorphism in LOTS. Suppose f does not have dense range. Then there exists y_o, y_1 € Y such that (y_o, Y_1) *t Θ* and *(y^Yj)* Θ f(x) = Θ. If *(Y_Q9Y_1)* is clopen define g_Q:Y—>R, and g^Y—>E (real line):

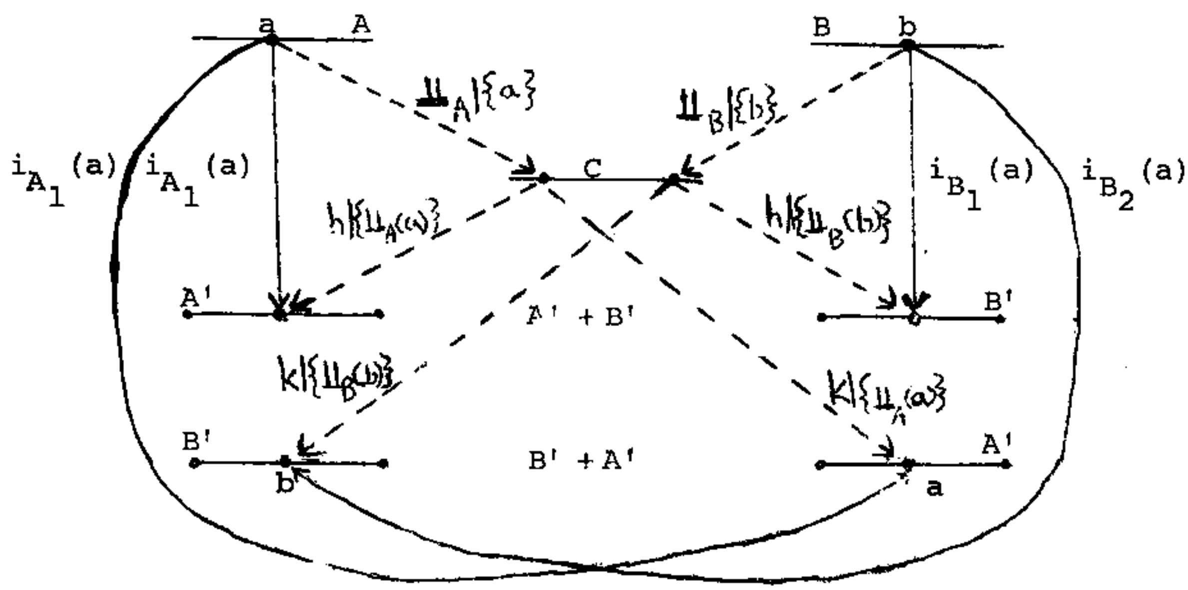
$$g_o(y) = \begin{cases} 0 & \text{for } y \,1\, y_o \\ 1 & \text{for } y > y_S \end{cases} \qquad g_x(y) = \begin{cases} 0 & \text{for } y < y_\pm \\ 1 & \text{for } y \wedge y\text{-}L \end{cases}$$

Thus g_o, g_1 *e* LOTS and $fg_Q = fg_1$, but $g_Q \wedge g_1$. a?his is a contradiction. If (y_o, y_1) is not closed,, $(\overline{y_o, y_1}) \wedge \wedge_o$• So pick $y^r, y^f S y'''$ ϵ (y_o, y_1) such that y' < y" < y»". since every ordered space is normal, the proof of TJrysohn's lemma permits us to construct h^ : [y^f ,y<^u]—^1 = [0,1] and h^ : [y^f,y"]_^I such that h^(y') = 0 = $h_1'(y')$, $h_o'(y")$ - 1/2, h^(y'») = 1 = hj_(y"), and h_o', h_1' e LOTS. Then extend h_o^f to h_o :Y—>I and h_1 to h_1 : Y—>I:

exists a unique $g: D \to C$ such that $\amalg_A g = g_A$ and $\amalg_B g = g_B$, and there exists a unique $f: E \to C$ such that $\amalg_A f = f_A$ and $\amalg_B f = f_B$. If $f(e_0) \geq g(d_0)$, then $a_0 = \Pi_A f(e_0) \geq \Pi_A g(d_0) = a_1$. If $f(e_0) < g(d_0)$, then $b_1 = \Pi_B f(e_0) \leq \Pi_B g(d_0) = b_0$. In either case there is a contradiction. Thus A and B have no product.

1.4. PROPOSITION. <u>Let</u> A <u>and</u> B <u>be non empty ordered spaces</u>. <u>Then</u> A <u>and</u> B <u>have no co-product in</u> LOTS.

<u>Proof</u>. Choose $a \in A$ and $b \in B$, and suppose there is a co-product C of A and B in LOTS. Let A' be formed by adding to A the end points if necessary. Form B' similarly. Let <u>$A' + B'$</u> be the topological sum of A' and B' with the orders



induced by A' and B' and such that for all $a \in A'$ and $b \in B'$ $a < b$. Define $B' + A'$ similarly. Let $i_{A_1}: A \to A' + B'$,

i_{A_2} ${}_2A$—»$B^! + A^T$, i_{B_1}:B->$A^f + B^f$, and i_{B_2}:B—>$B^! + A^T$ be the natural

embeddings. Let 11_A •-A—> C and $1L_B$:B—>C be the co-product maps;

Then there exists unique h:C—*$A^? + B^T$ and k:C—>$B^! + A^f$ such that

$i_{A_1} = h\amalg_A$, $i_{B_1} = h\amalg_B$, $i_{A_2} = k\amalg_A$, and $i_{B_2} = k\amalg_B$. Now

i_{A_2} (a) > i_{B_2}(b) and i_{A_1}(a) < i_{B_1} (b) . If $1L_A$ (a) < II_B-(b), then

i_{A_2} (a) = kU_A(a) $\leq$ kli_B(b) = i_{R_2} (b) . If JJ_A(a) > U_B(b), then

i_{A_1} (a) = hll_A(a) $\geq$ hll_B(b) = i_{R_1} (b) . In either case there is a

contradiction. Thus A and B have no co-product.

1.5 1.3 and 1.4 can be easily generalized as follows.

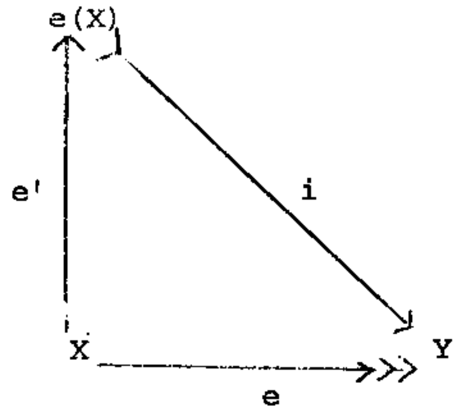
Let {A } ^ be a collection of non empty ordered spaces subscripted by the set G. Its product exists iff all but one of the A_α's is a one point space. Its co-product exists iff $\bar{5} = 1$.

Let Y be an ordered space, and let yeY. • Call y a <u>left limit point</u> if ye (- OD^y), and call y a <u>right limit point</u> if ye (y,OD) . Then y is called a <u>one (two) sided limit point</u> if y is <u>either (both)</u> a left or (and) a right limit point. A <u>gap</u> in Y is a pair $[A,B)_Y$ of non empty clopen subspaces such that A U B = Y and A < B i.e., for all aeA, beB a < b. If A has a sup y^1 and B has an inf y% $\{A,B\}_Y$ is called a <u>jump</u>. This jump can also be denoted by the ordered pair fy^T , v^{tf}3• The points y^T and y^{ff} are called jump points. If both y^1 and y" are one sided limit points,, $\{y^!, y^{fl}\}$ is called a two sided jump. $\{A,B\}_Y$ is called a cut if A has no sup and B has no inf. We also refer to the "hole" u between A and B as this cut.

Clearly, generalized ordered spaces need not be orderable. For example (0,1) U -{2} is not an orderable subspace of R. Let X be a generalized ordered space. If $\{A^!, B^T\}_X$ is a pair of non empty clopen, i.e., open and closed, subspaces of X with $A^T < B^T$ and $A^1 \cup B^1 = X$, we also call this a gap. Similarly, we define jumps and cuts as we did in the ordered case. However, if A^f has no sup but B has an inf x^f we call $\{A^1, B^!\}_v$ a left cut, which is also denoted by $\{\hat{}x^1\}$. If A^f has a sup x but B^1 has no inf we call $[A', B'\}_X$ a right cut, which is also denoted by [x,-]. Right cuts and let cuts are called half cuts, as are the "holes" they determine.

1.6 THEOREM. Let e:X-*»Y be an epimorphism in LOTS. Then e is an extremal epi iff for all yeY\e(x) there exists a. unique y'ee(X) such that y and y' form a. two sided jump in Y. Hence if e is an extremal epi and e(X) is ordered. then e is ein onto map.

Proof. Let e:X-39>Y be an extremal epi. If e(X) is ordered, then define e^1 :X-->e(X) such that for all xeX e^1(x) = e(x)



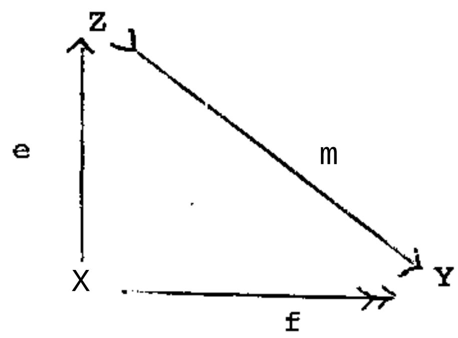
and $i:e(X)\rightarrowtail Y$ is the inclusion map. Thus the diagram commutes, and i is a mono. So i is an iso. Then since e' is onto, e is onto.

If $e(X)$ is not ordered, let $y e Y \backslash e(X)$. Suppose y is a two sided limit point of Y. Then $Y' = Y \backslash \{y\}$ is ordered and there exist $e^T : X \rightarrow Y^*$ such that for all $x e X$ $e(x) = e'(x)$, and $i:Y^f \rightarrowtail Y$ is the inclusion map. Then i is a mono and $ie^f = e$ but i is not an iso. Contradiction.

Therefore, since $e(X)$ is dense in Y, y cannot be isolated, so it is a one sided limit point. Hence there exists a unique y^f such that y and y^T form a two sided jump, and we may assume $y < y^T$..

suppose $y' e Y \backslash e(X)$. Then $Y^1 = Y \backslash (y, y'\}$ is ordered and as in the argument above, we have $e^f : X \rightarrow Y^f$, the inclusion $i:Y' \rightarrow Y$, $ie^1 = e$, and i is a mono but not iso. Contradiction. So $y^f e e(X)$.

