RADIAL AVERAGING OF DOMAINS, ESTIMATES FOR DIRICHLET INTEGRALS AND APPLICATIONS

by

Moshe Marcus

Research Report 71-36

August, 1971

/nlc

8/10/71

ABSTRACT

RADIAL AVERAGING OF DOMAINS, ESTIMATES FOR DIRICHLET INTEGRALS AND APPLICATIONS

by

Moshe Marcus

Let & = $\{D_1,...,D_n\}$ be a family of domains in the plane, containing the origin. We define a radial averaging transformation R_A on S by which we obtain a starlike domain D^*. When & is such that the domains $D_1,...,D_n$ are obtained from a fixed domain D by rotation or reflexion, ft. becomes a radial symmetrization. One of the results we present is an inequality relating the conformal radius of D to the conformal radii of $D_1,...,D_n$ at the origin. This result includes, as particular cases, the radial symmetrization results of Szego [11] (for starlike domains), Marcus [7] (for general domains) and Aharonov and Kirwan [1]. The inequality for the conformal radii is obtained via an inequality for conformal capacities, which seems to be of independent interest.

A number of applications in the theory of functions are discussed. Here we introduce a definition of a class of functions {f}, analytic in the unit disk $|\xi| < 1$, which includes the Bieberbach-Eilenberg functions and some other classes of functions considered in the literature. For this class we obtain the estimate $|f'(0)| \le 1$ which is sharp.

Other applications concern certain geometric features of the domain D_f obtained as the image of $|\xi| < 1$ by an analytic function $z = f(\xi)$.

RADIAL AVERAGING OF DOMAINS, ESTIMATES FOR DIRICHLET INTEGRALS AND APPLICATIONS

by

Moshe Marcus

Introduction.

Let D be a domain in the complex plane z, containing the origin. Let o_φ denote the ray issuing from the origin with argument cp. Then, we denote by R(cp;D) the measure of a_φ n D, this measure being defined on the basis of the logarithmic metric, ds = |dz|/|z|.

Let & = $\{D_1, \ldots, D_n\}$ be a family of domains containing the origin. We define a transformation ft_A acting on families of domains [&} , such that ft. : £ —»D , where D is a starlike domain. D is obtained from & by means of a weighted geometric average of the quantities R(<p;D.), (j = l,...,n). The weights are determined by the set A = {a.}?* This transformation is called a radial averaging transformation.

The transformation ft. is extended in a natural manner to families of condensers {C.}[1]?., such that the origin is a point of potential 1 for each condenser in the family. We denote $R_A(\{Cj\})$ = C*.

With this notation we obtain the following result:

$$I(C^*) \le \sum_{j=1}^{n} a_j \, I(C_j)\,,$$

where I(C) denoted the capacity of the condenser C.

From this inequality we derive an inequality relating the conformal radius of D^* (with respect to the origin) to the conformal radii of D^...,D_n.

In the case that D_1..,...,D_n are obtained from a fixed domain D by simple transformations, such as rotations or reflexions with respect to a line through the origin, the radial averaging transformation becomes a radial symmetrization of D.

The symmetrization result thus obtained, includes as particular cases the radial symmetrization results of Szego [11] (for star-like domains), Marcus [7] (for general domains) and Aharonov and Kirwan [1].

A result concerning a process of continuous symmetrization, and a result on a symmetrization based on an integral averaging, are also obtained.

Finally, these results are applied to certain problems in the theory of functions. Here we introduce a definition of a class of functions {f}, analytic in the unit disk ||| < 1, which includes the Bieberbach-Eilenberg functions as well as some other classes of functions considered in the literature. For this class we obtain the estimate $|f'(0)| \leq 1$, which is sharp.

Other applications concern certain geometric features of the domain D_f obtained as the image of the unit disk by an analytic function f.

The plan of the paper is as follows:

In Section 1, we discuss a linear averaging transformation related to R_A and obtain certain integral inequalities.

In Section 2 we obtain the basic results concerning capacity and conformal radius in relation to radial averaging transformations.

In Section 3_5 various symmetrization results are obtained.

In Section 4 the extension of the Bieberbach-Eilenberg class of functions, mentioned above, is discussed.

In Section 5 some additional applications are considered.

The radial averaging transformation presented in this paper is based on the logarithmic metric. Similar transformations based on various other metrics are discussed in [2], where these transformations are considered also in higher dimensional spaces.

The author wishes to thank Professor Nehari for a number of stimulating conversations concerning this paper.

§1. Linear Averaging Transformations.

Let n be a set in the plane (x,y). We denote:

(1.1) $M x_o, n) = meas.((x = x_Q\} nfl)$,

the measure being Lebesgue measure.

Definition 1.1. Let f be a function defined in the half strip $M = \{(x,y)|0 \le x \le 1, 0 \le y]$• We shall say that feB(M) if:

(i) feC(M)

(ii) $0 \le f \le 1$ in M

(iii) On any half line $\{(x_Q,y)|0 \le y\}$, such that $0 \le x_Q \le 1$, f obtains every value A in the open interval (0,1), at least once, but not more than a finite number of times.

(iv) $\lim_{y -*-+oo} f(x,y) = 1$, uniformly with respect to x, $0 \le x \le 1$.

For any real function f defined in M, we denote:

(1.2)
$$n_A(f) = \{(x,y) jf(x,y) < A\} \; 0 \; M$$
$$\Omega'_\lambda(f) = \{(x,y) |f(x,y) \le A\} \; n \; M$$
$$\Omega_{\mu,\lambda}(f) = \{(x,y) |\mu < f(x,y) < \lambda\} \cap M = Q_A(f) - \Omega'_\mu(f).$$

For feB(M) we denote:

(1.3)
$$<t(x,A;f) = \text{^}(x,n_{A'}(f)), \quad 0 < A \le 1$$
$$\ell(x,O;f) = \ell(x,\Omega'_O(f)).$$

Definition 1.2. Let 55 = {i^,...,^} c B(M) and let $A = \{a_j\}_1^n$ be a set of positive numbers such that $\sum_{j=l}^{n} a_J = 1$. Set:

(1.4) $\ell^*(x,\lambda) = \sum_{j=1}^{n} \ell(x,\lambda;f_j)$, $(0 \le x \le 1, 0 \le \lambda \le 1)$.

Then for $(x,y) \in M$ we define

$$(1.5) \qquad f^*(x,y) = \mathcal{L}_\lambda(\mathfrak{F}) = \begin{cases} 0 & \text{if } 0 \le y \le \ell^*(x,0) \\ \lambda & \text{if } y = \ell^*(x,\lambda), \quad 0 < \lambda < 1 \\ 1 & \text{if } \ell^*(x,1) \le y . \end{cases}$$

Note that for every fixed x, $0 \le x \le 1$, $\ell^*(x,\lambda)$ is a strictly monotonic increasing function of λ, $(0 \le \lambda \le 1)$. Hence f^* is well-defined in M.

We now prove:

Lemma 1.1. Let $\mathfrak{F}$ and λ be as in Definition 1.2. Then $f^* \in B(M)$. If in addition f_j is Lipshitz in $\Omega_{a,b}(f_j)$, $j = 1,\ldots,n$, (where a, b are fixed numbers, $0 < a < b < 1$), then f^* is Lipshitz in $\Omega_{a,b}(f^*)$.

Proof. It is easily verified that for $\lambda > 0$ the set $\Omega_\lambda(f^*)$ is open (relative to M) and that for $0 \le \lambda < 1$ the set $\bar{\Omega}_\lambda(M, f^*)$ is compact. This implies the continuity of f^* in M and the fact that f^* satisfies condition (iv). It is obvious that f^* satisfies also conditions (ii) and (iii). Hence $f^* \in B(M)$.

We proceed now with the proof of the second assertion of the lemma. To simplify the notation we set $\ell(x,\lambda;f_j) = \ell_j(x,\lambda)$. By our assumption, there exists a constant k such that:

(1.6) $|f_j(P) - f_j(P')| \le k|P-P'|$, $\forall P,P' \in \Omega_{a,b}(f_j)$, $j = 1,\ldots,n$

where $|P-P'|$ denotes the distance between the two points.

Let $a < a' < b' < b$. We shall show that f^* satisfies a Lipshitz condition with constant k in $\Omega_{a',b'}(f^*)$.

Let δ_j denote the distance between $\Omega_{a',b'}(f_j)$ and the boundary of $\Omega_{a,b}(f_j)$. Set $\delta_0 = \min(\delta_1, \ldots, \delta_n)$. (Note that $\delta_0 > 0$.)

