ABSOLUTE CONTINUITY ON TRACKS AND' MAPPINGS OF SOBOLEV SPACES

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M. Marcus and V. J. Mizel

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Abstract

The present paper is concerned with the circumstances under which a function $g(x, t_1, \dots, t_m)$ provides, via composition, a mapping between Sobolev spaces. That is, we examine conditions which ensure that for every system of functions $u_1, \dots, u_m \in W_{1,q}(\Omega)$ (where $W_{1,q}(\Omega)$ is the class of L_q functions with L_q summable strong first derivatives on the domain $\Omega \subset R^n$), the composite function v given by $v(x) = g(x, u_1(x), \dots, u_m(x))$ belongs to $W_{1,p}(\Omega)$ with preassigned $1 \leq p < \infty$. Our overall approach in this paper is patterned after a classical chain rule result of Vallée Poussin [8, p. 467] for real functions on a real interval.

By introducing a (seemingly new) definition for absolute continuity of a function $g(t_1, \dots, t_m)$ on the track of an absolutely continuous curve and exploring its properties, we have been able to attain an exact analogue of the above result of Vallée Poussin in the case of functions $g(t_1, \dots, t_m)$ defined on R^m. This result is thereafter utilized in obtaining necessary and sufficient conditions in order that for given functions $u_1, \dots, u_m \in W^{loc}_{1,1}(\Omega)$ the composite function $v = g(u_1, \dots, u_m)$ belong to $W^{loc}_{1,1}(\Omega)$. This last result leads in a relatively straightforward manner to con-

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ditions for g to map $W_{n\,1,q}(ft)^m$ to $W_{n\,1,P}(ft)$. We also obtain a different set of conditions on g under which $g(t_o, t_1, \ldots, t_n)$ takes $W_{2,q}(ft)$ into $W_{1,p}(ft)$ via the composition

$v(x) = g(u(x), d_1u(x), \ldots, S_nu(x))$.

On the other hand for functions $g_\wedge(x, t_1, \ldots, t_m)$ $x e O$, we have obtained fully analogous results only when the function g satisfies a local Lipschitz condition on $fi \times R_m$.

The entire approach relies heavily on a characterization of the spaces $W_{1,p}(ft)$ due to Gagliardo [2].

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Introduction.

The present paper is concerned with the circumstances under which a function $g(x,t_1,\dots,t_m)$ provides, via composition, a mapping between Sobolev spaces. That is, we examine conditions which ensure that for every system of functions $u_1,\dots,u_m \in W_{1,q}(\Omega)$, where $W_{1,q}(\Omega)$ is the class of L_q functions with L_q summable strong first derivatives on the domain $\Omega \subset R_n$, the composite function v given by $v(x) = g(x,u_1(x),\dots,u_m(x))$ belongs to $W_{1,p}(\Omega)$, with preassigned $1 \le p < \infty$. Our overall approach in this paper is patterned after a classical chain rule result of Vallée Poussin [8,p.467] for real functions on a real interval. He showed that when g and u are both absolutely continuous functions then the composite function $g \circ u$ will be absolutely continuous if and only if $g'(u(x))u'(x)$ is summable (when the product is properly interpreted), and that then the chain rule

$$\text{(c)} \qquad \frac{d}{dx}\, g(u(x)) = g'(u(x))u'(x)$$

is valid almost everywhere. In this direction Serrin has shown [5] that for $g : R_1 \to R_1$ locally absolutely continuous and $u \in W_{1,1}^{loc}(\Omega)$, one has $v(x) = g(u(x)) \in W_{1,1}^{loc}(\Omega)$ if and only if $g'(u(x))\nabla u(x) \in L_1^{loc}(\Omega)$.

By introducing a (seemingly new) definition for absolute continuity of a function $g(t_1,\ldots,t_m)$ on the track of an absolutely continuous curve and exploring its properties, we have been able to attain an exact analogue of the above result of Vallée Poussin in the case of functions $g(t_1,\ldots,t_m)$ defined on R_m: if g is absolutely continuous and has a total differential almost everywhere on the track of the absolutely continuous curve $\underset{\sim}{u} = (u_1,\ldots,u_m)$, then the composite function $v(x) = g(u_1(x),\ldots,u_m(x))$ will be absolutely continuous if and only if $\sum \frac{\partial g}{\partial t_i}(\underset{\sim}{u}) \frac{\partial u_i}{dx}$ is summable (when the products are properly interpreted), and then the chain rule analogous to (c) holds. This result is thereafter utilized in obtaining necessary and sufficient conditions in order that for given functions $u_1,\ldots,u_m \in W^{loc}_{1,1}(\Omega)$ the composite function $v = g(u_1,\ldots,u_m)$ belong to $W^{loc}_{1,1}(\Omega)$. This last result leads in a relatively straightforward manner to conditions for g to map $W_{1,q}(\Omega)^m$ to $W_{1,p}(\Omega)$. We also obtain a different set of conditions on g under which $g(t_0,t_1,\ldots,t_n)$ takes $W_{2,q}(\Omega)$ into $W_{1,p}(\Omega)$ via the composition $v(x) = g(u(x),\delta_1 u(x),\ldots,\delta_n u(x))$.