Conversely, let e be an epi, m a mono,, and



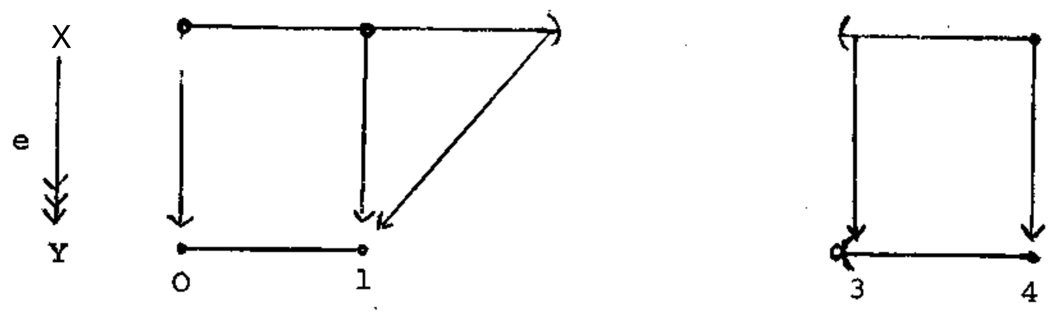
f € LOTS such that the diagram commutes.

If e(X) is onto, then m is onto. So m is an iso.

If e(X) is not onto, let yeY\e (X) . Let y^T be as in the hypothesis. We may assume $y < y^f$. Since m is mono and $y^!$ee(X), there exists a unique $z^!$ eZ such that $m(2^?) = y^f$. Since e(X) is dense in Y, m(Z) is dense in Y. Thus z^1 is not a left limit point since m is mono and y^1 is not a left limit point. Moreover, $z^!$ ^ o, and so z^f has a predecessor z. Necessarily, m(z) = y. So m is onto and, therefore, an iso. Thus e is an extremal epi.

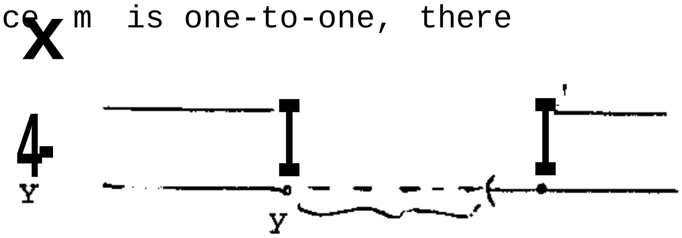
1.7. EXAMPLE. There exists an extremal epi that is not onto. Let X = [0,2) + (3,4],Y = [0,1] + [3,4] define e : ^%Y as follows:

e(x) = x for xe[0,1] U (3,4] and e(x) =1 for ze(l,2).
Then e(X) = [0,1] + (3,4] is an unordered subspace of Y.



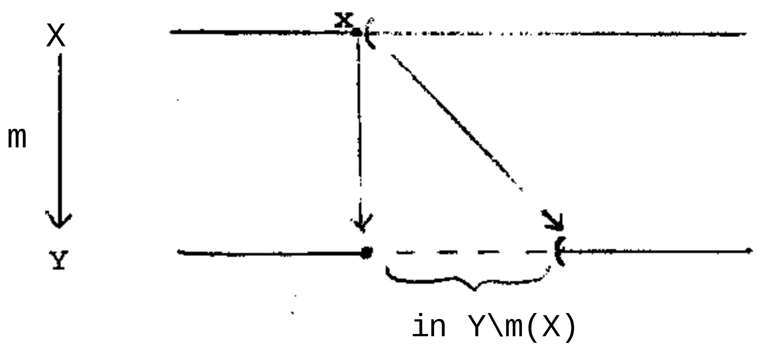
1.8 LEMMA. *If $m: X \to Y$ is a one-to-one order preserving function and X and Y are ordered spaces, then m is continuous iff $m(X)$ is ordered.*

Proof. Let m be continuous. Suppose $m(X)$ is not ordered. Then there exists $y \in m(X)$ and a half, say right, cut $\{y, \cdot\}$ in $m(X)$. Since m is one-to-one, there



exists a unique $x \in X$ such that $m(x) = y$. Since $y \notin \overline{m(X) \cap (y, \infty)}$, $x \notin \overline{(x, \infty)}$. Thus x has a successor x', and $m(x') = \min(m(X) \cap (y, \infty))$ which is impossible since it has no min. Hence $m(X)$ is ordered.

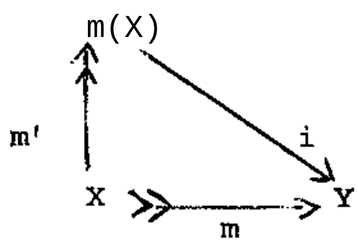
Conversely, let m be discontinuous. Then we may assume there exists $x \in X$ such that $x \in \overline{(x, \infty)}$ but $m(x) \notin \overline{m((x, \infty))}$. Since m is mono and (x, ∞) has



no minimum, $m((x, \infty))$ has no minimum. Thus $\{m(x), \to\}$ is a right cut in $m(X)$, and hence $m(X)$ is not ordered.

1.9. THEOREM. _Let_ $m: X \rightarrowtail Y$ _be a monomorphism in_ LOTS. _Then_ m _is an extremal mono iff_ X _is embedded as the largest ordered subspace of_ $\overline{m(X)}$.

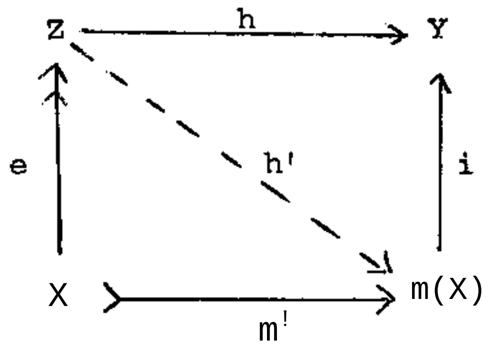
Proof. Let $m: X \rightarrowtail Y$ be an extremal mono. Then by the lemma $m(X)$ is an ordered space. Thus the diagram commutes where



i is the inclusion map and for all $x \in X$ $m(x) = m'(x)$. Hence m' is an epi and, therefore, an iso. So X is embedded in Y.

Now let Y' be an ordered space such that $m(X) \subseteq Y' \subseteq \overline{m(X)}$. Then $im' = m$ where $m': X \to Y'$ is defined such that for all $x \in X$ $m(x) = m'(x)$, and $i: Y' \rightarrowtail Y$ is the inclusion map. Hence m' is an epi, so it is an iso. Thus $m(X) = Y'$. So $m(X)$ is the largest ordered space in $\overline{m(X)}$.

Conversely, let m:X>-^Y be a mono such that m(X) is the largest ordered subspace of $\overline{m(X)}$, and let the diagram



commute where i is the inclusion, im^f = m, and e is an epi.

We want to show the existence of h^T :Z—>m(X) such that $ih^!$ = h. To do this it is sufficient to show that h is a mono and h(Z) c $\overline{m(X)}$.

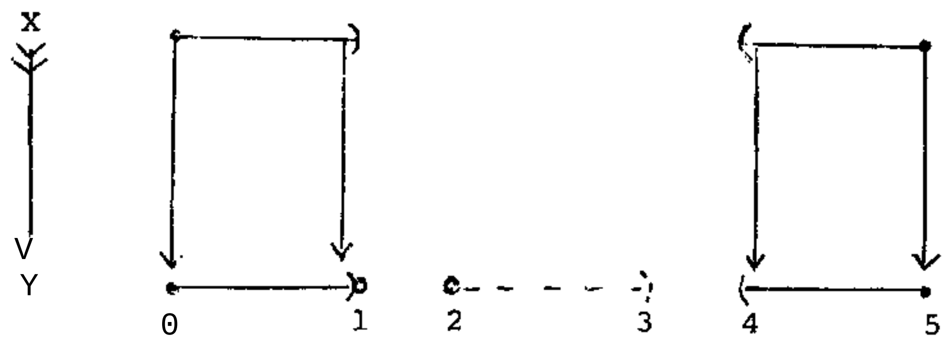
However, m(X) ch(Z) = h($\overline{e(X)}$) c$\overline{he(X)}$ = $\overline{m(X)}$. Since m is mono, e is mono. So e(X) is an ordered dense subspace of Z.

Suppose there exists $z,z^!$€Z such that z < z^T and h(z) = $h(z^!)$ = y for some yeY. Then since e(X) is an ordered dense subspace of Z, $\{z,z^T\}$ is a two sided jump of Z in zXefX), and y is a two sided limit point in $\overline{m(X)}$\jn(X). But then m(X) U {y} is ordered, and m(X) c m(X) U fy} c $\overline{m(X)}$. Contradiction. Thus h is a mono.

Then by the lemma h(Z) is ordered. Hence h(Z) = m(X) since m(X) c h(Z) c$\overline{m(X)}$. So there exists h^f:Z—^m(X) defined by h^T (z) = h(z) for all zeZ, i.e., $ih^!$ = h. Thus h^T is one-to-one and onto, i.e., it is an iso. Then m^T = $h^!e$ since i is a mono and im^T = he = ih^re. Moreover, since m^T is also an iso, e is an iso. Thus m is an extremal mono.

1.10. Denote by G LOTS the category of generalized ordered spaces and order preserving maps. Then the last two theorems seem to indicate that both G LOTS and the subcategory in LOTS of maps with ordered range would lend themselves more naturally to a categorical treatment than would LOTS.

1.11. EXAMPLE. There is an extremal mono in LOTS with unordered range. Let X = [0,1) + (4,5] and Y = [0,1] + [2,3) + (4,5].



Let m:X;»->Y be the inclusion map. Then m(X) is the greatest ordered subspace of $\overline{m(X)}$ = [0,1] + (4,5].

SECTION 2. <u>Ordered and Generalized Ordered Extensions</u>.

2-1. Let X,B 8 (GLOTS) LOTS. Then B is an <u>(generalized)</u> <u>ordered extension</u> of X iff X can be embedded into B by a map in (GLOTS)LOTS. Let peB\x. Then a <u>neighborhood of</u> p <u>in</u> X is the intersection of a neighborhood of p in B with X. B is called an <u>ordered compactification</u> of X if B is a compact ordered extension of X in which X is dense. Note that <u>no unordered generalized ordered space is compact</u>. B ∈ G LOTS is a *generalized ordered realcompactification* of X if B is a realcompact generalized ordered extension of X in which X is dense.

The set of all ordered compactifications of an ordered space X can be partially ordered as follows. Let b_1X and b_2X be two ordered compactifications of X. Then $b_2X \geq b_1X$ iff there exists a unique $f: b_2X \to b_1X \in$ LOTS such that f is the identity on X.

Note that X is compact iff it has both end points and no cuts.

2.2. THEOREM. (Fedorchuk). *The partially ordered set of ordered compactifications of an ordered space* X *is order isomorphic to the set of all subsets of the set* U *of all cuts of* X.

Proof. If U is empty, i.e. X has no cuts, there exists only one ordered compactification of X obtained by adding to X any end points it doesn't possess. Assume that $U \neq \emptyset$, and let bX be an ordered compactification of X. Choose a nonterminal point $y \in bX\setminus X$. Then y defines a cut of X by the subsets $X_y^- = \{x \in X \mid x<y\}$ and $X_y^+ = \{x \in X \mid x>y\}$. Thus every nonterminal point $y \in bX\setminus X$ defines a cut u of X such that $X_y^- = X_u^-$, $X_y^+ = X_u^+$, i.e. u can be considered as containing y. It is easy to see that for a given cut u of X, there is either one or two points of $bX\setminus X$ in u. Thus the ordered compactification bX defines a division of tt into two disjoint subsets $lu_1(bX)$ and $lu_2(bX)$, where $U_i(bX)$ consists of those cuts ueU containing i points of bx\x i = 1,2.