We now keep j fixed. Let $P_1 = (x_1, y_1) \in \Omega_{a',b'}(f_j)$ and $f_j(P_1) = \lambda_1$. Denote by $K_\delta(P_1)$ the open disk of radius δ, centered at P_1. If $0 < \delta < \delta_0$, then (by (1.6)):

$$(1.7) \qquad f_j(P) < \lambda_1 + k\delta, \qquad P \in K_\delta(P_1) \cap M.$$

Hence, if $|x_2 - x_1| < \delta < \delta_0$ ($a < x_2 < b$ and $\lambda_1 + k\delta \le \lambda_2 < b$), then

$$(1.8) \qquad \psi_j(x_2, \lambda_2) \ge \psi_j(x_1, \lambda_1) + [\delta^2 - (x_2 - x_1)^2]^{1/2}.$$

Since this holds for every j, we obtain (under the same assumptions):

$$(1.9) \qquad \psi^*(x_2, \lambda_2) \ge \psi^*(x_1, \lambda_1) + [\delta^2 - (x_2 - x_1)^2]^{1/2}.$$

From (1.9) it follows that:

$$(1.10) \qquad |f^*(P) - f^*(P')| \le k|P - P'|, \qquad \forall P, P' \in \Omega_{a',b'}(f^*).$$

Indeed, if this is not true, there exist points $P_i = (x_i, y_i)$, $(i = 1, 2)$, in $\Omega_{a',b'}(f^*)$ such that:

$$(1.11) \qquad |P_1 - P_2| = \delta < \delta_0, \qquad |f^*(P_1) - f^*(P_2)| > k\delta.$$

Suppose $f^*(P_1) < f^*(P_2)$; then $f^*(P_1) + k\delta < f^*(P_2)$. Choose λ_1, λ_2 so that $f^*(P_1) < \lambda_1$, $f^*(P_2) > \lambda_2$ and $\lambda_1 + k\delta < \lambda_2$. Then, by the definition of f^* we have $y_1 < \psi^*(x_1, \lambda_1)$ and

y_2 > £*(x_2,A_2). On the other hand inequality (1.9) holds for these values of $^{X}T \geq {}^{X}{}_{2} > \wedge I * \wedge 9$' Hence we obtain:

$$^{y}2 > {}^{y}l + t^{62} - ({}^{x}i - {}^{X}2)^{2]1/2} \quad i\_e* \quad I^{P}1"^{P}2^{(} > {}^{5}{}'$$

which is a contradiction to (1.11).

Definition 1.3. Let feB(M) and denote:

$$\Omega(f) = \Omega_1(f) - \Omega_0'(f) = \{(x_5y) | 0 < f(x,y) < 1\} \text{ PI } M;$$

$$Y_A(f) = \{(x,y) | f(x,y) = A\} \text{ 0 } M, \quad 0 < A < 1.$$

Suppose that f€C^1(Q(f)). Let P_Q = (x_Q,y_0) be an interior point of Cl(f) and f(P_Q) = A_Q. We shall say that P_Q is a regular point of f, if Sf/5y *j4* 0 at all the points of the set $Y_{\wedge O}$x PI {x = x_Q} and if this set is contained in the interior of fi(f). Otherwise we shall say that P_Q is a critical point and A_Q a critical value of f on x = x_O.

Lemma 1.2. Let feB(M) n C^1(n(f)). Suppose that A_Q, (0 < $7\backslash_Q$ < 1), is not a critical value of f on x = x_{σ} (0 < x_Q < 1). Then *l(x,~k;±) e*C^1 in a neighborhood of (x_Q,A_O).

Proof. Since feB(M), $y_{\wedge 0}$v intersects the line x = x_n at a finite number of points {$p._1$, . . . ,$P._k$}. Let P_j. = (x_O ,y_j.) and suppose that $y_1 < y_2 <\ldots< y_k$. Then the sequence

$$\{\frac{\partial f}{\partial y}(P_j)\}_{j=1}^{k}$$

has alternating signs.

Let y_j = y_j(x,A) be the inverse function (with respect to y) of A = f(x,y), in a neighborhood of p_j. Then for (x_1,λ_1)

sufficiently near to (x_0,λ_0), the intersection of γ_{λ_1} with the line $x = x_1$ consist precisely of the points $\{(x_1, y_j(x_1,\lambda_1))\}_{j=1}^{k}$. Hence for (x,λ) in some neighborhood of (x_0,λ_0) we have:

$$(1.12)\qquad \ell(x,\lambda;f) = \beta \sum_{j=1}^{k} (-1)^{j+1} y_j(x,\lambda)$$

where $\beta = \mathrm{sign}(\frac{\partial f}{\partial y}(P_1))$. Since $y_j(x,\lambda) \in C^1$ in a neighborhood of (x_0,λ_0) the assertion of the lemma is proved.

Note that for (x,λ) in a neighborhood of (x_0,λ_0) we have:

$$(1.13)\qquad \frac{\partial \ell}{\partial \lambda} \ \sum_{j=1}^{k} \ \text{[illegible]}$$

Definition 1.4. We shall say that a function f belongs to $B'(M)$ if $f \in B(M)$ and in addition:

(v) $f \in C^1(\Omega(f))$;

(vi) For every x_0, $(0 < x_0 < 1)$ and every a,b $(0 < a < b < 1)$, $\partial f/\partial y$ vanishes at most at a finite numbed of points on $\{x = x_0\} \cap \Omega_{a,b}(f)$.

Applying the transform $\mathcal{L}_\Lambda$ to functions in $B'(M)$ we obtain the following basic result:

Theorem 1.1. Let $\mathcal{F} = \{f_1,\dots,f_n\} \subset B'(M)$ and let $\Lambda = \{a_1,\dots,a_n\}$ be a set of positive numbers such that $\sum_{j=1}^{n} a_j = 1$. Let $G(t)$ be a function defined for $t \geq 0$ such that $G(t)$ is non-negative continuous, convex and non-decreasing. If $f^* = \mathcal{L}_\Lambda(\mathcal{F})$ then:

$$(1.14)\qquad \iint_{\Omega(f^*)} G((1+|\nabla f^*|^2)^{1/2})\,dxdy \leq \sum_{j=1}^{n} a_j \iint_{\Omega(f_j)} G((1+|\nabla f_j|^2)^{1/2})\,dxdy.$$

Proof. From properties (iv) and (v) of the class B'(M) and Lemma 1.1 it follows that f^* is Lipshitz in every compact subset of $\Omega(f^*)$. Hence the left side of (1.14) is meaningful.

Let $0 < a < b < 1$. We shall prove:

$$(1.15) \quad \iint_{a,b} G((1+|\nabla f^*|^2)^{1/2})\,dxdy < \sum \iint_{a,b} G((1+|\nabla f_i|^2)^{1/2})\,dxdy .$$

The inequality (1.14) follows from (1.15) if we let $a \to 0$ and $b \to 1$.

Given x_0, $0 < x_0 < 1$, we denote by $\{\lambda_1, \ldots, \lambda_{\nu-1}\}$ the set of all critical values of the functions $\{f_i\}_{i=1}^{n}$ on the line $x = x_0$, such that $a < \lambda_k < b$ $(k = 1, \ldots, \nu-1)$. This set is finite by property (vi) of the class B'(M). We set $\lambda_0 = a$ and $\lambda_\nu = b$ and we assume that $\lambda_0 < \lambda_1 < \ldots < \lambda_{\nu-1} < \lambda_\nu$. By property (iii) of the class B(M), the set

$$T_j^m = \{(x_0, y) \mid \lambda_m < f_j(x_0, y) < \lambda_{m+1}\}$$

consists of a finite number of open intervals. Denote these intervals, ordered by increasing y, by $\{T_{j,k}^{m}\}$ $(m = 0, \ldots, \nu-1;$ $j = 1, \ldots, n)$. Each interval $T_{j,k}^{m}$ is free of critical points of f_j; in a neighborhood of $T_{j,k}^{m}$, the function $\lambda = f_j(x,y)$ has an inverse $y = y_{j,k}^{m}(x,\lambda)$. By Lemma 1.2, $\xi_j(x,\lambda) = \xi(x,\lambda; f_j) \in C^1$ in a neighborhood of the interval $I_m = \{(x_0,\lambda) \mid \lambda_m < \lambda < \lambda_{m+1}\}$ and:

$$(1.16)\qquad \begin{cases} \ell_j(x,\lambda) = \pm \displaystyle\sum_{i=1}^{k(j,m)} (-1)^{i+1} y^m_{i,j}(x,\lambda); \\[2ex] \dfrac{\partial \ell_j}{\partial \lambda} = \displaystyle\sum_{i=1}^{k(j,m)} \left|\dfrac{\partial y^m_{i,j}}{\partial \lambda}\right| . \end{cases}$$

Hence $\ell^*(x,\lambda) \in C^1$ in a neighborhood of I_m and $\partial \ell^*/\partial \lambda > 0$ there.