On the other hand for functions $g(\underset{\sim}{x},t_1,\ldots,t_m)$ $\underset{\sim}{x} \in \Omega$, we have obtained fully analogous results only when the function g satisfies a local Lipschitz condition on $\Omega \times R_m$. For the convenience of the reader we discuss these "Lipschitz condition" results prior to the "absolute continuity" results because the analysis in the latter topic is much more delicate. (As is proved in Section 3, every function on R_m which is locally Lipschitz is automatically absolutely continuous on the tracks of <u>all</u> absolutely continuous curves.)

The entire approach relies heavily on a characterization of the spaces $W_{1p}(\Omega)$ due to Gagliardo [2], while in the study of the absolute continuity results we utilize not only the above mentioned result of Vallée Poussin, but also Tonelli's results on absolutely continuous curves [4,p.123], Roger's work on tangent cones [3], results of Banach for real functions on real intervals [4,p.282 and p.113] and some work of Federer [1,p.211 and p.245].

The present paper is completely restricted to situations in which a chain rule analogous to (c) holds in Ω. In a subsequent paper we propose to examine conditions under which $g(\underset{\sim}{x},t_1,...,t_m)$ provides a mapping between Sobolev spaces even though a chain rule is not available, and in addition to examine continuity properties of such mappings.

The plan of the paper is as follows. Section 1 is devoted to background material. Section 2 deals with functions $g(\underset{\sim}{x},t_1,...,t_m)$ which are locally Lipschitz. In Section 3 we introduce the notion of absolute continuity on a track in R_m and discuss its properties, and in Section 4 we apply these results to deal with the case of functions $g(t_1,...,t_m)$ which are absolutely continuous in this new sense.

§1. Preliminaries, We adopt the following notation and conventions. The vector space R_m will always be considered with the Euclidean norm, denoted by | | • £ denotes k-dimensional Lebesgue measure, and Ji_1 denote 1-dimensional Hausdorff measure. Finally, an R_m-valued function v is said to be absolutely continuous on an interval of the real.line provided that the infinite sum $\Sigma|v(a_i^t)-v(a_l)|$ can be made arbitrarily small by making the total length $23|a_I^!-a_l|$ of the disjoint subintervals $[[a_l, Q_I^t]\}$ sufficiently small.

A basic feature in our discussion of mappings from one Sobolev space into another is a characterization of the spaces $W_{1,p}$ in terms of absolute continuity, due to Gagliardo [2]. This characterization will be described below. However we shall first mention some necessary classical results on absolute continuity due to Vallée Poussin [8,p.467] (see also Serrin and Varberg [6,pp.517-518]) and Tonelli [4,p.123].

Lemma 1.1 (Vallée Poussin) . Let w be an absolutely continuous real-valued [respectively, R_m-valued] function on an interval I of the real line. Let N be an $£_1$-null set on R_1 [respectively, an Ji_1-null set on R_m] and set $M = w^{-1}(N)$ fl I. Then $\dot{w} = 0$ $£_1$-a.e. in M.

Proof; It suffices to treat the case where w is real-valued, since $N \subset R_m$ being $!i_1$-null implies that its projection on each axis is

$\mathcal{L}_1$-null, and $\dot{w}_i = 0$ $\mathcal{L}_1$-a.e. in M, $i = 1,\ldots,m$, implies $\dot{w} = 0$ $\mathcal{L}_1$-a.e. in M. Therefore we suppose w to be real.

We may assume that N is a Borel set and hence that M is measurable. Suppose that the assertion of the lemma is not true. Let $e_n = \{\sigma : \sigma \in M$ and $|\dot{w}(\sigma)| \geq \frac{1}{n}\}$ $(n=1,2,\ldots)$. Then $\mathcal{L}_1(\bigcup_{n}^{\infty} e_n) > 0$. Hence for some n, say $n = n_0$, we have $\mathcal{L}_1(e_{n_0}) > 0$. Denote e_n^+ (resp. e_n^-) the subsets of e_n where $\dot{w} > 0$ (resp. $\dot{w} < 0$). Then at least one of the two sets $e^+_{n_0}$, $e^-_{n_0}$ has positive measure. We may assume that $\mathcal{L}_1(e^+_{n_0}) > 0$.