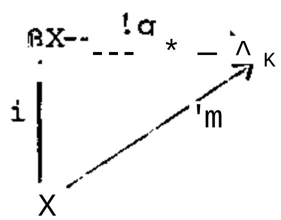
We now set up a correspondence between each ordered compactification bX of X and the set $lu_2(bX)$ c \\, and show the mapping lu_2 is an order isomorphism between the set of all ordered compactifications of X and the set of all subsets of U, ordered by inclusion. Since each nonterminal point of the growth of an ordered compaetification lies in a cut of X, we have $U_2(b_1X) = U_2(k>_2X)$ implies $b_1X = b_2X$' i-e- the mapping lu_2 is one-to-one. Let U^T c U. Consider the ordered set B obtained from X as follows: (1) by the addition$_3$ if necessary, of the end points; (2) by the addition of one point to each cut ueU\U$^!$; (3) by the addition of an ordered pair of points to each cut ueU^T . It is easy to see that B is an ordered compactification of X and that $lu_2(B) = U^T$. Thus the mapping lu_2 is onto. We show lu_2 is an order preserving mapping. Let b_1X and b_2X be two ordered compactifications of X, with $b_2X \geq$ i^X, i.e. there exists f:b_2X—> b^ G LOTS such that f is the identity on X. f maps the "cut points" in b_2x\x to the corresponding cut points in b_1X\X. Hence U_2 (t>$_2$X) 2 1⁄2 (^{b}n^x)$_1$ • Now we show lu_2^{-1} is order preserving. Let U_2 (fc>$_2$X) 2 $^u{}_2$ ($^b{}_1{}^x$) • Then l^X is obtained from b_2X by identifying those ordered pairs of cut points of b_2x\x which fill the growth from the set $U_2(b_2X)\backslash u_2(b_1X)$. Hence there exists an onto map $f: b_2X \to b_1X \in$ LOTS which is the identification on X and thus $b_2X \geq b_1X$. Hence the theorem is proved.

2.3 COROLLARY. (Fedorchuk). *For every ordered space* X *there exists a greatest ordered compactification* βX. βX *is obtained by the addition of an ordered pair of points to each cut in* X *and by the addition, if necessary of the end points.*

2.4. Let X be an ordered space. Then $\overline{\overline{\beta X}} \leq 2^{\overline{\overline{X}}}$ since the points in $\beta X \setminus X$ are determined by cuts of X which in turn are determined by pairs of subspaces in X.

2.5. PROPOSITION. *The category of compact* LOTS *is an epireflective subcategory of* LOTS.

Proof. Consider the following diagram where $m: X \to K \in$ LOTS, K compact and i is the inclusion map. Since X is dense in βX, i is an epi. So



if there exists $q: \beta X \to K$ such that $qi = m$, then q is unique. Define q as follows. For $x \in X$ let $q(x) = m(x)$. For points of

BX\X look at the ordered pair of points x_{a_1}, x_{a_2} formed in the cut $\{A_\alpha, B_\alpha\}_X$. This cut induces a cut $f^m(A_a \wedge^m(B_{r t} \wedge_m/X)$. Let $q(x_{\alpha_1}) = \sup(m(A_\alpha))$ and $q(x_{\alpha_2}) = \inf(m(B_\alpha))$. Let $q(0) = 0$ and $q(1) = 1$. It is easy to show q is order preserving and continuous. Hence the category of compact LOTS is an epireflective subcategory of LOTS.

2.6. Consider the set <u>$0^*(X)$</u> of order preserving bounded maps from an ordered space X to £. We form the topological product $P^\wedge = R^{0^*(X)}$ and embed X into $P^\wedge$ by Tikhnov's method: for $x \in X$ $f(x) = y = (y_\alpha)_{\alpha \in G}$ where G is an indexing set for $0^*(X)$, and each $y_\alpha = f_\alpha(x)$ for $f_\alpha \in 0^*(X)$. P^* is a partially ordered set with the following order relation: $y = (t_\alpha)_{\alpha \in G} \leq (t'_\alpha)_{\alpha \in G} = y'$ iff $t_\alpha \leq t'_\alpha$ for each $\alpha \in G$.

2.7. THEOREM. (Fedorchuk). <u>Let</u> $f: X \to P^\wedge$ <u>be the embedding described above</u>. <u>Then</u> $cl_{P^*} f(X)$ <u>considered with the order relation induced by</u> $P^\wedge$ <u>is isomorphic to</u> βX.

<u>**Proof**</u>. Since each $f_\alpha \in O^*(X)$, the order induced on X from $P^\wedge$ coincides with the initial order of X, i.e. the embedding $f: X \to f(X)$ is an isomorphism. Clearly cl X is compact, and we shall prove that it is an ordered compactification of X.

We first show that any two points y' and y'' in $cl_{P_*} X$ are comparable in the order relation induced by P_*. Let $y' \in f(X)$. If $y' = (t'_\alpha)$ cannot be compared with $y'' = (t''_\alpha)$, then there exist subscripts α_1 and α_2 such that $t'_{\alpha_1} < t''_{\alpha_1}$, $t'_{\alpha_2} > t''_{\alpha_2}$. We choose a neighborhood Vy'' of y'' consisting of all points $y = (t_\alpha)$ for which $t_{\alpha_1} > t'_{\alpha_1}$, $t_{\alpha_2} < t'_{\alpha_2}$. We show that Vy'' doesn't intersect X. Since X is linearly ordered and $y' \in X$, we have $X \subset (-\infty, y'] \cup [y', \infty)$. Vy'' is the intersection of the two neighborhoods $V_1 y''$ and $V_2 y''$, where $V_1 y'' = \{y=(t_\alpha) \mid t_{\alpha_1} > t'_{\alpha_1}\}$ and $V_2 y'' = \{y=(t_\alpha) \mid t_{\alpha_2} < t'_{\alpha_2}\}$. Clearly, $(-\infty, y'] \subset P_* \backslash V_1 y''$ and $[y', \infty) \subset P_* \backslash V_2 y''$. We have $X \subset (-\infty, y'] \cup [y', \infty) \subset (P_* \backslash V_1 y'') \cup (P_* \backslash V_2 y'') = P^* \backslash (V_1 y'' \cup V_2 y'') = P \backslash Vy''$. Thus $X \cap Vy'' = \emptyset$. But $y'' \in cl_{P_*} X$. Contradiction. Hence every point $y'' \in cl_{P_*} X$ can be compared with every point $y' \in X$. Now let $y', y'' \in cl_{P_*} X \backslash X$. From what has just been proven $X \subset (-\infty, y'] \cup [y', \infty)$, and, repeating the above argument, we find that y'' can be compared with y'. Thus $cl_{P_*} X$ is linearly ordered.

Now we shall show that the interval topology on $cl_{P_*} X$ coincides with the subspace topology. Since $cl_{P_*} X$ is compact, it is sufficient to show that the identity map from $cl_{P_*} X$ with the subspace topology to $cl_{P^*} X$ with the interval topology is continuous. Let $y \in cl_{P^*} X$ and let Vy be an interval neighborhood of y, i.e. V_y = fy^T eel$_{p^*}$ x|y$_1$ < y^T < y$_2$L where y$_1$,^y$_2$€cl$_{p^*}$ X.

Let ΙΙα be the a-coordinate projection map. Ihen there exists subscripts a_n and a_9 such that ΙΙ (y$_n$) < ΙΙ (y) and

n_ (y) < IL (y~). Then Vy = (y^1 eel x|ll (y<) > ΙΙ (y,) } 0

{y^T eel$_{P^*}$x|ll$_{a_2}$ (y$^!$) < TI$_{a_2}$ (y$_0$) } is a neighborhood of y in $cl_{P^\wedge}$ X with the subspace topology such that V^{T}y c: Vy. Hence the identity map on $cl_{P^*}X$ is continuous. As seen above $cl_{P^*}X$ induces the original order on X, and, therefore, it is an ordered compactification of X.

Now let i_X:X—>foX and i_XJ,:X—> $cl_{p^\wedge}X$ be the embedding maps. By Proposition 2.5$_5$ there exists a unique q:BX—^ XGLOTS such that $qi_X = iL_X$. Since iJ_X is dense, q is dense, and since BX is compact, q(BX) is closed. Hence q(BX)'= cl_{P^*} X, i.e. q is onto. Moreover, since if $\{A,B\}_X$ is a cut in X, then the existence of the map f^α eO* (X) which is 0 on A and 1 on B implies that q must be one-to-one. Hence q is an isomorphism, i.e. BX is isomorphic to cl_{P^*} X.

2.8. Fedorchuk noted [F_2] that BX has characteristic properties similar to those of the Stone-Cech compactification:

(1) In order for X to be 0*-embedded in an ordered compactification bX, i.e. every map in 0*(X) "has an extension to a map in 0* (bX)$_9$ it is necessary and sufficient that bX = BX.

(2) In order that any two convex nonintersecting closed subsets in X should have nonintersecting closures in an ordered compactification bX it is necessary and sufficient that bx = BX.

(3) If A is an ordered subspace of X, then BA = $cl_{\beta X} A$.

Let S = $(H_\alpha)_{\alpha \in G}$ be a collection of nonempty subsets of a topological space X. The collection is said to be regularly decreasing if for each H_a e £ there exists L_P e S such that

2.9. THEOREM. (Fedorchuk). Let XSLOTS. Let aX be the set of all maximal regularly decreasing filters, the elements of which are convex open subsets of X. Then there is a linear order on the set aX, with respect to which aX JLS isomorphic to *HX.*

We order the set aX as follows. Let £^{f}$_9$£^{lf} *e* aX. We put S^f < £" iff there exists intervals H$^!$ *eT$^]$* and H^M€S" such that

$H^f < H^{f'}$. This is a linear order on aX. The rest of the proof is omitted.

2,10. THEOREM. (Fedorchuk). If the ordered space X has weight Y, then BX is the inverse limit of the directed family of the ordered compactifications of X of weight Y. (The partial order and boundary maps for this family are defined in the second paragraph of 2.1).

Proof omitted.

In 2.11-2.16 we will consider the usefulness of the operator 6 in showing its role in solving the metrizability problem for compact ordered spaces and in determining when certain kinds of ordered spaces are isomorphic.

The following example gives insight for Theorems 2.10, 2.15 and 2.16 as well as how BX may be visualized for $X \in$ LOTS.