To simplify the notation we shall denote:

$$G((1 + \sigma^2)^{1/2}) = \tilde{G}(\sigma), \qquad (-\infty < \sigma < \infty) .$$

Let:

$$T^m = \{ (x_0, y) \mid \ell^*(x_0,\lambda_m) < y < \ell^*(x_0,\lambda_{m+1}) \} .$$

We claim that:

$$(1.17)\qquad \int_{T^m} \tilde{G}(|\nabla f^*|)dy \le \sum_{i=1}^{k(j,m)} \int_{T^m_{i,j}} G(|\nabla f_{i,j}|)dy, \qquad (m = 0, \ldots, \nu-1).$$

Note that, since $\lambda = f^*(x,y)$ is the inverse of $y = \ell^*(x,\lambda)$, $0 < \lambda < 1$, we have

$$(1.18)\qquad \int_{T^m} G(|\nabla f^*|)dy = \int_{\lambda_m}^{\lambda_{m+1}} G((1+|\nabla_{x,\lambda}\ell^*|^2)^{1/2}/\ell^*_\lambda)\,\ell^*_\lambda\, d\lambda$$

where $\ell^*_\lambda = \partial \ell^*/\partial \lambda$. Similarly:

$$(1.19)\qquad \int_{T^m_{i,j}} G(|\nabla f_{i,j}|)dy = \int_{\lambda_m}^{\lambda_{m+1}} G((1+|\nabla_{x,\lambda} y^m_{i,j}|^2)^{1/2}/|\partial y^m_{i,j}/\partial\lambda|)\,|\partial y^m_{i,j}/\partial\lambda|\, d\lambda .$$

Hence in order to prove (1.17) it is sufficient to show that:

$$(1.20)\quad G((1+|\nabla_{x,\lambda}\ell^*|^2)^{1/2}/\ell^*_\lambda)\ell^*_\lambda \le$$

$$\le \sum_{j=1}^{n} a_j \sum_{i=1}^{k(j,m)} G((1+|\nabla_{x,\lambda} y^m_{i,j}|^2)^{1/2}/|\frac{\partial y^m_{i,j}}{\partial\lambda}|)\,|\frac{\partial y^m_{i,j}}{\partial\lambda}|.$$

Now, by the triangle inequality (for norm in R) we have:

$$(1.21)\quad (1+|\nabla_{x,\lambda}\ell^*|^2)^{1/2} = ((\sum a_j \ell_{j,\lambda})^2 + |\sum_{j=1}^{n} a_j \nabla \ell_j|^2)^{1/2}$$

$$\le \sum a_j (1+|\nabla \ell_j|^2)^{1/2}.$$

Since G is non-decreasing and convex we obtain:

$$(1.22)\quad G((1+|\nabla\ell^*|^2)^{1/2}/\ell^*_\lambda) \le G(\sum a_j (1+|\nabla\ell_j|^2)^{1/2}/\ell^*_\lambda)$$

$$\le \frac{1}{\ell^*_\lambda}\sum_{1}^{n} a_j \ell_{j,\lambda} G((1+|\nabla\ell_j|^2)^{1/2}/\ell_{j,\lambda}),$$

where $\ell^*_\lambda = \partial\ell^*/\partial\lambda$. Here we used the equality $\ell^*_\lambda = \sum^{n} a_j \ell_{j,\lambda}$.

Similarly we have (by (1.16)):

$$(1.23)\quad (1+|\nabla\ell_j|^2)^{1/2} \le ((k(j,m))^2 + |\sum_{i=1} \nabla y^m_{i,j}|^2)^{1/2}$$

$$\le \sum_{i=1}^{k(j,m)} (1+|\nabla y^m_{i,j}|^2)^{1/2}$$

and hence:

$$(1.24)\quad G((1+|\nabla\ell_j|^2)^{1/2}/\ell_{j,\lambda}) \le G(\sum_{i=1}^{k(j,m)} (1+|\nabla y^m_{i,j}|^2)^{1/2}/\ell_{j,\lambda})$$

$$\le \frac{1}{\ell_{j,\lambda}} \sum_{i=1}^{k(j,m)} G((1+|\nabla y^m_{i,j}|^2)^{1/2}/|\frac{\partial y^m_{i,j}}{\partial\lambda}|)\,|\frac{\partial y^m_{i,j}}{\partial\lambda}|,$$

for $x = x_0$, $\lambda_m < \lambda < \lambda_{m+1}$.

Combining (1.22) with (1.24) we get (1.20) and hence (1.17). Finally, summing up (1.17) with respect to m $(m = 0, \dots, \nu-1)$ and then integrating with respect to x_0 $(0 < x_0 < 1)$ we get (1.15). This completes the proof of the theorem.

Corollary 1.1. Under the assumptions of the theorem:

$$(1.25) \qquad \iint_{\Omega(f^*)} |\nabla f^*|^p dxdy \le \sum_{j=1}^{n} a_j \iint_{\Omega(f_j)} |\nabla f_j|^p dxdy, \qquad (1 \le p < \infty).$$

Proof. First we observe that the theorem and all the arguments presented above are valid also for sets of functions of the form $\{\alpha f_1, \dots, \alpha f_n\}$, where $0 < \alpha$ is a constant and $\{f_1, \dots, f_n\} \subset B^p(M)$. Naturally, in this case f^* will be replaced by αf^*.

Let $G(t) = t^p$, $(p \ge 1)$. Then $G(t)$ satisfies all the conditions of the theorem. Applying (1.15), with $G(t)$ as above, to the set of functions $\{\alpha f_j\}_1^n$, we obtain:

$$(1.26) \qquad \iint_{\Omega_{a,b}^*} (1+\alpha^2 |\nabla f^*|^2)^{p/2} dxdy \le \sum_{j=1}^{n} a_j \iint_{\Omega_{a,b}^j} (1+\alpha^2 |\nabla f_j|^2)^{p/2} dxdy .$$

Note that the domains of integration are bounded and that $|\nabla f^*|$ and $|\nabla f_j|$ are bounded in these domains. Hence, dividing both sides of (1.26) by α^p and letting $\alpha \to \infty$ we obtain:

$$(1.27) \qquad \iint_{\Omega_{a,b}^*} |\nabla f^*|^p dxdy \le \sum_{j=1}^{n} a_j \iint_{\Omega_{a,b}^j} |\nabla f_j|^p dxdy.$$

Letting $a \to 0$ and $b \to 1$ we get (1.25).

Note. The assumption that G is non-negative was made in order that the integrals in (1.14) would be meaningful even if they are infinite. The inequality (1.15) holds even if $G(0)$ is negative. Indeed, in this case set $G^*(t) = G(t) - G(0)$. Then we obtain (1.15) with G replaced by G^*. But we observe that:

$$\text{area}(Q_{a,b}(f^*)) = \sum_{1}^{n} a_{ij}\, \text{area}(Q_{a,b}(f_j)) .$$

Hence the terms with $G(0)$ cancel and we obtain (1.15).

By the same argument, if area $(Q(f_j))$ is finite for $j = 1,\ldots,n$, then (1.14) holds even if we remove the assumption that G is non-negative.

§2. Radial Averaging Transformations.

In this section we define radial averaging transformations and examine their effect on capacities of condensers and conformal radii of domains.

Definition 2.1. A condenser C in the plane is a system (Ω, E_0, E_1), where Ω is a domain, E_0 and E_1 are non-empty disjoint closed sets and $E_0 \cup E_1$ is the complement of Ω with respect to the extended plane, (i.e. the complex plane z including the point at ∞). If E_0 and E_1 are connected, Ω is called a ring.

In this section we shall assume also that E_0 is compact and that E_1 contains the point at infinity. The condenser C will also be denoted by (D, E_0) where $D = \Omega \cup E_0$.

Definition 2.2. A real function $f(x,y)$ will be called admissible for the condenser $C = (\Omega, E_0, E_1)$ if:

(i) f is continuous in the extended plane,

(ii) $f \equiv 0$ on E_0 and $f \equiv 1$ on E_1,

(iii) f is Lipshitz on every compact subset of Q.

The conformal capacity of C is defined by:

$$(2.1) \qquad I(C) = \inf. \; JJ|7f|^2 dxdy_5$$
$$n$$

the infinum being taken over all admissible functions.

We shall say that a domain Q has the ~~segment property~~ if, for every boundary point P_Q of Q, there exists a segment or an arc of a circle, with one endpoint at P_O, contained in the complement of fi. (When P_Q is the point at infinity, this means that there is a half line lying outside Q.) If C = (fi, E_Θ, E_1) is a condenser such that fi has this property we shall say that C has the segment property. If C has the segment property, then there exists a unique admissible function CO which is harmonic in fl. This function will be called the ~~potential function of~~ C. In this case we have:

$$(2.2) \qquad I(C) = \hat{jj} \, |vco|^2 dxdy.$$

For a proof of these assertions see for instance Hayman [4] (p. 62-7).

Lemma 2.1. Given a condenser C = (n, E_n, E_1), there exists a sequence of condensers $\{C_j\}_1^\infty$, $C_j = (Q_j, E_{n_j}, E_{1,j})$ such that:

(a) $\{Q_j\}_1^\infty$ is a monotonic increasing sequence of domains possessing the segment property;

(b) $\bar{Q}_j$ is a compact subset of fi and $\bigcup_1^n Q_j = Q$, (j = 1,2,...);

(c) E_{Q_j} contains a neighborhood of E_j (j = 1,2,...);

(d) $\lim_{j \to \infty} I(C_j) = I(C)$.

This assertion is easily verified by a standard argument.

We note also that if $C = (\Omega, E_0, E_1)$ and $C' = (\Omega', E_0', E_1')$ are two condensers such that $\Omega \subset \Omega'$ then $I(C') \leq I(C)$. This follows immediately from Definition 2.2. We shall refer to this as the monotonicity property of the capacity.