Hence we have the following situation: there exists a measurable subset e of M such that $0 < \mathcal{L}_1(e) < \infty$ and such that $\dot{w}(\sigma) > a$ for all $\sigma \in e$, where a is a fixed positive number. We may also assume that $\bar{e}$ is compact and that it is contained in the interior of I. Let $\sigma_0 \in I - e$ be a point on the left of e. Then

$$w(\sigma) = \int_{\sigma_0}^{\sigma} \dot{w}(\tau)\,d\tau + 2c,$$

where $2c = w(\sigma_0)$.

Let $\chi = \chi_e$ be the characteristic function of e and set:

$$g_1(\sigma) = \dot{w}(\sigma)\chi(\sigma) \text{ and } g_2(\sigma) = \dot{w}(\sigma) - g_1(\sigma).$$

Denote:

$$w_i(\sigma) = \int_{\sigma_0}^{\sigma} g_i(\tau)\,d\tau + c, \quad (i=1,2), \quad \sigma \in I.$$

Hence, $w(a) = w_1(a) + w_2(a)$ and $w_1(a)$ is a monotonic non-decreasing function.

We shall obtain the required contradiction by showing that the range of $w_1(a)$ over e has positive measure while the range of $w_2(a)$ over e is a null set.

Given $\varepsilon > 0$, let O be an open set and C be a closed set such that $C \subset e \subset O \subset I$ and $\ell_1(O-C) < \varepsilon$. The range of $w_1(a)$ over O contains the interval $(c, c+p)$ where $p = a\ell_1(e) > 0$. On the other hand, the measure of the range of $w_1(a)$ over the open set $O - C$ is at most $\int_{O-C} g_1(t)dt$ and this tends to zero when $\varepsilon \to 0$. Hence the range of $w_1(a)$ over e has positive measure.

With ε and O as above, consider now the range of $w_2(a)$ over O. The measure of this range is at most equal to

$$\int_O |g_2(\tau)|d\tau = \int_{O-e} |g_2(\tau)|d\tau,$$

since $g_2 = 0$ on e. Again, this integral tends to zero when $\varepsilon \to 0$. Hence the range of $w_2(a)$ over e is a null set. This completes the proof of the lemma.

We have as a corollary the first half of the following result.

<u>Lemma 1.2</u> (<u>Vallée Poussin</u>). Let w and s be absolutely continuous real-valued functions on intervals I and J, respectively. If $s(J) \subset I$ and $w \circ s$ is absolutely continuous then

(1.1) ^Ti" (a) = w^f (s(a))$\dot{s}$(a) $\mathcal{L}_1$-a.e. on j,

provided that we interpret the right side as zero whenever $\dot{s}$(a)= o, irrespective of whether w' (s(a)) is defined. Conversely, if w is absolutely continuous and if with the above convention w' (s(a))$\dot{s}$(a) is summable on J, then w*a is absolutely continuous and (1.1) holds.

Proof: We give here only the proof of the first half. By absolute continuity w^1 is defined for all points of I with the exception of a null set N and $\dot{s}$ is defined for all points of J with the exception of a null set N'. Now (1.1) is clearly valid for those a in J - N' for which s(a) e I - N. On the other hand by Lemma 1.1 $\dot{s}$(a) = o $\mathcal{L}_1$-a.e. on M = s^{-1}(N). Hence to establish (1.1) it suffices to show that also $\dot{\overline{w \circ s}}$ (a) = 0 $\mathcal{L}_1$-a.e. on M. However

$$w \circ s(M) \subset w(N),$$

and the fact that N" = w(N) is an $\mathcal{L}_1$-null set follows directly from the definition of absolute continuity for w. Thus by Lemma 1.1, $\overline{w \circ s}'$ (OF) = 0 $\mathcal{L}_1$-a.e. on $(w \circ s)^{-1}$ (N") ^ M, which completes the proof.

Recall that an R_m-valued function $\underset{\sim}{w}$ of bounded variation on a real interval I is referred to as a rectifiable curve in R_m and that any real function s on that interval for which s(a') - s(a) is the total variation of $\underset{\sim}{w}$ over [a,a'] for all *o < o** , is referred to as a length function for $\underset{\sim}{w}$. Moreover the range $\underset{\sim}{w}(I)$ is called the track of the curve $\underset{\sim}{w}$.