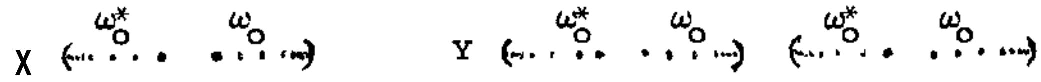
2.11 EXAMPLE. There is an ordered space whose cardinality and weight are K_0 but its greatest ordered compactification has cardinality and weight c and is not metrizable. Let C' be the rationals in the unit interval I. Both the weight and cardinality of C' equal K_0, BO is constructed from I by replacing each irrational point in I by an ordered pair of points. Another way to construct BC' is to identify corresponding rational points in $I \times \{0,1\}$ (lexicographic product). $B\omega_1$ is compact and separable but both the cardinality and weight of $B\omega_1$ equal c. Hence $B\omega_1$ is not metrizable.

2.12. An ordered space X is said to be *minimal* if it has no end points and no two sided jumps.

2.13. THEOREM. (Fedorchuk). *If X and Y are minimal ordered spaces, and BX is isomorphic to BY, then X is isomorphic to Y.*

Proof. Let $f: BX \to BY$ be an isomorphism. Then f maps end points to end points and two sided jumps to two sided jumps. Now $BX \setminus X$ and $BY \setminus Y$ consists of two sided jumps with the possible exception of the two end points. Since X and Y are minimal, f maps $BX \setminus X$ onto $BY \setminus Y$. Hence $f|X$ is a one-to-one onto map from X to Y, i.e. an isomorphism. So X is isomorphic to Y.

2.14. EXAMPLES. *Minimal ordered spaces X and Y may be homeomorphic while BX and BY are not isomorphic.* For let X be the discrete space $\omega_0^* + \omega_0$ and let Y be the discrete space $\omega_0^* + \omega_0 + \omega_0^* + \omega_0$ where λ^* is the ordinal λ with the reverse order. Then $BX = (\omega_0+1)^* + (\omega_0+1)$, which has two limit points, and $BY = (\omega_0+1)^* + (\omega_0+1) + (\omega_0+1)^* + (\omega_0+1)$, which has four limit points.



EX (05+D* o>+1 BY (05+D* 05+1 (05+D* OL+1

-.P.. • • « . St —°- • • ^.•°- - -•p ••. . . . .°*

Conversely, minimal ordered spaces X and Y may not be homeomorphic, while Bx and BY are homeomorphic. For let X be the discrete space OJ* + £ (w^4og)n, and let Y be the

OD
n=1

nondiscrete space og + S (oo_o+l+ug) R. Then BX =

OD
n=l

OS* + (E ((o?+l) + (co+l)%))+l. and BY = Y+l, which can

OD
n=l
on

easily be shown to be homeomorphic.

2.15. THEOREM. (Fedorchuk). A compact ordered space B is metrizable iff there exists EL separable space X with no two sided jumps such that B = fitx and BX\X <C K_o .

Proof. Sufficiency. Let X be a separable space with no two sided jump points such that BX\X <. N_Q- It is sufficient to show that BX is second countable. Let X = (x^x^ . . .x^, . . .} a countable dense subset of X and BX\X = {y1^y2^•••sYn> •••}• We renumber the points of X_o U Bx\X = [z^z2, . . . , zn, . . .}. Since X has no two sided jump points, it is easy to verify that all sets of the type [0^), (z^z_j), or $(z_j,,1)$, where $z_i < z_j$, form a countable base for RX.

Necessity. Let B be a metrizable compact ordered space. Consider the ordered space X obtained by deleting all two sided jumps of B. Since B has a countable base, there are at most a countable number of two sided jump points in B. Hence $\overline{\overline{B\backslash X}} \leq H_o$. We shall show that X is dense in B. Suppose it is false. Then there exist a nonempty open interval C of B contained in B\X such that C consists of two sided jumps and hence is perfect and totally disconnected. Thus since B is compact metric, C contains the Cantor set whose cardinality is c. Contradiction.

Now BX = B since &X is formed by placing an ordered pair of points to each cut in X. But these are precisely the points removed from B to form X since X is dense in B. Moreover, the construction of X and its density in B implies that X has no two sided jumps. Since B is compact metric, it has a countable base, and hence X is separable.

2.16. THEOREM. (Fedorchuk). <u>A compact ordered space</u> B <u>is metrizable iff there exists an ordered space</u> X <u>with a point countable base such that</u> B = RX <u>and</u> $\overline{\overline{BX\backslash X}}$ <C fc$\leq_o$.

Proof. The necessity is obvious, since we can take X as B. Now let X be an ordered space with point countable base such that B = BX and $|BX\setminus X| \leq \aleph_0$. First we show BX satisfies the first axiom of countability. Suppose this is false. Then there exists $y \in BX\setminus X$ which doesn't have a countable neighborhood base. Hence there is a monotonic (for example, increasing) net $\{x_\alpha \mid x_\alpha \in X, \alpha < \omega_q\}$ converging to y, where ω_q is a regular uncountable ordinal, and the set $\{x_\alpha \mid x_\alpha \in X, \alpha < \omega_q\}$ is a closed subset of X. Consider the part $\{x_\alpha \mid \alpha < \omega_1\}$ of this net. Since X has a point countable base, it is first countable. Hence the net $\{x_\alpha \mid \alpha < \omega_1\}$ converges to some point $y' \in BX\setminus X$. Since $|BX\setminus X| \leq \aleph_0$, then at most a countable number of intervals $[x_\alpha, x_{\alpha+1}]$ contain points of $BX\setminus X$. Hence there exists an ordinal $\alpha_0 < \omega_1$ such that the interval $[x_{\alpha_0}, y')$ is contained in X as a closed subset. Now $[x_{\alpha_0}, y')$ contains as a closed subspace the nonparacompact space $\{x_\alpha \mid \alpha_0 \leq \alpha < \omega_1\}$ of order type ω_1. Thus X is not paracompact. Then X doesn't have a point countable base [F_1 or B]. Contradiction. Hence BX is first countable.

A. Mishchenko [M] proved that a compact Hausdorff space with a point countable base is metrizable. Hence to prove that BX is metrizable it is sufficient to show that it has a point countable base. Let C = (v) be a point countable for X whose members are convex. For each $V \in \mathcal{C}$ we denote by V' the maximal interval of βX such that $V' \cap X = V$. Then $\mathcal{C}' = \{V'\}$ remains point countable at all points of X. Now let $y \in \beta X \setminus X$ and (x_n) be a sequence in X converging to y. Consider the set $\mathcal{C}_y$ consisting of all intervals $V' \in \mathcal{C}'$ containing y. Each such interval V' contains some point x_n. Hence $\mathcal{C}_y \subset \bigcup_{n=1}^{\infty} \mathcal{C}_{x_n}$. But each $\mathcal{C}_{x_n}$ is countable. Therefore, $\mathcal{C}_y$ is also countable, and thus $\mathcal{C}'$ is point countable throughout all of βX. Since βX is first countable and $|\beta X \setminus X| \le \aleph_0$, then we obtain a point countable base for all of βX by adding to $\mathcal{C}'$ a countable set consisting of the elements of a neighborhood base for each point in $\beta X \setminus X$. Hence βX is metrizable.

2.17. _For any Hausdorff space, sequential compactness implies countable compactness which in turn implies pseudo-compactness. Conversely, for ordered spaces pseudo-compactness implies sequential compactness_. To prove this last statement note that if an ordered space X has a sequence with no convergent subsequence, then one can find a monotonic subsequence which is a copy of $\mathbb{N}$ (by mapping the sequence in an order preserving not necessarily continuous fashion into $\mathbb{R}$). Since X is normal, Tietze's extension theorem shows that any closed subspace of X is C^* embedded in X.

Recall that υ is the Hewitt realcompact operator [G-J].

2.18. THEOREM. Let X be an ordered space. TTien £X is orderable iff X jls sequentially compact. If X is sequentially compact, then j3X = vX = BX.

Proof. Assume X is not sequentially compact. Then it is easy to show there is a monotone (for example,, increasing) sequence $\{x_n\}$ which does not converge in X. Hence this sequence is a closed set isomorphic to the natural numbers (ST. By Tietze[1] s extension theorem, fx } is C-embedded in X. Thus, cl $\{x_n\}$ = j3($\{x_n\}$) is isomorphic to /BIN. Hence $\{x_n\}$ does not converge in its closure and hence not in j3X. So j3X is not sequentially compact. By 2.17 /3X is not orderable.

Conversely, assume X is sequentially compact. It is sufficient to show that X is C-embedded in BX. First show that if A and B are disjoint closed subsets of X then (BX\X)n cl J^ PI cl B = 0. So choose peBx\x, and let A and B be disjoint closed subsets of X. We may assume that p is a left limit point of BX. Since X is sequentially compact no sequence in X can converge to p. Suppose $p \in cl_{BX}A \cap cl_{BX}B$. Then there exists an increasing sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in A$ for n odd and $x_n \in B$ for n even. Then $\{x_n\}$ converges to $x_o < p$, where $x_o \in X$, since X is sequentially compact. Since A and B are closed, $x_o \in A \cap B$. Contradiction. Hence $p \notin cl_{BX}A \cap cl_{BX}B$. Thus (BX\X) $\cap$ $Cl_{BX}A \cap Cl_{BX}B = \emptyset$.

Now show for every $f \in C(X, I)$ there exists $x_p \in (-\infty, p) \subset X$ such that $f|[x_p, p)$ is constant. Since X is sequentially compact and $[x,p)$ is closed in X for every $x < p$ in X, • $f([x,p))$ is sequentially compact in ! and hence compact. So the nested intersection $\bigcap_{x<p} f([x,p))$ is non-empty. Choose r_p in this intersection. Then $f^{-1}(r_p)$ is closed in X and $p \in cl_{RX} f^{-1}(r_p)$. For every $n \in$, the closed set $(x \in (-\infty, p) \mid |f(x) - r_p| \geq \frac{1}{n})$ is disjoint from $f^{-1}(r_p)$. Hence by the above paragraph,, this set has an upper bound $x_n \in (-\infty, p)$. Thus $\sup_{n \in N} x_n = x_p$ exists in X and is less than p. Thus $f|[x_p, p) = r_p$.

Thus f extends to $f_p : X \cup \{p\} \to R$ such that $f_p(p) = r_p$. Similarly,, we can extend f to $f_{p'} : X \cup \{p'\} \to R$ for each $p' \in \beta X \setminus X$. Let f' be the induced extension of the $f_{p'}$s. Obviously, f' is order preserving. Moreover f' is continuous since any net in βX converging to $p' \in \beta X \setminus X$ is eventually in $[x_{p'}, p']$ and thus the image of the net is eventually equal to $r_{p'} = f'(p')$. . Hence X is C-embedded in βX and thus $\mathcal{E}X = \nu X = \beta X$. So βX is orderable.

2.19. After writing this paper I was told that M. Venkatajaman, M. Rajogopolan, and T. Soundararajan has also shown in a paper not yet published that if βX is orderable,, then X is countably compact. However, the first half of the proof above is more concise than is their proof.