<u>Definition</u> 2.3. Let $\mathcal{D} = \{D_1, \ldots, D_n\}$ be a family of open sets in the complex plane z, with non-empty intersection. Suppose that the closed disk $|z - z_0| \leq \rho$ (for some positive ρ) is contained in $\bigcap_1^n D_j$. Let:

$$(2.3) \qquad K_j(\varphi) = \{r \mid z = z_0 + re^{i\varphi} \in D_j, \rho < r < \infty\}, \quad (0 \leq \varphi < 2\pi).$$

Set:

$$(2.4) \qquad \ell_j(\varphi) = \int_{K_j(\varphi)} \frac{dr}{r} \quad \text{and} \quad R_j(\varphi) = R(\varphi; D_j; z_0) = \rho \exp . \ell_j^{\rho}(\varphi).$$

(Note that $R_j(\varphi)$ does not depend on ρ.)

Let $A = \{a_1, \ldots, a_n\}$ be a set of positive numbers such that $\sum_1^n a_j = 1$. We define:

$$(2.5) \qquad R^*(\varphi) = \prod_{j=1}^{n} R_j(\varphi)^{a_j}, \quad (0 \leq \varphi < 2\pi)$$

and

$$(2.6) \qquad D^* = \mathfrak{A}_A(\mathcal{D}; z_0) = \{z = z_0 + re^{i\varphi} \mid 0 \leq r < R^*(\varphi), 0 \leq \varphi < 2\pi\}.$$

We shall say that $\mathfrak{A}_A$ is a <u>radial averaging transformation</u> on $\mathcal{D}$ with center z_0.

It is easy to verify that D^* is a domain which is starlike with respect to z_0.

Definition 2.4. Suppose that $\{E_j\}_1^n$ is a family of compact sets with non-empty intersection and $z_0 \in \bigcap_{j=1}^{n} E_j$. For each E_j we define $R_j^*(\varphi)$ and $\rho_j(\varphi)$ as before (where p is any positive number). Then we define:

$$(2.7) \qquad R_j(\varphi) = R(\varphi; E_j; z_0) = \lim_{p \to 0} p \exp . \rho_j(\varphi),$$

and

$$(2.8) \qquad E^* = \mathcal{R}_A(\{E_j\}_1^n; z_0) = \{z = z_0 + re^{i\varphi} \mid 0 \le r \le R^*(\varphi), 0 \le \varphi < 2\pi\},$$

with $R^*(\theta)$ as in (2.5).

It is easily verified that E^* is a connected compact set, starlike with respect to z_0. (Of course, in some cases, E^* may consist only of the point z_0.)

Definition 2.5. Let $\{C_j\}_1^n$ be a family of condensers, $C_j = (\Omega_j, E_{0,j}) = (D_j, E_{0,j})$. Suppose that the intersection of the sets $E_{0,j}$ $(j = 1,\dots,n)$ is non-empty and let z_0 be a point of this intersection. Let A be as in Definition 2.3 and set:

$$(2.9) \qquad \begin{aligned} D^* &= \mathcal{R}_A(\{D_j\}_1^n; z_0), \\ E_0^* &= \mathcal{R}_A(\{E_{0,j}\}_1^n; z_0). \end{aligned}$$

Then we denote:

$$(2.10) \qquad C^* = \mathcal{R}_A(\{C_j\}_1^n; z_0) = (D^*, E_0^*).$$

We denote also $\Omega^* = D^* - E_0^*$.

Note that if D^* is not the entire plane and E_0^* contains more than one point, then the condenser C^* possesses the segment property. In fact, Q^{v} ' is a ring which is starlike with respect to z_0.

The following theorem gives a relation between the capacities of $C_1, \ldots, C_n$ and the capacity of C^*.

<u>Theorem</u> 2.1. Let $\{C_j\}_1^n$ be a family of condensers as in Definition 2.5. Let $C^* = ft_A(\{C_j\}; z_0)$. Then:

$$(2.11) \qquad I(C^*) \le \sum_{j}^{n} a_j I(C_j) \cdot$$

~~Proof~~. By Lemma 2.1 and the monotonicity property of the capacity it is sufficient to prove the theorem in the case that the condenser C_j possesses the segment property and z_0 is an interior point of $E_{0,j}$, $(j = 1,\ldots,n)$. Therefore we restrict our attention to this case. Without loss of generality we may assume that $z_0 = 0$ and that the disk $|z| \le 1$ is contained in each of the sets $E_{0,j}$, $(j = 1,\ldots,n)$.

Consider the mapping $w = \ln z$ of the domain $|z| > 1$ cut along the positive real axis onto the half strip $\{w = u + iv \mid 0 < v < 2\pi, 0 < u\}$.

Let ω_j be the potential function of C_j and denote $f_j(u,v) = \omega_j(e^w)$, $(j = 1,\ldots,n)$. Then f_j is periodic in v with period 2π and it is easily verified that $f_j \in B'(M)$ where $M = \{(u,v) \mid 0 \le v \le 2\pi, 0 \le u\}$. Let $f^* = \Sigma_A(\{f_j\}_1^n)$ in the half strip M. Obviously $f^*(u,0) = f^*(u,2\pi)$ and we extend f^* periodically in v (with period 2π) to the half plane $u \ge 0$. The extension will also be denoted by f^*.

By Lemma 1.1, f^* (extended as above) is continuous in the half plane $u \ge 0$ and is Lipshitz in every compact subset of

$\Omega = \{(u,v) \mid 0 < f^*(u,v) < 1, 0 \le u\}$. Also $f^* \in B(M)$. By Definitions 1.2 and 2.5, $\Omega^{(*)}$ is the image of $\Omega^{(*)}$ by the (multiple valued) function $w = \ln z$.

We denote $\omega(x,y) = f^*(\ln z)$, $(z = x + iy)$. Because of the periodicity of f^*, the function ω is well defined in $|z| \ge 1$. We extend ω to the entire plane by setting $\omega = 0$ in $|z| < 1$ and $\omega = 1$ at ∞. Then, from the properties of f^* described above, it follows that ω is an admissible function for the condenser C^*. Hence:

$$I(C^*) \le \iint_{\Omega^{(*)}} |\nabla \omega|^2 dxdy.$$

By Corollary (1.1), with $p = 2$, and the invariance of the Dirichlet integral under conformal mapping:

$$\iint_{\Omega} |\nabla w^*|^2 dxdy \le 2^n a_j \iint_{j} |\nabla \omega_j|^2 dxdy.$$

Combining these inequalities and taking into account formula (2.2) we obtain (2.11).

Let D be a domain in the plane and $z_0 \in D$. We denote by $r(z_0;D)$ the conformal (or inner) radius of D at z_0.

If $f(\zeta)$ is an analytic function in the unit disk $|\zeta| < 1$, such that $f(0) = z_0$, and if the image of this unit disk by $z = f(\zeta)$ is contained in D, then:

(2.12) $$|f'(0)| \le r(z_0;D).$$

Equality holds if and only if $z = f(\zeta)$ maps $|\zeta| < 1$, (1-1) onto D.

If D is a bounded domain which possesses the segment property, $z_0 \in D$, and C_ϵ denotes the condenser (D, K_ϵ), where K_ϵ is the disk $|z-z_0| < \epsilon$, then:

(2.13) $\overline{I(}\quad \frac{r(z\ ;D)}{\quad} + O(1),$

where $0(1)$ is a quantity which tends to zero when $\epsilon \to 0$.

If D is any domain in the plane and $z_0 \in D$, there exists a sequence of subdomains $\{D_n\}_1^\infty$ such that $z_0 \in D_n \subset D_{n+1}$, $\overline{D}_n \subset D$, D_n is bounded and has an analytic boundary (i.e. its boundary consists of a finite number of disjoint, simple, closed analytic curves), $(n = 1,2,...)$, $\bigcup_1^\infty D_n = D$ and finally $\lim_{n\to\infty} r(z_0;D_n) = r(z_0;D)$. ($\{D_n\}_1^\infty$ will be called an <u>approximating sequence for the inner radius of D at z_0</u>.)

Note that if $\{B_k\}_1^\infty$ is a monotonic increasing sequence of subdomains of D such that $\bigcup_1^\infty B_k = D$ and if $z_0 \in B_k$, then $\lim_{k\to\infty} r(z_0;B_k) = r(z_0;D)$. This is a simple consequence of the previous result and the monotonicity property of the conformal radius (i.e. if $z_0 \in D' \subset D''$ then $r(z_0;D') \leq r(z_0;D'')$).

For the definition of inner radius and proofs of the above statements, see Hayman [4] (p. 78-84). Formula (2.13) is due to Polya-Szegö [9].

We use now Theorem 2.1 and formula (2.13) in order to obtain:

<u>Theorem</u> 2.2. Let $\mathcal{D} = \{D_1,...,D_n\}$ be a family of domains in the plane with non-empty intersection. Let $z_0 \in \bigcap_1^n D_j$ and let $A = \{a_j\}_1^n$ be a set of positive numbers such that $\sum_1^n a_j = 1$.

If $D^* = \Omega_A(\mathcal{D};z_0)$ then:

(2.14) $$\prod_{1}^{n} r(z_0, D_j)^{a_j} \le r(z_0, D).$$

Proof. It is sufficient to prove the theorem in the case where D_j $(j = 1,\dots,n)$ is a bounded domain with analytic boundary. This is clear in view of the existence of approximating sequences for the inner radius of D_j at z_0 $(j = 1,\dots,n)$, as described above. Therefore, we restrict our attention to this case.