Let X be a generalized ordered space. In a similar manner to 2.6 we let $O^*(X)$ be all order preserving maps from X to R and define an embedding $f: X \to P^*$, where $P^* = K^{O^*(X)}$. Then similar to the proof of Theorem 2.7, $cl_{P^*} f(X)$ is the greatest ordered compactification BX in the sense that X is O^*-embedded in BX. Note that a generalized ordered space is compact iff it has its end points and has neither cuts nor half cuts. Hence a compact generalized ordered space is ordered. Similar to the proof of Theorem 2.2, BX is constructed by the addition of an ordered pair of points to each cut, by the addition of a single point to each half cut, and by the addition, if necessary, of the end points. Then similar to the proof of Proposition 2.5 we have, the category of compact LOTS is an epireflective subcategory of GLOTS.

D. J. Lutzer pointed out to me that for a topological space X, βX is orderable iff X is a sequentially compact generalized ordered space. Noting that 2.17 also holds for generalized ordered spaces, there is a proof of this almost identical to that of Theorem 2.18. Similarly, we also have that if X is a sequentially compact generalized ordered space then $\beta X = BX = uX$.

Let X be a topological space. A point *peX* is called a P-point iff every map f rX-^ 1 is constant in a neighborhood of p. If X is an ordered space then *peX* is a p-point iff no monotone sequence in x\(p) converges to p [G-J, problem 5.0]. This characterization can be extended to generalized ordered spaces- Hence for a topological space X, /3X is orderable iff X is a generalized ordered space and every point of (BX\X is a P-point of BX.

2.20. THEOREM. Let X bg. a. generalized ordered space and let $\bar{\bar{X}}$ bg. nonmeasurable. Then uX c <BX^ and uX = X U T,, where T = {xeRX\x | for every pair of disjoint closed subspaces A,B of X $xecl_{fiX}Ancl_{gx}B$).

Proof. Let X and T be as in the hypothesis. To show uX = XU.T we prove that there is a bijective correspondence between the real free Z-ultrafilters on X and their limits— the points of T, and that every f£C(X,,l) can be continuously extended over X UT. First we show that no real free Z-ultra-filter on X can converge to a point in (SX\(XUT) • Let p£(BX\x. We may assume that p is a left limit point. Suppose there is a real free Z-ultrafilter A_p on X converging to p. First suppose that p is not a P-point. Then there exists an increasing sequence $\{x_n\}_{n=1}^{\infty}$ in X converging to p. Then $p \notin T$ since $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=0}^{\infty}$ are disjoint closed sets in X both of whose closures in BX contain p. Moreover, A_p must contain the collection $\{[x_n, p) \mid n \in N\}$. Hence A_p doesn't have the countable intersection property and therefore, it is hyper-real. Contradiction. Thus if a real free Z-ultrafilter converges to $p \in RX \setminus X$, then p must be a P-point. Also all the points of T are P-points.

Now suppose p is a P-point in $BX \setminus (X \cup T)$. Since $p \notin T$, there exist nonempty disjoint closed sets A and B in X such that $p \in cl_{BX} A \cap cl_{BX} B$. We can construct an increasing net $\{x_\xi \mid \xi < \omega_\alpha\}$ in X converging to p, where ω_α is an initial regular ordinal and for all $\xi < \omega_\alpha$ $x_{2\xi} \in A$ and $x_{2\xi+1} \in B$. (Note for a limit ordinal λ, $2\lambda = \lambda$). Since p is a P-point, $\omega_\alpha > \omega_0$. [The remainder of this paragraph is from the proof in G-H of Theorem 10.3(2)]. For each limit ordinal $\lambda < \omega_\alpha$ the limit of the subnet $\{x_\xi \mid \xi < \lambda\}$ is a cut or left cut u_λ. Then the increasing net $\{u_\xi \mid \xi < \omega_\alpha\}$ of cuts and left cuts, obtained as above, "converge" to p. The intervals $J_\xi = (u_\xi, u_{\xi+1})$ are clopen and their union $J = \bigcup_{\xi < \omega_\alpha} J_\xi$ is a clopen interval with sup p. Hence $J \in A_p$. Consequently if C and D are any two complementary subsets of ω_α then $J_C = \bigcup_{\xi \in C} J_\xi$, $J_D = \bigcup_{\xi \in D} J_\xi$ are clopen and exactly one of them is in A_p. Now denote by $\mathcal{J}$ the set of intervals $\{J_\xi\}_{\xi<\omega_\alpha}$; every subset of $\mathcal{J}$ is of the form $\mathcal{J}_C = \{J_\xi \mid \xi \in C\}$. Define a finitely additive two valued measure m on the family of all subsets of $\mathcal{J}$ by putting $m(\mathcal{J}_C) = 1$ iff $J_C \in A_p$. Since for each $\xi < \omega_\alpha$ $(\omega_\alpha \setminus \{\xi\}) \in A_p$, points have zero measure i.e. $J_\xi \notin A_p$. Moreover $m(\mathcal{J}) = 1$. Since ω_α is nonmeasurable, the measure m cannot be countably additive. Hence there exists a countable family $\{\mathcal{J}_{C_n} \mid n \in N\}$ of subsets of $\mathcal{J}$ of measure 1, whose intersection $\mathcal{J}_{C}$ is of measure zero. Then $J_C \notin A_p$. Therefore $j \setminus J_C \in A_p$. Hence $\{J_{C_n} \cap (j \setminus J_C) \mid n \in N\}$ is a countable family of zero sets of A_p having empty intersection. Thus A_p is hyper-real. Contradiction.

Hence no real free Z-ultrafilter on X converges to any point in $RX \setminus (X \cup T)$.

Before we show that for each point in T there is a unique real free Z-ultrafilter converging to it, we show that any map $f \in C(X, R)$ can be continuously extended to $X \cup T$. Let $p \in T$. First we show f is constant on a neighborhood of p in X, and to do this we first show that f is bounded on a neighborhood of p in X. If not $Z^+_{n+1} = \{x \in X \mid |f(x)| \geq n+1\}$ and $Z^-_n = \{x \in X \mid |f(x)| \leq n\}$ are closed disjoint subsets in X for each $n \in \mathbb{N}$. Since $p \in cl_{\beta X} Z^+_{n+1}$, each $x_n = \sup_{\beta X}(Z^-_n \cap (-\infty, p))$ is less than p. Hence, since p is a P-point, $\sup_{n \in \mathbb{N}}{}_{\beta X}\{x_n\} = x' < p$. Then, since p is a left limit point, (x', p) is a non-empty neighborhood of p in X on which f has the value of ∞ or $-\infty$ which is not in $\mathbb{R}$. Contradiction. Thus, f is bounded on a neighborhood of p. Now we show that f is constant on a neighborhood of p in X. Since f is bounded on a neighborhood of p, $\overline{f([x, p))}$ is compact for $x \in (x', p)$. Hence the nested family $\{\overline{f([x, p))} \mid x \in (x', p)\}$ has a nonempty intersection. Thus there exists $r_p \in \bigcap_{x \in (x', p)} \overline{f([x, p))}$. Moreover, $r_p \in \bigcap_{x \in (x', p)} f([x, p))$, since otherwise we could construct a map unbounded in a neighborhood of p in X. Thus $A = f^{-1}(r_p)$ is a nonempty closed subspace of X and $p \in cl_{\beta X} A$. In addition, for each $n \in \mathbb{N}$, $Z_n = \{x \in (x', p) \mid |f(x) - r_p| \geq \frac{1}{n+1}\}$ is closed in X and disjoint from A. So $p \notin cl_{\beta X} Z_n$. For Z_n empty define $\sup_{\beta X} Z_n = x'$. Then for each $n \in \mathbb{N}$ $\sup_{\beta X} Z_n = x_n < p$. Hence, $\sup_{n \in \mathbb{N}}{}_{\beta X} x_n = x'' < p$, and (x'', p) is nonempty. Thus $f^{-1}((x'', p)) = r_p$. Therefore, f is constant on a neighborhood in X of each point of T, and hence f extends to $X \cup T$.

To finish the proof we show p is the limit in βX of a unique real free Z-ultrafilter in X. Note there is a free Z-ultrafilter converging to p. By the preceding paragraph p is not the limit of a hyper-real Z-ultrafilter in X, since every map is bounded in a neighborhood of p [see G-J]. Suppose A_p and A'_p are distinct Z-ultrafilters in X converging to p. Then there exists disjoint zero sets $Z \in A_p$, $Z' \in A'_p$ and $p \in cl_{\beta X} Z \cap cl_{\beta X} Z'$. Contradiction to $p \in T$. Therefore, each point in T is the limit of a unique Z-ultrafilter in X, in fact, a free real Z-ultrafilter. Hence there is a one-to-one correspondence between the free real Z-ultrafilters on X and their limits, the points of T. Hence $uX = X \cup T$.

2.21. COROLLARY. Let X be a generalized ordered space, and let T be the set of P-points in $\beta X \setminus X$. Then $X \cup T = uX$ iff whenever A and B are disjoint closed subspaces of X, $T \cap cl_{\beta X} A \cap cl_{\beta X} B = \emptyset$.

2.22. Recall that a cardinal $\aleph$ is regular iff it is not the supremum of less than $\aleph$ cardinals, each less than $\aleph$. An ordinal is regular if it is a regular cardinal. Let ω_α be a regular initial ordinal, whose cardinal is $\aleph_\alpha$, where α is an ordinal number. Then a monotone net $\{x_\xi\}_{\xi<\omega_\alpha}$ in a linearly ordered space X is called a Q-net [G-H], if for every nonzero limit ordinal A < to_α , the limit in (BX of the segment $t^x£\}_{\xi<\lambda}$ is in (BX\X. In particular every to_o- sequence and every tog - sequence are Q-nets. If to_α is (non) measurable $\{x_\xi\}_{\xi<\wedge_\alpha}$ is called a (non) measurable Q-net. A point in Bx\x is a Q-point if it is the" limit of a Q-net in X.

Let p£(Bx\x. Then p is a non Q-point iff for every pair A and B of disjoint closed subspaces of X, p \ cl_{oV}A H cl--B. The proof of Theorem 2.20 shows that if there exists a distinct pair A and B of disjoint closed subspaces in X such that $peel_{B}$^A ncl_{BX}B, then p is a Q-point. Conversely, if p is a Q-point then there is an ordinal to_a and a Q-net $(x_*j_{\xi<\omega_\alpha}$ in X converging to p. Let A = $O_2£\wedge<t0_\alpha$ and B = ^2£+l^<t0 $_\alpha$. Then clearly A and B is a pair of distinct disjoint closed subspaces in X and p€cl_{RX}A fl cl^B.

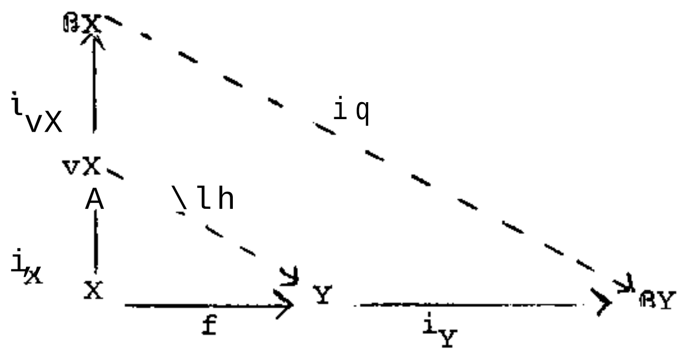
Gillman and Henriksen [G-H, pp. 359-360] proved that if X is a linearly ordered space with no measurable Q-net, then uX = X UT, where T is the set of non Q-points in Bx\x. Hence if $\bar{\bar{X}}$ is nonmeasurable this statement is Theorem 2.20.