We shall use the following notations:

$$r_j = r(z_0, D_j) \quad (j = 1,\dots,n); \quad r^* = r(z_0, D^*); \quad \tilde{r} = \prod_{1}^{n} r_j^{a_j}.$$

Given $\varepsilon > 0$ let $\alpha = \varepsilon/\tilde{r}$ and $\varepsilon_j = \alpha r_j$. Finally denote $C_{j,\varepsilon_j} = (D_j, K_{\varepsilon_j})$, where K_{ε_j} denotes the disk $|z - z_0| < \varepsilon_j$, and $I(C_{j,\varepsilon_j}) = I(j,\varepsilon)$. Then by (2.13):

(2.15) $$\frac{2\pi}{I(j,\varepsilon)} = \log\frac{r_j}{\varepsilon_j} + O_j(1) = \log\frac{1}{\alpha} + O_j(1),$$

where $O_j(1)$ tend to zero as $\varepsilon \to 0$.

Let $C^*_\varepsilon = GM(\{C_{j,\varepsilon_j}\}; \{a_j\}; z_0)$. Since $\prod \varepsilon_j^{a_j} = \alpha\tilde{r} = \varepsilon$ we find that (for sufficiently small ε): $C^*_\varepsilon = (D^*, K_\varepsilon)$. Hence:

(2.16) $$\frac{2\pi}{I(C^*_\varepsilon)} = \log\frac{r^*}{\varepsilon} + O(1) = \log\frac{r^*}{\alpha\tilde{r}} + O(1),$$

where $O(1)$ tends to zero when $\varepsilon \to 0$.

By (2.11), (2.15) and (2.16) we obtain:

$$I(C^*_\varepsilon) = \frac{2\pi}{\log\frac{r^*}{\alpha\tilde{r}} + O(1)} \le 2\pi \sum_{j=1}^{n} a_j/\left(\log\frac{1}{\alpha} + O_j(1)\right)$$

$$\le 2\pi \sum a_j/\left(\log\frac{1}{\alpha} + O'(1)\right)$$

$$= 2\pi/\left(\log\frac{1}{\alpha} + O'(1)\right),$$

where $\theta'(1) = -\max\limits_{1\le j\le n} |\theta_j(1)|$. Hence:

$$\log \frac{r^*}{\tilde{r}} + \theta(1) \ge \log \frac{1}{\alpha} + \theta'(1).$$

Since $\xi = \alpha\tilde{r}$ we finally obtain:

$$\log \frac{r^*}{r} \ge 0 \quad r^* \ge \tilde{r}.$$

This completes the proof of the theorem.

§3. Radial Symmetrization and Related Results.

The results that we present in this section are applications of Theorem 2.2. We begin with a definition of radial symmetrization which extends the definitions introduced in [11], [7] and [1].

Definition 3.1. Let D be a domain in the plane and let $z_0 \in D$. Denote $R(\varphi;D;z_0)$ as in (2.4). Let $A = \{\alpha_j\}_1^n$ be a set of positive numbers such that $\sum_1^n \alpha_j = 1$; let $a = \{a_j\}_1^n$ be a set of integers different from zero and let $\beta = \{\beta_j\}_1^n$ be a set of real numbers. We denote:

(3.1) $$R^*(\varphi) = \prod_{j=1}^{n} R(a_j\varphi + \beta_j)^{b_j}, \quad (0 \le \varphi < 2\pi),$$

where $b_j = \alpha_j / |a_j|$, and

(3.2) $$D^* = \{z = z_0 + re^{i\varphi} \,|\, 0 \le r < R^*(\varphi), 0 \le \varphi < 2\pi\}.$$

The transformation $D \to D^*$ will be called a (generalized) radial symmetrization and will be denoted by $S(A,a,\beta;z_0)$.

The following lemma will be needed:

Lemma 3.1. Let D be a domain in the plane, such that D contains the origin. Let k be a positive integer. Consider a k-fold covering of D such that each point $z \in D$, $z \neq 0$, is covered by k points with arguments $\varphi + 2\pi j$, $0 \leq j \leq k-1$, $0 \leq \varphi < 2\pi$. Let $T_k D$ denote the image of this covering by $w = z^{1/k}$. Then:

$$r(0;T_kD) = r(0;D)^{1/k}. \tag{3.3}$$

Proof. Let $\{D_n\}$ be an approximating sequence for the conformal radius of D at 0, as described in Section 2. Then

$$\lim_{n\to\infty} r(0;D_n) = r(0;D),$$

$$\lim_{n\to\infty} r(0;T_kD_n) = r(0;T_kD).$$

The second formula follows from the fact that $\{T_k D_n\}_{n=1}^{\infty}$ is a monotonic increasing sequence of subdomains of T_kD such that $\bigcup_{n=1}^{\infty} T_k D_n = T_k D$.

Hence it is sufficient to prove the lemma for a domain D which is bounded and has analytic boundary.

Let K_ε be the disk $|z| < \varepsilon$, suppose that $K_\varepsilon \subset D$ and denote $C_\varepsilon = (D, K_\varepsilon)$. If $C_\varepsilon^k = (T_kD, K_{\varepsilon_k})$, where $\varepsilon_k = \varepsilon^{1/k}$, it is easily seen that:

$$kI(C_\varepsilon) = I(C_\varepsilon^k).$$

Hence, by formula (2.13) we obtain the required result.

Theorem 3.1. Let D be a domain in the plane and let $z_0 \in D$. Let D^* be as in Definition 3.1. Then:

(3.4) $r(z_0;D) \leq r(z_0;D_j)^{1/b}$, where $b = \sum_1^n b_j$.

Proof. Without loss of generality we may assume that $z_0 = 0$. Denote by $D_j^{(1)}$ the domain $\frac{1}{a_j} D$. If $a_j < 0$, denote by $D_j^{(2)}$ the domain obtained from $D_j^{(1)}$ by reflexion with respect to the real axis. If $a_j > 0$ let $D_j^{(2)} = D_j^{(1)}$. Finally denote by D_j the domain obtained from $D_j^{(2)}$ by a rotation of angle $-\theta_j$.

By (3.3) we have:

(3.5) $r(0;D_j) = r(0;D)^{1/|a_j|}$, $(j = 1,\dots,n)$.

Furthermore, it is clear from our construction that:

(3.6) $R(\alpha;D_j;0) = R(a_j\alpha + \beta_j)^{1/|a_j|}$.

Applying Theorem 2.2 to the set $\mathcal{D} = \{D_j\}_1^n$, with D_j as above, inequality (3.4) follows immediately from (3.5) and (3.6).

Corollary 3.1. Let $z = f(\zeta)$ be an analytic function in $|\zeta| < 1$ such that $f(0) = 0$. Let D be the image of $|\zeta| < 1$ by the mapping $z = f(\zeta)$. Define D^* as in (3.2) with $z_0 = 0$. Suppose that D^* is not the entire plane and let $z = F(\zeta)$ be an analytic function in $|\zeta| < 1$ which maps this disk (1-1) onto a domain containing D^*. Then:

(3.7) $|f'(0)| \leq |F'(0)|^{1/b}$,

with b as in (3.4).

Proof. Using (2.12) and the subordination principle, this result follows immediately from the theorem.

Remark. Theorem 3.1 includes as particular cases the radial symmetrization results of Szegö [n] (for starlike domains, with $a_j = 1/n$, $\alpha_j = 1$, $\beta_j = \frac{2\pi j}{n}$, $j = 1,\ldots,n$), Marcus [7] (for general domains with a_j, α_j, β_j as above), and Aharonov and Kirwan [1] (for general domains with $a_j = \frac{1}{n}$, $\alpha_j = \pm 1$, β_j arbitrary).

The method of the present paper is different from the methods employed in the previous papers, in that the symmetrization results are obtained as a particular case of a more general class of transformations, namely, the radial averaging transformations.

By varying continuously the exponents in the weighted geometric mean (2.5), we can obtain a process of continuous symmetrization with properties similar to those of the continuous symmetrization of Polya-Szegö [9] (p. 200-202). (In [9] the process refers to Steiner symmetrization.) The following theorem provides a result of this type.

Theorem 3.2. Let D be a domain in the plane and let $z_0 \in D$. Consider the transformation $S = S(A,\bar{\alpha},\bar{\beta};z_0)$ with $a_j = \pm 1$. Using the notation of Definition 3.1 set:

$$R_t^*(\varphi) = R^*(\varphi)^t R(\varphi)^{1-t}$$

(3.8)

$$S_t D = \{z = z_0 + re^{i\varphi} \mid 0 \le r < R_t^*(\varphi),\ 0 \le \varphi < 2\pi\}, \quad (0 \le t \le 1).$$

(Note that $S_1 D = SD = D^*$.)

Then, for $0 \le t_1 < t_2 \le 1$ we have:

$$(3.9) \qquad r(z_0;D) \le r(z_0;S_{t_2}D)^{t_1/t_2}\, r(z_0;D)^{1-t_1/t_2} \le r(z_0;S_{t_1}D) .$$

<u>Remark</u>. Suppose that D is bounded and starlike with respect to z_0. Then $S_0D = D$ as well as $S_1D = D^*$ and $R^*_t(\varphi)$ depends continuously on t, for $0 \le t \le 1$. Hence, in this case the transformation S_t may be called a continuous symmetrization connecting D and D^*.