Let X be a generalized ordered space. Denote by 0(X) all order preserving maps from X to E. If Y is a generalized ordered space containing X, then X is said to be 0-embedded in Y if every map in 0(X) can be extended to a map in θ(Y). Let us call vX the greatest generalized ordered real-compact extension of X in the sense that vX is the greatest subspace of BX in which X Jjs θ-embedded. Then clearly, vX = X U T, where T is the set of all P-points in ftx\x. Hence, if $\overline{X}$ is nonmeasurable, then uX c vX c (BX. !£, JJQ. addition, the last condition in Corollary 2.21 holds, then uX = vX. Now let aX be the family of maximal filters, described in Theorem 2.9, on the generalized ordered space X. Call a maximal filter in aX real iff it has the countable intersection property. Then there is a one-to-one correspondence between the real maximal filters in aX and their limit points, the points of vX. Moreover, similar to Theorem 2.9, there is an order on this subfamily of aX for which it is isomorphic to vX. Now let JP = R$^{\rho'(X)}$. Then as in 2.6 and Theorem 2.7 we can find an embedding f:X—>P such that cl$_p$f(X) is isomorphic to vX.

We call a generalized ordered space ordered realcompact iff X = vX.

2.23 PROPOSITION. The category of ordered realcompact G LOTS is an epireflective subcategory of G LOTS.

Proof. Consider the diagram where i_{vX}, i_X, and i_Y are



inclusion maps, $f: X \to Y \in$ GLOTS, and Y is ordered and real-compact. Then by 2.19 there exists a unique $q: \beta X \to \beta Y$ such that $q i_{\upsilon X} i_X = i_Y f$. We shall show there exists $h: \upsilon X \to Y$ such that $h i_X = f$. It is sufficient to show that $q i_{\upsilon X}$ maps onto Y. Suppose this is false. Then there exists $x \in \upsilon X$ and $y \in \beta Y \setminus Y$ such that $q i_{\upsilon X}(x) = y$. We may assume that x is a left sided limit point of υX. Then there exists an increasing chain net $\{x_\alpha\}$ in X converging to x, and $q i_{\upsilon X}(\{x_\alpha\})$ is an increasing chain net in Y converging to y. Since y is not a P-point in βY, we can choose an increasing cofinal subsequence $q i_{\upsilon X}(\{x_{\alpha_i}\})$ converging to y. Hence $\{x_{\alpha_i}\}_{i \in N}$ is an increasing sequence in X converging to x. But this is impossible since x is a P-point of υX. Hence $q i_{\upsilon X}$ maps onto Y. So we can define $h: \upsilon X \to Y$ such that for all $x \in \upsilon X$, $h(x) = q i_{\upsilon X}(x)$. Hence $h i_X = f$, and since i_X is an epi, h is unique, and we are done.

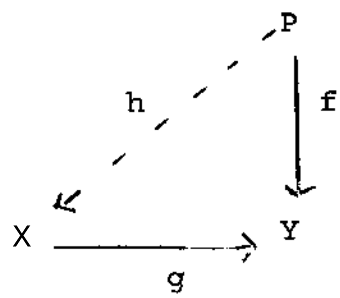
2.24. EXAMPLES. _There exists a topologically realcompact ordered space X such that $\beta X \setminus X$ contains a P-point of βX._ Consider the space ω_1 of all ordinals less than the first uncountable ordinal. For each limit ordinal $\alpha < \omega_1$ replace α by ω_0^*. Call this space A_1. This is the required space since one can easily show $(A_1$ is discrete and the greatest element 1 of Rui_1 is in KJoi$_1$\u£$_1$• Hence,, since $\bar{oc}_1 = K_1$ is a nonmeasurable cardinal, uX = X. Therefore, 1^ uX.

<u>If there exists a measurable cardinal</u>5 <u>then there exists an ordered space</u> X <u>such that</u> uX <u>is not an ordered extension of</u> X. Let 0^ be the space of all ordinals less than the first measurable ordinal. For each limit ordinal $a < cc_m$ replace a by ω_o^* . Call this space $\omega_m^!$ • Then $\omega_m^!$ is discrete of cardinality K_m, the first measurable cardinal. The greatest point 1 of $B\omega_m$ is in ^^iA6^ an<^ for any subset $A \subseteq X$ whenever $\bar{K} = K_m$, then $1 \in cl_{o^*X}A$. Then there is a free real Z-ultrafilter in o^ converging to 1 in ^^- However, the map $f \in O(\omega_m^!)$, where f is 0 on the points of ω^m from ω^m and f is 1 otherwise, has no extension to ω^m U {1}. Hence 1^ uX, and uX is not an ordered extension of X.

2.24. Note that although an extension Y of an ordered space X may not be an ordered extension, Y may still be orderable. A new ordering may make Y ordered while inducing an unordered generalized order on X. For example let X= (0,1)+(2,3] and Y= [0,1] + (2,3]. Reorder Y as (2,3] 4- [0,1]. Then with the new ordering Y is ordered and X is the unordered subspace (2,3] + (0,1) .

SECTION 3. Projectives.

3.1. Let C be a category and let P be a class of morphisms in C. An object P of C is called p-projective iff for each morphism $f:P \to Y$ and for each P-morphism $g:X \to Y$ there exists a morphism $h:P \to X$ such that the diagram



commutes. A morphism f is called P-essential (P^*) iff $f \in P$ and $fg \in P$ implies $g \in P$ for each morphism $g \in C$ $f:P \to X$ is called a P-protective cover of X iff P is P-projective and $f \in P^*$.

Note that P^* is closed under compositions. Since the essential morphisms and projective covers generated by P and its closure under compositions are identical, it is convenient to choose P to be closed under composition.

It will be shown in this section that in LOTS if P is the class of closed onto maps P^* is the class of irreducible maps. Also we shall show that for every $X \in$ LOTS there exists a unique P-projective cover $IL_X: <kX \to X$, but that no class intersecting the complement of P has this property.

3.2. PROPOSITION. *Let* X *and* Y *be ordered spaces, and let* $f: X \to Y$ *be an onto order preserving function. Then* f *is continuous iff* $f^{-1}(y)$ *is closed for every* $y \in Y$.

Proof. Necessity is obvious. Now assume $f^{-1}(y)$ is closed for every $y \in Y$. It is sufficient to show that $f^{-1}((-\infty, y])$ and $f^{-1}([y, \infty))$ are closed for each $y \in Y$. Let $y_0 \in Y$. Now $f^{-1}((-\infty, y_0]) = f^{-1}((-\infty, y_0)) \cup f^{-1}(y_0)$, which is a ray in X from the nonempty closed convex set $f^{-1}(y_0)$ to $-\infty$. Hence $f^{-1}((-\infty, y_0])$ is closed. Similarly, $f^{-1}([y_0, \infty))$ is closed.

3.3. PROPOSITION. *Let* $f: X \to Y$ *be an onto map in* LOTS. *If* $f^{-1}(y)$ *is compact for every* $y \in Y$, *then* f *is a closed map*.

Proof. We prove the contrapositive. Assume f is not a closed map. Then there exists A closed $\subset X$ such that $f(A)$ is not closed. Hence there exists $y \in Y$ such that $y \in \overline{f(A)} \setminus f(A)$. We may assume that y is a right limit point of $\overline{f(A)}$, i.e. $y \in \overline{(y, \infty) \cap f(A)}$. Hence, $f^{-1}(y, \infty)$ has no inf, and, therefore, $f^{-1}(y)$ has no sup. Thus $f^{-1}(y)$ is not compact.

3.4. For topological spaces X and Y recall that a map $f: X \to Y$ is *irreducible* iff f is onto and for all proper closed subspaces A in X, $f(A) \neq Y$. An ordered space Y is called an <u>ordered absolute</u> iff whenever $f: X \to Y$ is irreducible in LOTS, then f is an isomorphism.

3.5. PROPOSITION. *Let $f: X \to Y$ be an onto map in LOTS. Then f is irreducible iff for all $y \in Y$ (1) $|f^{-1}(y)| \leq 2$ and (2) if $|f^{-1}(y)| = 2$, then $f^{-1}(y)$ is a two sided jump in X.*

Proof. Let (1) and (2) hold. Suppose there exists a proper closed subspace A in X such that $f(A) = Y$. Then there exists $x \in X \setminus A$ and $x' \in A$ such that $f^{-1}f(x) = \{x, x'\}$, which is a two sided jump in X. Then there exists $x_0 < x$ such that $(x_0, x]$ is a neighborhood of x contained in the open set $X \setminus A$. Necessarily, $|(x_0, x]| \geq \aleph_0$. Hence, since (1) holds, there exists $y \in Y$ such that $f^{-1}(y) \subset (x_0, x]$. This is a contradiction since $(x_0, x] \subset X \setminus A$, but $f(A) = Y$. Therefore, f is irreducible.

Conversely, let f be irreducible. Suppose there exists $y \in Y$ such that $|f^{-1}(y)| > 2$. Then there exists a proper open subinterval $A \subset f^{-1}(y)$. Hence $X \setminus A$ is closed and $f(X \setminus A) = Y$. Contradiction. So (1) holds.

Now let $|f^{-1}(y)| = 2$ for some $y \in Y$. Then there exists $x, x' \in X$ such that $f^{-1}(y) = \{x, x'\}$. Obviously $\{x, x'\}$ is a jump. Suppose it is not a two sided jump. Then one of the points, say x, is isolated. So $X\setminus\{x\}$ is closed, and $f(X\setminus\{x\}) = Y$. Contradiction. Therefore, (2) also holds.

3.6. Note that *if* $f: X \to Y$ *is irreducible in* LOTS, *then for* $y \in Y$ *whenever* $|f^{-1}(y)| = 2$, y *is a two sided limit point*. *Whenever* $|f^{-1}(y)| = 1$, *both* y *and* $f^{-1}(y)$ *are either isolated points*, *left limit points*, *right limit points*, *or two sided limit points in* Y *and* X *respectively*. Note also that Propositions 3.3 and 3.5 imply f *is a closed map*.

3.7. THEOREM. (Fedorchuk). *Let* X *be an ordered space*. Then there exists $\pi_X: \alpha X \to X \in$ LOTS such that αX is an *ordered absolute and* π_X *is irreducible*. *If* $f: X \to Y$ *is irreducible in* LOTS, *then there exists an isomorphism* $h: \alpha X \to \alpha Y$ *of ordered* absolutes such that $f\pi_X = \pi_Y h$.

Proof. Let X_a be the set of all two sided limit points of X. Consider the ordered space αX obtained by replacing each point $x \in X_a$ by an ordered pair of points $\{x_0, x_1\}$, which is clearly a two sided jump in αX. Define $\pi_X: \alpha X \to X$ as follows, $\pi_X(x) = x$ for $x \in X\setminus X_a$ and $\pi_X(x_0) = \pi_X(x_1) = x$ for $x \in X_a$.