<u>Proof</u>. By (3.8):

$$R^*_{t_1}(\varphi) = R^*_{t_2}(\varphi)^{t_1/t_2}\, R(\varphi)^{1-t_1/t_2}, \quad (0 \le t_1 < t_2 \le 1) .$$

Hence by Theorem 2.2$_5$ we obtain the second inequality in (3.9). In particular, for $t_1 = t$ and $t_2 = 1$ we get:

$$r(z_0;D^*)^t\, r(z_0;D)^{1-t} \le r(z_0;S_tD).$$

Finally, by Theorem 3.1:

$$r(z_0;D) \le r(z_0;D^*).$$

Combining these inequalities we obtain (3.9).

The following theorem supplies a symmetrization based on an integral formula.

<u>Theorem</u> 3.3. Let $A(\psi)$ be a bounded, monotonic increasing function in the interval $[0,2\pi]$, such that:

$$(3.10) \qquad \int_0^{2\pi} dA(\psi) = 1 ,$$

the integral being a Stieltjes integral.

Let $z = f(\zeta) = a_1\zeta + a_2\zeta^2 + \cdots$ be an analytic function in the unit disk $|\zeta| < 1$ and let D be the image of this disk by $z = f(\zeta)$. Denote as usual, $R(\varphi) = R(\varphi;D;0)$ and set:

$$(3.11)\qquad \begin{cases} \tilde{R}(\varphi) = \exp\left[\int_0^{2\pi} \ln R(\varphi + \psi)\, dA(\psi)\right], \\ \tilde{D} = \{z = re^{i\varphi} \mid 0 \leq r < \tilde{R}(\varphi),\ 0 \leq \varphi < 2\pi\}, \end{cases}$$

the integral being a Lebesgue-Stieltjes integral. Then:

$$(3.12)\qquad |a_1| \leq r(0;D) \leq r(0;\tilde{D}).$$

Proof. Without loss of generality we may assume that $f(\zeta)$ is analytic in $|\zeta| \leq 1$ and that $f'(\zeta) \neq 0$ for $|\zeta| = 1$. In the general case, the result will follow by approximating the function $f(\zeta)$ by functions $f(\rho\zeta)$ with $0 < \rho < 1$.

Under the above mentioned assumption, $R(\varphi)$ is a bounded continuous function of φ (and periodic with period 2π). Hence the integral in (3.11) may be interpreted as a Riemann-Stieltjes integral. Therefore, we have:

$$(3.13)\qquad \ln \tilde{R}(\varphi) = \lim_{n\to\infty} \sum_{j=0}^{n-1} \gamma_{j,n} \ln R\left(\varphi + \frac{2\pi}{n} j\right),$$

uniformly with respect to φ, where $\gamma_{j,n} = A\left(\frac{2\pi}{n}(j+1)\right) - A\left(\frac{2\pi}{n} j\right)$.

Let:

$$\Gamma_n = \sum_{j=0}^{n-1} \gamma_{j,n} \quad \text{and} \quad \delta_{j,n} = \gamma_{j,n}/\Gamma_n .$$

We observe that by (3.10) $\lim_{n\to\infty} \Gamma_n = 1$. Denote:

$$R_n^*(\varphi) = \prod_{j=0}^{n-1} R\left(\varphi + \frac{2\pi}{n} j\right)^{1/n}$$

$$D_n^* = \{z = re^{i\varphi} \mid 0 \le r < R_n^*(\varphi), 0 \le \varphi < 2\pi\} .$$

Given $\varepsilon > 0$, for sufficiently large n we have (by (3.13)):

$$|\tilde{R}(\varphi) - R_n^*(\varphi)| < \varepsilon , \quad 0 \le \varphi < 2\pi.$$

From this and Theorem 3.1 it follows that:

$$|a_1| \le r(0;D) \le r(0;D_n^*) \le r(0,\tilde{D}) + \varepsilon_n$$

where $\varepsilon_n \to 0$ when $n \to \infty$. This completes the proof of the theorem.

Remark. Considering the special case $\lambda(\psi) = \frac{1}{2\pi}\psi$, we observe that by the geometric-arithmetic mean inequality we have:

$$(3.14) \qquad \tilde{R} = \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \ln R(\varphi)d\varphi\right] \le \frac{1}{2\pi}\int_0^{2\pi} R(\varphi)d\varphi .$$

Let R_0 be the radius of the disk whose area equals the area of D. Then by Hölder's inequality:

$$\frac{1}{2\pi}\int_0^{2\pi} R(\varphi)d\varphi \le \left[\frac{1}{2\pi}\int_0^{2\pi} R^2(\varphi)d\varphi\right]^{1/2} = R_0 .$$

By (3.12), $|a_1| \le K$. Since $\tilde{R} \le R_0$, this estimate is stronger than the corresponding estimate obtained by Schwarz symmetrization.

The following result is also a consequence of Theorem 2.2:

<u>Theorem</u> 3.4. Let D be a domain in the plane and let $z_0 \in D$. Denote:

$$D^*_t = \{z = z_0 + re^{i\varphi} \mid 0 \le r < R(\varphi)^t, 0 \le \varphi < 2\pi\}, \tag{3.15}$$

where $R(\varphi) = R(\varphi; D; z_0)$. Then:

$$r(z_0;D)^t \le r(z_0;D^*_t), \quad (0 \le t \le 1). \tag{3.16}$$

<u>Proof</u>. Let $\mathcal{D} = \{D_1, D_2\}$ where $D_1 = D$ and D_2 is the disk $|z-z_0| < r(z_0;D)$. Let $\Lambda = [t, 1-t]$ and $D^* = R_\Lambda(\mathcal{D}; z_0)$. Note that

$$R(\varphi;D^*;z_0) = r(z_0;D)^{1-t} R(\varphi)^t,$$

so that:

$$r(z_0;D^*) = r(z_0;D)^{1-t} r(z_0;D^*_t).$$

But by (2.14):

$$r(z_0;D) \le r(z_0;D^*).$$

Hence we obtain (3.16).

§4. <u>An Extension of the Bieberbach-Eilenberg Class of Functions</u>.

The following notation will be employed throughout this section:

The unit disk $|\zeta| < 1$ in the complex plane $\mathbb{C}_\zeta$ will be denoted by $E = E_\zeta$.

The class of functions $f(\zeta)$, holomorphic in E, such that $f(0) = 0$ will be denoted by M.

If $f \in M$, the image of E by $z = f(\zeta)$ will be denoted by

D_f; furthermore, we set $R_f(\varphi) = R(\varphi; D_f; 0)$ (see 2.4).

A function $f(\zeta) \in W$ is a Bieberbach-Eilenberg (or B.E.) function if it satisfies the following condition:

$$f(\zeta_1)\cdot f(\zeta_2) \neq 1, \quad \text{for all } \zeta_1, \zeta_2 \in E. \tag{4.1}$$

Various classes of functions, defined by conditions analogous to (4.1) have been introduced and studied in the literature. See for instance Goodman [3], Lebedev [6] and Jenkins [5].

The following class of functions was introduced in [5]. Let θ be a fixed real number. Then we say that $f \in K(e^{i\theta})$ if $f \in W$ and

$$f(\zeta_1)\cdot f(\zeta_2) \neq e^{i\theta}, \quad \text{for all } \zeta_1, \zeta_2 \in E. \tag{4.2}$$

A classical result on B.E. functions states that if $f(\zeta)$ is such a function, then:

$$|f'(0)| \le 1, \tag{4.3}$$

with equality if and only if $f = \eta\zeta$, $(|\eta| = 1)$.

Jenkins [5] has shown that this result holds also for $f \in K(-1)$ and $f \in K(1)$. Goodman [3] obtained the same result for the classes of functions introduced by him there.

Let $f, g \in W$ and suppose that:

$$f(\zeta_1) g(\zeta_2) \neq 1, \quad \text{for all } \zeta_1, \zeta_2 \in E. \tag{4.4}$$

Then it is known that:

$$|f'(0)\cdot g'(0)| \le 1, \tag{4.5}$$

and again, the inequality is sharp in the same sense as before. (This result was obtained, in a different form, by Nehari [8].)

In order to unify and extend conditions (4.1), (4.2) and (4.4) we introduce:

<u>Definition 4.1.</u> Let P denote the matrix $(p_{k,j})_{k=1,\dots,n-1;\ j=1,\dots,n}$ where $p_{k,j}$ are integers and $n \geq 2$.

Let $\Theta = (\theta_1,\dots,\theta_{n-1})$ be a vector whose components are real numbers.

(i) Let $\mathcal{D} = \{D_j\}_1^n$ be a set of domains in the z-plane such that each D_j contains the origin. We shall say that $\mathcal{D} \in \tilde{C}_n(P;\Theta)$ if the following condition holds:

$$e^{-i\theta_k}\prod_{j=1}^{n} z_j^{p_{k,j}} > 0, \quad (k = 1,\dots,n-1) \Rightarrow |z_1 \cdots z_n| \leq 1, \tag{4.6}$$

for all $(z_1,\dots,z_n) \in D_1 \times \dots \times D_n$.

If D is a domain containing the origin and $\{D_j\}_1^n \in \tilde{C}_n(P;\Theta)$, where $D_1 = \dots = D_n = D$, we shall say that $D \in C(P;\Theta)$.

(ii) Let $\{f_j\}_1^n \subset \mathcal{S}$. We shall say that $\{f_j\}_1^n \in C_n(P;\Theta)$, iff $\{D_{f_j}\}_1^n \in \tilde{C}_n(P;\Theta)$.

If $f \in \mathcal{S}$ and $\{f_j\}_1^n \in C_n(P;\Theta)$, where $f_1 = \dots = f_n = f$, we shall say that $f \in C(P;\Theta)$.

<u>Remark</u>. We observe that $C((1,1);0)$ is the class of B.E. functions while $C((1,-1);\theta) = K\{e^{i\theta}\}$. Also $C_2((1,1);0)$ is the class of pairs satisfying (4.4).

Denote by P_j' the determinant of the matrix obtained from P by deleting the j-th column. Let $P_j = (-1)^{j+1}P_j'$.

With this notation we have:

<u>Theorem</u> 4.1. Let P and θ be as in Definition 4.1 and suppose that $P_j \neq 0$, $(j = 1,\dots,n)$. If $\mathcal{D} = \{D_1,\dots,D_n\} \in C_n(P;\theta)$ then:

(4.7)
$$\prod_{1}^{n} r(0;D_j) \le 1.$$

Equality holds if $D_j = E_z$, $(j = 1,\dots,n)$ where E_z is the disk $|z| < 1$.

In particular, if $f \in C(P;G)$ then:

(4.8)
$$|f'(0)| \le 1,$$

and equality holds for $f = \eta z$, $(|\eta| = 1)$.

<u>Proof</u>. We prove the first assertion of the theorem. The second one follows immediately from the first assertion.

Consider the system of equations:

(4.9)
$$P \cdot \begin{pmatrix} \varphi_1 \\ \cdot \\ \cdot \\ \cdot \\ \varphi_n \end{pmatrix} = \begin{pmatrix} \theta_1 \\ \cdot \\ \cdot \\ \cdot \\ \theta_{n-1} \end{pmatrix}.$$

If $(z_1,\dots,z_n) \in D_1 \times \dots \times D_n$ and if (4.9) is satisfied by $\varphi_1,\dots,\varphi_n$, where φ_j is one of the values of arg z_j (we set $\varphi_j = 0$ when $z_j = 0$), then, by (4.6), $|z_1 \cdots z_n| \le 1$.

The general solution of (4.9) is given by

(4.10)
$$\varphi_j = \varphi_j(\varphi) = P_j\varphi + \psi_j,$$

where $\{\psi_j\}_1^n$ is a particular solution of (4.9). (We may choose $\psi_n = 0$; then $\psi_1, \dots, \psi_{n-1}$ will be uniquely determined since $\rho_n \neq 0$.)

Choose ρ, $0 < \rho < 1$, such that $|z| \le \rho$ is contained in $\bigcap_{j=1}^{n} D_j$ and denote $D'_{j,\rho} = \{z \mid z \in D_j, |z| > \rho\}$. Let $D^*_{j,\rho}$ be the image of $D'_{j,\rho}$ by the (multiply valued) function $w = \ln z$, $(w = u + iv)$. Denote:

$$L_j(v) = \{(u,v) \mid u \in D^*_{j,\rho}\},$$
(4.11)
$$L_j^{\pm}(v) = \{(u,v) \mid u \in D^*_{j,\rho}, \pm u > 0\}.$$

Let $\ell_j(v)$, $\ell_j^+(v)$, $\ell_j^-(v)$ denote the linear measures of $L_j(v)$, $L_j^+(v)$ and $L_j^-(v)$ respectively.

For given φ, let $v_j = \gamma_j(\varphi)$. By (4.6), if the interval $a_j < u < b_j$ is contained in $L_j^+(v_j)$, $(j = 1, \dots, n-1)$, then the interval $-\sum_1^{n-1} b_j < u < -\sum_1^{n-1} a_j$ does not intersect $L_n^-(v_n)$. Hence:

$$\ell_n^-(v_n) + \ell_j^+(v_j) \le |\ln \rho|, \quad (j = 1, \dots, n-1).$$

Similarly, we have:

(4.12) $$\ell_k^-(v_k) + \ell_j^+(v_j) \le |\ln \rho|, \quad (k = 1, \dots, n;\ j = 1, \dots, n;\ k \neq j).$$

Summing up (4.12) over j (with k fixed) we get:

$$\ell_k^-(v_k) + \frac{1}{n-1} \sum_{\substack{j=1 \\ j \neq k}}^{n} \ell_j^+(v_j) \le |\ln \rho|, \quad (k = 1, \dots, n);$$

and now summing up over k we obtain;

(4.13) $$\sum_{k=1}^{n} \ell_k(v_k) \le n|\ln \rho|.$$

It follows from (4.13) that:

(4.14) $$\prod_{j=1}^{n} R_j(\varphi_j) \le 1,$$

with φ_j as in (4.10) and $0 \le \varphi \le 2\pi$.

Now, as in the proof of Theorem 3.1, we construct a domain $\tilde{D}_j$ $(j = 1,\ldots,n)$ such that:

(4.15) $$\begin{cases} R(\varphi;\tilde{D}_j;0) = R_j(P_j\varphi + i\lambda_j)^{1/|P_j|}, \\ r(0;\tilde{D}_j) = r(0;D_j)^{1/|P_j|}, \quad (j = 1,\ldots,n). \end{cases}$$

Let $a_j = |P_j|/\gamma$ where $\gamma = \sum_1^n |P_j|$. Then, by (4.15):

(4.16) $$R^*(\theta) = \prod_1^n R(\varphi;\tilde{D}_j;0)^{a_j}$$

$$= \Big[\prod_1^n R_j(P_j\varphi + i\lambda_j)\Big]^{1/\gamma} \le 1, \quad 0 \le \varphi \le 2\pi.$$

Applying Theorem 2.2 to the set $\Omega = \{\tilde{D}_j\}_1^n$, with $R^*(\theta,\Omega)$ as above, and using (4.15) we obtain (4.7).

<u>Remark</u>. Suppose that, in Definition 4.1, condition (4.6) is replaced by:

(4.6)' $$e^{-i\theta_k}\prod_{j=1}^{n} z_j^{p_{kj}} > 0, \quad (k = 1,\ldots,n-1) \Rightarrow |z_1^{t_1}\cdots z_n^{t_n}| \le 1,$$

where t_j $(j = 1,\ldots,n)$, are fixed positive numbers.

If inequality (4.7) is replaced by

(4.7)' $$\prod_1^n r(0;D_j)^{t_j} \le 1,$$

then Theorem 4.1 is valid also in this case.

The proof, except for minor modifications, is the same as before. Instead of (4.14) we have in this case:

$$\prod_{j=1}^{n} R_j(\varphi_j)^{t_j} \le 1, \qquad (4.14)'$$

with φ_j as in (4.10) and $0 \le \varphi \le 2\pi$.

§5. Further Applications.

In this section we consider the following problem. Let $\{\varphi_1, \ldots, \varphi_m\}$ be a set of m distinct numbers in the interval $[0, 2\pi)$. Denote by α_φ the ray with argument φ, issuing from the origin. Let $f \in S$ and let D_f be defined as in the previous section. Then, the problem is to obtain sharp lower bounds for the measure of the intersection of $\bigcup_{j=1}^{n} \alpha_{\varphi_j}$ with D_f.

Results of this type (which will be referred to as "covering theorems") were obtained in [11], [7] and [1], by means of radial symmetrization. We state below a rather general covering theorem which, as we shall show, implies the results mentioned above.

The following notation will be convenient for the statement of our result. Let $0 < M \le \infty$ and consider the disk $|z| < M$ cut along the segments (or rays) $z = re^{i\varphi_j}$, $K \le r \le M$, $(j = 1, \ldots, m)$. We denote this domain by $D_{M,K}(\varphi_1, \ldots, \varphi_m)$. Let $1 \le M$ and let K be so chosen, that the conformal radius of this domain, with respect to the origin will be equal to 1. We denote this value of K by $K_M(\varphi_1, \ldots, \varphi_m)$, and we denote the domain $D_{M,K}(\varphi_1, \ldots, \varphi_m)$ with K as above by $D_M(\varphi_1, \ldots, \varphi_m)$.

If $\varphi_j = \gamma_j(\varphi) = \varphi + 2\pi j/m$ $(j = 1,\ldots,m)$, we denote: $K_{M,m}(\varphi) = K_M(\varphi_1,\ldots,\varphi_m)$. The value of $K_{M,m}$ can be computed from the equation:

$$1 = K\cdot 4^{1/m}/(1 + K^m/M^m)^{2/m},$$

(see [7], p. 624). Hence we have:

(5.1) $$K_{M,m} = M[(2M^m - 1) - 2\sqrt{M^m(M^m - 1)}]^{1/m}, \quad (M \geq 1).$$

In particular, $K_{\infty,m} = (1/4)^{1/m}$.