Clearly π_X is order preserving and onto. So by Proposition 3.2, π_X is continuous, and so by Proposition 3.5, π_X is irreducible.

Now let $g: Z \to \alpha X$ irreducible in LOTS. Since αX has no two sided limit points, then by 3.6, g must be one-to-one. Then since g is onto, it is an isomorphism. Hence αX is an ordered absolute.

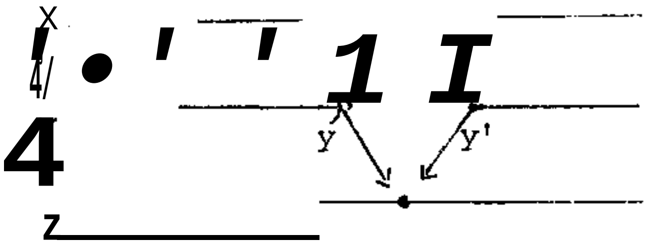
The rest of the proof will be given in 3.17.

3.8. Let $f: X \to Y$ be a non-closed map in LOTS. Then there is a clopen ray R in X with no initial point whose image in $f(X)$ is a ray whose initial point is in $cl_Y(f(R)) \setminus f(R)$.

3.9. We now begin to show that the projectives are the ordered absolutes for the class P defined below.

LEMMA. _In LOTS let P be the closed onto maps. Then the P essential morphisms (P^*) are the irreducible maps._

Proof. First note that P is closed under composition. Let $f: Y \to Z$ be irreducible in LOTS, $g: X \to Y \in$ LOTS, and $fg \in P$. Then by 3.6, f is closed, and hence $f \in P$. We must show $g \in P$. First suppose g is not onto. Then there exists $y \in Y \setminus g(X)$.

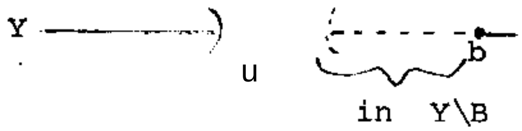


Since f is irreducible and fg is onto, $f^{-1}f(y)$ is a two sided jump consisting of y and some $y' \in Y$, and hence $f(y)$ is a two sided limit point. We may assume that $y < y'$. Now $g^{-1}((-\infty,y]) = g^{-1}((-\infty,y))$ is clopen, and $fgg^{-1}((-\infty,y)) = (-\infty,f(y))$. However, $f(y) \in \overline{(-\infty,f(y))} \setminus (-\infty,f(y))$, which implies fg is not closed. Contradiction. Hence g is onto.

Now suppose g is not closed. Then there exists a ray, say $(-\infty,u)$, where u is a cut, and there exists $y \in Y$ such that $y \in \overline{g((-\infty,u))} \setminus g((-\infty,u))$. Then since y is a left limit point and f is irreducible, $f(y) \in \overline{fg((-\infty,u))} \setminus fg((-\infty,u))$. Thus since $(-\infty,u)$ is closed, fg is not closed. Contradiction. Hence g is closed. So $g \in P$, and hence $f \in P^*$.

Conversely, let $f: Y \to Z \in P$ not be irreducible. Then there is a proper closed subspace B in Y such that $f(B) = Z$. We will find an ordered space $B' \supseteq B$ such that B' is also a proper subset of Y. If B is ordered let $B' = B$. If B is not ordered, then there exists a half cut, say a left cut $\{C,D\}_B$.

Let b = min D. Since B is closed, C has no sup in Y. Hence $C^!$ = {yeY|y≤c for some ceC} and D^f = X\$C^!$ determine a cut

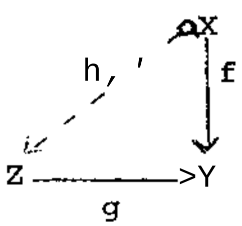


u in Y. So u < b, and ($\overline{u,b}$) ;> ^o in Y. Hence there exists y,$y^!$,y^{ft}eY\B such that u<y<y"<y'<b. Let $B^!$ = (-GD ,y] U [y^1 , OD) . Then since y"eB$^!$, B^f is a proper closed ordered subspace of Y containing B. Hence f($B^!$) = Z.

Now let i:$B^!$—->B be the inclusion map. Thus fieP, since B^1 is closed. However, i is not onto. So i^^* an<3 hence **f 4 P***.

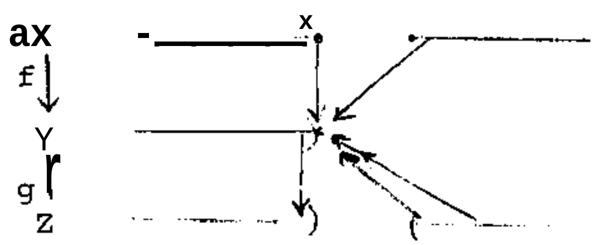
3.10. THEOREM. Let X 8 LOTS. Then IL^s aX—^ x is a P-proiective cover, where the P-morphisms are the closed onto maps.

Since IL_X is irreducible, then by Lemma 3.9, IL_X-€P*. So we need only show that *{XX* is P-projective. So let f: αX→Y∈LOTS, and let g:Z—>YeP



Then we must find $h: X \to Z \in LOTS$ which makes the diagram commute.

Claim: If $x \in X$ is a left (right) limit point such that $\min(\max) f^{-1}f(x) = x$, then $\min(\max) g^{-1}f(x)$ exists and is a left (right) limit point. Let x be a left limit point in X such that $\min f^{-1}f(x) = x$. Then



$f(x) \in \overline{(-\infty, f(x))} \setminus (-\infty, f(x))$. Thus $f(x)$ is a left limit point.

Suppose $g^{-1}f(x)$ has no minimum. Then since g is continuous, $g^{-1}f(x)$ has no inf in Z. Hence $(-\infty, g^{-1}f(x)) = \{c \in Z \mid c < z \text{ for all } z \in g^{-1}f(x)\}$ has no sup, and it is non empty since g is onto. In addition, since g is onto,

$$f(x) \in \overline{g(-\infty, g^{-1}f(x))} \setminus g(-\infty, g^{-1}f(x)),$$

i.e. g is not closed. Contradiction. Hence $g^{-1}f(x)$ has a min.

Moreover, since $f(x)$ is a left limit point and g is onto, then $\min g^{-1}f(x)$ is a left limit point. Use the dual argument if x is a right limit point. Hence the claim is proved.

Now we begin to construct h: <xX~—*Z. Decompose X into the collection (f~^{1}(y) |yef(cxX)}. For *yef(aX)* such that $\overline{\overline{f"^{1}(y)}}$ >_ 2 , there is a gap u_y in f"^{1}(y), which is either a cut or a jump since <xX has no two sided limit points. Let $u_y^{\sim}$ = {x€f"^{1}(y) |x< u_y}, and let u_y^{*} = {xef~^{1}(y)|x>u_y}. Now for each gef(<xX), let z_y'' = inf g~^{1}(y) if it exists, and let z_y^{+} = sup *g*~^{1}(y) if it exists. Choose a point $z_y$eg^{-1}(y) . Define h:OX—> Z as follows. For yef(*X) and $\overline{\overline{f""^{1}(y)}}$ $\geq$ 2 then

(1) If both sup and inf g~^{1}(y) exists, then for all xeu$''_y$ h(x) = $z_y^{\sim}$ *s* for all xcu^ h(x) = z^ .

(2) If inf of g~^{1}(y) exists but sup doesn't exist, then for all xef~^{1}(y) h(x) = $z_y^{\sim}$.

(3) If sup of g~^{1}(y) exists but inf doesn't exist, then for all x€f"^{1}(y) h(x) = z_y^{+} .

(4) If neither sup nor inf of *g*~^{1}(y) exists, then for all x€f"^{1}(y) h(x) = z_y.

For *ye£(o&)* and $\overline{\overline{f\sim^{1}(y)}}$ = 1, there exists a unique X G & X such that f(x) = y. Then

(1) If x is a left sided limit point, then by the claim *?T*$_y$ exists- So h(x) = $z_y^{\sim}$

(2) If x is a right sided limit point, then by the claim z_y exists. So $h(x) = z_y$.

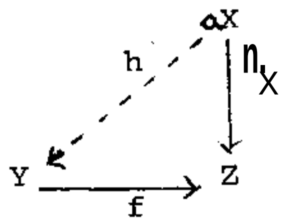
(3) If x is isolated, $h(x) = z_y$.

This definition is complete and the function is well defined since αX has no two sided limit points.

Clearly h is order preserving and $gh = f$. To prove h is continuous it is sufficient to show that whenever a monotone, say increasing, chain net $\{x_\alpha\}$ in αX converges to $x \in \alpha X$, then $h(\{x_\alpha\})$ converges to $h(x)$. If x is not a left limit point, then $\{x_\alpha\}$ eventually is equal to x, and hence $h(\{x_\alpha\})$ is eventually equal to $h(x)$. If x is a left limit point and $\min f^{-1}(y) = x$, where $y = f(x)$, then by the claim z_y^- exists, is a left limit point, and $h(x) = z_y^-$. Clearly $f(\{x_\alpha\})$ is an increasing chain net converging to y. Thus since g is onto, $g^{-1} f(\{x^\alpha\})$ is an increasing chain net of convex sets converging to z_-^y, and hence $h(\{x^\alpha\})$ converges to $z_-^y = h(x)$. If $x \neq \min f^{-1}(y)$, where $y = f(x)$, then $f^{-1}(y) \geq 2$, and $x \in U_+^y$ or $x \in U_-^y$. So if $x \in U_-^y(U_+^y)$, then $\{x^\alpha\}$ is eventually in $U_-^y(U_+^y)$. Hence $h(\{x^\alpha\})$ is eventually equal to $h(x)$. Dually, we can show that if $\{x^\alpha\}$ is a decreasing chain net converging to $x \in \alpha X$, then $h(\{x^\alpha\})$ converges to $h(x)$. Hence, h is continuous,, and h:<xX—»ZeLOTS. Therefore <x X is P-projective, and IL_X.:fcX—>X is a P-projective cover.

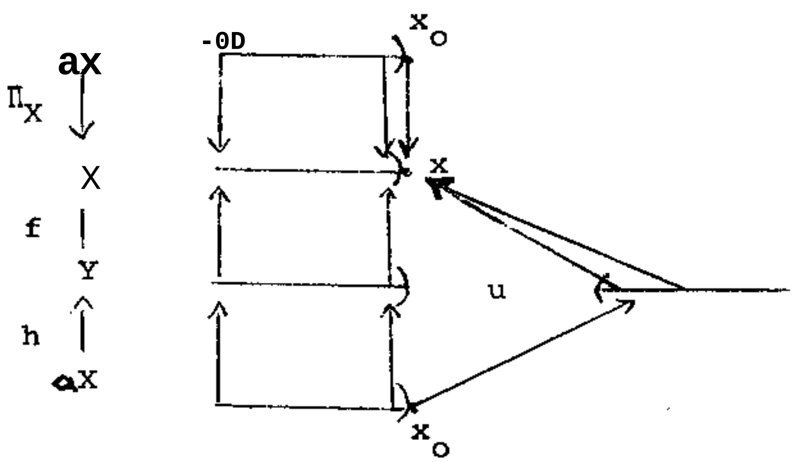
3.11. PROPOSITION. Let TI_X: t\X-> X e LOTS and let P be a class of morphisms for which n_v:OsX—T> X JL'S a. P-projective cover. If f:Y—^*XeP,* then f is a closed onto map.

Proof. Consider the diagram below. Let TL_X.:aX—>X be a



P-projective cover,, and let f:X—>YGP. Then there exists h:ftX—>Y such that fh = TL_X . Then since II_X is onto^ f must be onto.

Now suppose f is not closed. Then there exists a ray, say • (u,aD),, where u is a cut in Y and there exists a point xeX such that f((-0D,u)) = (-QD,X) and xe $\overline{(-OD ^X)}$\(-QD ^X) .



Since f must be onto,, $f^{-1}(x)$ is nonempty and $\inf f^{-1}(x) = u$. Since $\overline{\overline{f^{-1}(x)}} \le 2$, $\inf f^{-1}(x) = x_o$ exists, and it is a left limit point (since x is such a point). However, $h(x_o) > u$ and $h((-\infty, x_o)) \subseteq (-\infty, u)$, which implies h is not continuous. Contradiction. Hence' f must be closed, and the Proposition is proved.

3.12. From now on let P be the closed onto maps in LOTS.

We will now look at connectivity properties of ordered absolutes.

An ordered space X is called ordered extremely disconnected if for any open interval V c X, $\overline{V}$ is open. Obviously,, if X is extremely disconnected, then X is ordered extremely disconnected.

3.13. EXAMPLE. There is an ordered space X which is ordered extremely disconnected but not extremely, disconnected. Let $X = \omega_o + 1$. Then clearly the closure of any open interval in X is open. However $V = \{2n \mid n \in \omega_o\}$ is an open set, but $\overline{V}$ is not open in X.

3.14. THEOREM. (Fedorchuk). Let X be an ordered space. Then the following are equivalent:

(1) X *is an ordered absolute*.

(2) X *has no two sided limit points*.

(3) X *is order extremely disconnected*.

Proof. First we show that not (3) implies not (2) implies not (1). So assume X is not order extremely disconnected. Then there exists an open interval $V \subset X$ and $x \in X$ such that $x \in \overline{V} \cap \overline{(X \setminus \overline{V})}$. Hence x is a two sided limit point. By splitting x in two, we obtain a nonisomorphic irreducible map onto X. Thus X is not an ordered absolute.

Now we show that not (1) implies not (2) implies not (3). So assume X is not an ordered absolute. Then since αX is obtained by splitting the two sided limit points in X, then there exists a two sided limit point $x \in X$. Hence (x,∞) is an open interval in X, but $\overline{(x,\infty)}$ is not open. Therefore, X is not ordered extremely disconnected.

3.15. The next two propositions give equivalences for P-projectives and P-projective covers, most of which are true in any category. With slight alteration these propositions were stated for another topological category with perfect onto maps as P-morphisms by H. Herrlich [H_2, Theorem 4.3].

PROPOSITION. Let X 8 LOTS. Then the following are equivalent:

(1) X jLs p-projective.

(2) Any P-morphism f:Y—>X is a retraction

(3) For all feP* f:Y—>X, f is an isomorphism, i.e. X jLs an ordered absolute.

(4) X is ordered extremely disconnected.

Proof. For all categories (1) is equivalent to (2) which in turn implies (3). By Theorem 3.14,, (3) is equivalent to (4), and by Theorems 3.7 and 3.10,, (3) implies (1).

3.16. PROPOSITION. Let f:P—» XeP. Then the following are equivalent;

(1) f:P—>X is a. P-projective cover.

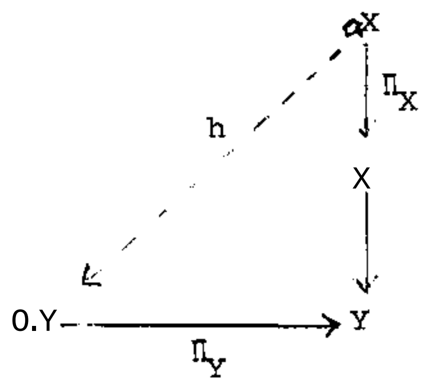
(2) f€P* and for all g such that fgeP, g i§. an isomorphism.

(3) P JLS. P-protective, and if g^heP such that gh = f and the domain of g jLs P-projective, then h jjs joi isomorphism

(3*) P is P-projective and if g^h e LOTS such that gh = f, h JLS onto, and the domain of g is P-projective, then h is ail isomorphism.

Proof. In every category (1), (2), and (3) are equivalent. Obviously (3*) implies (3). To finish the proof we show (1) implies (3*). So let f:P—> X be a P-projective cover,, h:P—^Y an onto map, Y P-projective, g:Y—> X, and gh = f. Since f is irreducible, it is at most a two-to-one map, so h is at most a two-to-one map. Now it is sufficient to show that h is a one-to-one map. Suppose this is false. Then there exists p,p^T€P such that f(p) = gh(p) = gh(p^T) = f(p). Hence p and p$^!$ form a two sided jump, and since g is at most a two-to-one map, h(p) is a two sided limit point. But this is impossible since P is an ordered absolute. Thus (1) implies (3*).

3.17. Note that in Proposition 3.16 that (3*) implies that geP. Also Lemma 3.9 and Proposition 3.15 imply the last part of Theorem 3.7. For let f:X—^ YeP*. Since n_Y eP* c P and αX is P-projective, there exists



h:AX—>*T such that II_Y h = fll_X . Thus h is in P*, since P* is closed under composition. Then it is easy to show heP*. (In fact in any category if gk^P* and g€P*> then k€P*). Then since <XY is an ordered absolute and h is irreducible, h is

an isomorphism. Hence the last part of Theorem 3.7 is proved.

Note that the proof above shows that any f:X—>YeP can be lifted to f' : OX—> <xY.

Also note that for each X S LOTS,, the P-protective cover H_X :<XX—>X JLS unique up to isomorphism, since in any category the projective covers are unique.

3.18. In the last part of this section we look at more properties of ordered absolutes. In particular we look at the importance of minimal ordered spaces and the functor (B in relation to ordered absolutes.

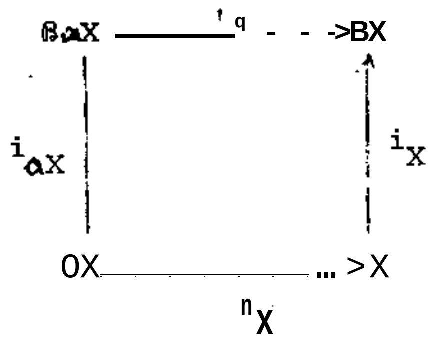
THEOREM. (Fedorchuk) . Let X be an ordered absolute and bx an ordered compactification of X. Then bx i^ an ordered absolute iff bx = ftx.

Proof. Necessity. Let bx be an ordered absolute. Clearly the canonical map f:(BX—>bX is irreducible. So f is an iso, and hence (BX = bX.

Sufficiency. Let bX = (BX. Since X is an ordered absolute, it has no two sided limit points. Since (BX is constructed by the addition of an ordered pair of points to each cut of X, (BX has no two sided limit points either. Hence (BX is an ordered absolute.

3.19. COROLLARY. (Fedorchuk). <xBX = BAX.

Proof. By Theorem 3.16 B<xX is an order absolute. Now there exists a unique q:B&X—^ BX such that qi$_{\alpha X}$ = ^$_X$Π$_X$.



To prove the Corollary it is sufficient to show qcP*. Since BαX is compact, q is closed. Thus since q(B<xX) 2 X as a subspace of BX, then BX 2 q(fefcX) = q(BoX) 2 X = lBX- So q is onto. Since <xX is dense in S&X, qi v (A X) = X is a subspace of BX, and qi$_{\alpha X}$ is at most a two-to-one map, then no point of BGJX\AX maps to a point of X in BX, i.e., q(Be\X\<*X) = BX\X. Hence if A is closed in B<*X and q(A) = BX, then A *ZD* X. So Bx\X => A => X = BaX. Hence qeP.^. Since P-projective covers are unique BAX = <XBX.

3.20. B.V.S. Thomas (T) has formed categorical proofs of the last two theorems using definitions that I do not wish to consider in this paper.

3.21. THEOREM. (Fedorchuk). *If* X *and* Y *are ordered minimal spaces and* αX *is isomorphic to* αY, *then* X *is isomorphic to* Y.

Proof. Let $h: \alpha X \to \alpha Y$ be an isomorphism. We will show that $f = \Pi_Y h \Pi_X^{-1}$ is an isomorphism from X to Y. First we show that f is single valued. Let $x \in X$. If $|\Pi_X^{-1}(x)| = 1$, then $|f(x)| = 1$. If $|\Pi_X^{-1}(x)| = 2$, then $\Pi_X^{-1}(x)$ is a two sided jump in αX. Hence $h\Pi_X^{-1}(x)$ is a two sided jump in αY. Then if $\Pi_Y h \Pi_X^{-1}(x)$ consists of two points, it is a two sided jump in Y, which is impossible, since Y is minimal. Hence $|f(x)| = 1$, and so f is single valued. In a similar manner it is proved that f^{-1} is single valued, i.e., f is one-to-one, by using the fact that h^{-1} is an isomorphism. Since Π_Y, h, and Π_X^{-1} are onto, then f is onto. Hence f is an isomorphism, and X is isomorphic to Y.

BIBLIOGRAPHY

fB] Bennett, H.R., "A note on point-countability in linearly ordered spaces[11], (To appear in Proc. Amer. Math. Soc.)

[C] Cech, E., Topological Spaces, Academia (Czechoslovak Acad. Sci.), Prague, 1966.

[F_1] Fedorchuk, V., "Ordered spaces", Dokl. Akad. Nauk SSSR 169, £, (1966) 777-780.

[F_2] ______________, "Some problems in the theory of ordered spaces", Sibirskii Mathematicheskii Zhurmal 10, JL, (1969) 172-187.

[G-H] Gillman, L., Henriksen, M., "Concerning rings of continuous functions", Trans. Amer. Math. Soc. 7J7., (1954) 340-362.

[G-J] Gillman, L., and M. Jerison, Rings of Continuous Functions, Van Nostrand Co., Princeton, New Jersey, 1960.

[H_1] Herrlich, H., Topologische Reflexionen und Coreflexionen, Springer-Verlag, Berlin-Gottingen-Heidelberg, 1968.

[H_2] ______________, "Categorical Topology", Genral Topology and Its Applications JL, (1971) 1-15.

[M] Mishchenko, A., "Spaces with a point-countable basis", Dokl. Akad. Nauk SSSR 144, J5, (1962) 985-988.

[T] Thomas. B. V. S., "Projectives", (Personal Communication).