If $\varphi_j = \gamma_j(\varphi)$, $1 \leq j \leq m$, and $\varphi_j = \gamma_j(-\varphi)$, $m + 1 \leq j \leq 2m$, we denote: $K^*_{M,m}(\varphi) = K_M(\varphi_1,\ldots,\varphi_{2m})$. Note that if $\varphi = \gamma_j(0)$ or $\varphi = \gamma_j(\pi/m)$ the set $\{\varphi_1,\ldots,\varphi_{2m}\}$ contains only m distinct rays. In this case we identify $K^*_{M,m}(\varphi)$ with $K_{M,m}$.

With this notation we have:

Theorem 5.1. Let $f(\zeta) = a_1\zeta + a_2\zeta^2 + \cdots$ be analytic in the disk $|\zeta| < 1$ and let D_f be defined as before. Let $R^*(\varphi)$ be defined as in (3.1), with $R(\varphi) = R(\varphi; D_f; 0)$. Suppose that:

(5.2) $$R^*(\varphi) \leq M, \quad (0 \leq \varphi < 2\pi),$$

for a fixed M, $0 < M \leq \infty$. Suppose also that, for a given set of distinct rays with arguments $\{\varphi_1,\ldots,\varphi_m\}$, there exists a number Q such that

(5.3) $$R^*(\varphi_j) = Q, \quad (1 \leq j \leq m).$$

Set $M' = M/|a_1|^b$, where $b = \sum_1^n b_j$ (see (3.1)). Then:

(5.4) $$K_{M'}(\varphi_1,\ldots,\varphi_m)|a_1|^b \leq Q.$$

If $b = 1$, equality holds for every function f which maps the disk $|\zeta| < 1$, (1-1) onto the domain $|a_1| D_{M,1}(\varphi_1, \ldots, \varphi_m)$.

If $a_1 = 1$, equality holds for every function f which maps the disk $|\zeta| < 1$, (1-1) onto the domain $D_{M,Q}(\varphi_1, \ldots, \varphi_m)$.

Proof. Let D^* be defined by (3.2). Then by Theorem 3.1;

$$|a_1| \le r(0;D_f) \le r(0;D^*)^{1/b}.$$

By our assumptions $D^* \subset D_{M,Q}(\varphi_1, \ldots, \varphi_m)$. Hence:

$$|a_1|^b \le r(0;D_{M,Q}(\varphi_1, \ldots, \varphi_m)).$$

Let $M' = M/|a_1|^b$ and $Q' = Q/|a_1|^b$. Then:

$$1 \le r(0;D_{M',Q'}(\varphi_1, \ldots, \varphi_m)).$$

Hence we obtain (5.4). The assertions regarding equality are obvious.

It is clear from the proof that, in (5.4), $|a_1|$ could be replaced by $r(0;D_f)$, with $M' = M/r(0;D_f)^b$.

Remark. Let:

(5.5) $$R_m^*(\varphi) = \prod_{1}^{m} R(\varphi_j), \qquad (0 \le \varphi < 2\pi)$$

with $\varphi_j = \gamma_j(\varphi)$. Then, for every domain D, we have:

$$R_m^*(\varphi_1) = \cdots = R_m^*(\varphi_m), \qquad (0 \le \varphi < 2\pi).$$

Let:

(5-6) $$R_m^{**}(\varphi) = [R_m^*(\varphi) \cdot R_m^*(-\varphi)]^{1/2}.$$

Then, for every domain D:

$$R_m^{**}(\gamma_1(\varphi)) = \ldots = R_m^{**}(\gamma_m(\varphi)) = R_m^{**}(\gamma_1(-\varphi)) = \ldots = R_m^{**}(\gamma_m(-\varphi)).$$

Corollary 5.1. Let $f(\zeta) = \zeta + a_2\zeta^2 + \ldots$ be an analytic function in $|\zeta| < 1$. Suppose that:

(5.7) $R_m^*(\varphi) \le M, \quad (0 \le \varphi < 2\pi),$

for a fixed M, $M < \infty$. (Clearly this can hold only if $1 \le M$.) Then:

(5.8) $i\,y_m \le R^*(\varphi), \quad (0 \le \varphi < 2\pi).$

If instead of (5.7) we have:

(5.7)' $R_m^{**}(\varphi) \le M, \quad (0 \le \varphi < 2\pi),$

then:

(5.8)' $K'_{M,m}(\varphi) \le R_m^{**}(\varphi), \quad (0 \le \varphi < 2\pi).$

Proof. In view of the previous remark this is an immediate consequence of the theorem.

Corollary 5.2. Let $f(\zeta) = a_1\zeta + a_2\zeta^2 + \ldots$ be a B.E. function. Denote: $\lambda = 1/|a_1|$. Then:

(5.9) $|a_1|\,K_{\lambda,m}(\varphi) \le R_m(\varphi), \quad (0 \le \varphi < 2\pi).$

In particular:

(5.10) $|a_1|\,K_{M,m} \le \min.(R_m^*(0), R_m^*(\pi/m)).$

Proof. By (4.14) we have $R(\varphi) - R(-\varphi) \le 1$, $(0 \le \varphi < 2\pi)$. Hence:

$$R_m^{**}(\varphi) \le 1, \qquad (0 \le \varphi < 2\pi). \tag{5.11}$$

Applying the theorem with $R^*(\varphi) = R_m^{**}(\varphi)$ we obtain (5.9). Inequality (5.10) is a particular case of (5.9), since $R_m^{**}(\varphi) = R_m(\varphi)$ for $\varphi = 0$ and $\varphi = \pi/m$.

Corollary 5.3. Let $f = a_1\zeta + a_2\zeta^2 + \dots \in K(-1)$. Denote $\mu = 1/|a_1|$. Then:

$$|a_1| \, K_{\mu,2m} \le R_{2m}^*(\varphi) \qquad (0 \le \varphi < 2\pi). \tag{5.12}$$

Proof. By (4.14), $R(\varphi + \pi) - R(\varphi) \le 1$, $(0 \le \varphi < 2\pi)$. Hence:

$$R_{2m}^*(\varphi) \le 1 \qquad (0 \le \varphi < 2\pi).$$

Applying the theorem with $R^*(\varphi) = R_{2m}^*(\varphi)$ we obtain (5.12).

Remark. In all the corollaries above we have $b = 1$. Hence the case of equality in the various estimates is clear from the statement of the theorem.

Inequality (5.8) was obtained in [11], under the assumption that f is starlike and $M = \infty$, and in [7] for the general case.

If we assume that f is a simple function, then, for $m = 2,3$ and $M = \infty$, it is known that an inequality stronger than (5.8) is valid. In fact, in this case, (5.8) holds if $R(\varphi)$ is replaced by the length of the segment from the origin to the nearest boundary point of D_f, along the ray σ_φ.

For $m = 2$, this result is classical; for $m = 3$ it was obtained by Reich and Shiffer [10].

Inequality (5.10) of Corollary 5.2 was obtained in [1].

It should be noted that $R(\varphi) \le \text{meas.}\{\alpha_{\varphi} \cap D_f\}$ and, in fact, equality holds if and only if $\alpha_{\varphi} \cap D_f$ is a segment (or ray) minus a set of linear measure zero, (see [7], p. 625). Hence, by the arithmetic-geometric mean inequality we have:

$$\Big[\prod_{j=1}^{n} R(\varphi_j)\Big]^{1/n} \le \frac{1}{n}\sum_{j=1}^{n} \text{meas.}\{\alpha_{\varphi_j} \cap D_f\}$$

with equality if and only if $R(\varphi_1) = \dots = R(\varphi_n)$ and each set $\alpha_{\varphi_j} \cap D_f$ $(j = 1, \dots, n)$ is a segment (or ray) minus a null set.

REFERENCES

[1] Aharonov, D. and W. E. Kirwan, "A method of symmetrization and applications", Univ. of Maryland, TR 70-179, (to appear in Trans. AMS).

[2] Bandle, C. and M. Marcus, "Radial averaging transformations with various metrics", (to be published; a preliminary version appeared as: TR 71-32, Carnegie-Mellon University).

[3] Goodman, A. W., "Almost bounded functions", Trans. AMS 78 (1955), 82-97.

[4] Hayman, W. K., Multivalent Functions, Cambridge University Press, 1958.

[5] Jenkins, J. A., "On Bieberbach-Eilenberg Functions III", Trans. AMS 119 (1965), 195-215.

[6] Lebedev, N. A., "Applications of the area principle in problems for non-overlapping domains", Trudy Math. Inst. Steklov 60 (1961), 211-231.

[7] Marcus, M., "Transformations of domains in the plane and applications in the theory of functions", Pacific J. Math. 14 (1964), 613-626.

[8] Nehari, Z., "Some inequalities in the theory of functions", Trans. AMS 7jL(1953), 256-286.

[9] Polya, G- and G. Szego, Isoperimetric Inequalities in Mathe-- matical Physics, Princeton University Ptess, 1951.

[10] Reich, E. and M. Schiffer, "Estimates for the transfinite diameter of a continuum", Math. Zeitschr. 8j^(1964), 91-106.

[11] Szego, G., "On a certain kind of symmetrization and its applications", Ann. Mat. Pura Appl. 40(1955), 113-119.

DEPARTMENT OF MATHEMATICS
CARNEGIE-MELLON UNIVERSITY
PITTSBURGH, PENNSYLVANIA 15213 U.S.A.