Lemma 1.3 (Tonelli). If w : I-*R_m is a rectifiable curve in R_m and s is a length function for $\underset{\sim}{w}$ then

(i) s is absolutely continuous if and only if $\underset{\sim}{w}$ is absolutely continuous;

(ii) whenever E c i is measurable, then

$\mathcal{H}_1(\underset{\sim}{w}(E)) \leq \mathcal{L}_1(s(E))$;

(iii) $\dot{s}(a) = |\dot{\underset{\sim}{w}}(a)|$ $\mathcal{L}_1$-a.e. on I;

(iv) for each subinterval $[a_1, o_2]$ of I

$s(a_2) - s(a_1) \geq \int_{\sigma_1} |\underset{\sim}{w}(r)|\, dr$, with equality if and only if w is absolutely continuous on $[o_1, a_2]$.

We omit the proof of this lemma.

Remark: If w is absolutely continuous and if E c i is an Z_1-null set then by Lemma 1.3(i) s(E) is also an $\mathcal{L}_1$-null set, so that W.(w(E)) = 0.

We now proceed with the characterization of $W_{1,p}$.

Definition 1.1. Let *Cl* be a domain in R_n and $u = u(x)$ be an $\mathcal{L}_n$-measurable function on *Cl.* We shall say that u ~~belongs to~~ *A(Cl)* provided that, for almost every line r parallel to any coordinate axis x_i, i = I5...,n, u is absolutely continuous on each compact subinterval of r 0 *Cl.*

If UG A(£2) then it is known that u possesses partial derivatives $\frac{\wedge-}{ox_i}$, i=l,...,n, which are defined X_n-a.e. in *Q* and are Z_n-measurable,

Lemma J_L.4^. Suppose that u e *A(Q)* and M is a measurable subset of *Q* such that for almost every line *r* parallel to the coordinate axis x_I, U(MHT) is a null set. Then $\frac{\partial u}{ox_I} = 0$ $£_n$-a.e. in M.

Proof: Let r be a line parallel to the x_i-axis such that u is locally absolutely continuous on *r* fl *Q* and u(Mflr) is a null set. Then by Lemma 1.1, $\frac{\wedge u}{\quad_i}=0$ a.e. in M H r. Since M is measurable and this result holds for a.e. line r parallel to the x_i-axis it follows that $\frac{\partial u}{ox_I} =0$ $£_n$-a.e. in M.

The characterization of $W_{1,P}$(£2) is as follows.

Lemma J_L.J5 (Gagliardo). Let $1 \le p < \infty$. A function u defined on *Cl* is in $W_{1,p}$*(Q)* if and only if there exists a ueA(f2) such that:

(l) $\tilde{u} = u$ $£_n$-a.e. in *Q;*

(ii) $\frac{\partial \tilde{u}}{\partial x_i} \in L$ (O), (i=l,...,n);

(iii) $\tilde{u} \in L(\Omega)$.

Moreover $\frac{\partial u}{\quad_l}$ coincides a.e. in *Ci* with the corresponding distribution derivative $d_l u$, i = l,...,n.

Finally, if *Q,* is bounded and has the cone property then condition (iii) is superfluous.

Remarks JL.JL. 1. The result was not stated in this form in [2], but is an immediate consequence of Sections 1 and 2 of that paper.

2. It is easily seen from the above that $u e W_{l,p^v}$ (ft) implies that every $\tilde{u}$ in A(ft) which satisfies (i) also satisfies (ii) and (iii).

3. As a consequence of this lemma we have $u e W_n^{loc}$(ft) (i.e., $u e W_{!P}$ (ft^T) for every compact subdomain ft' cz ft) if and only if there exists a u e A(ft) such that

(i) $\tilde{u} = u$ $£_n$-a.e. in ft;

(ii') $\frac{Q u}{ox_i} e L_p^{loc}$(ft), i = l,...,n.

We end the present section by introducing a notion of smallness for sets in $I\!L_k$ which will be crucial for later developments.

Definition 1.j?.. Let S be a subset of $I\!L_k$. We shall say that S has the null intersection property (alternatively, the NI property) if S intersects the track of every absolutely continuous curve in R_k in an Jt_1-null set.

The following result gives a sufficient condition for a set S to have the NI property.

Lemma JL.J5. Let S be a set in K_k such that for a given orthogonal system of coordinates $t = (t_1,...,t_k)$, the orthogonal projection of S on each of the coordinate axes has $£_1$-measure zero. Then S has the null intersection property.

Proof: Let C be an absolutely continuous curve given by $\underset{\sim}{t} = \underset{\sim}{w}(a)$ where $\underset{\sim}{w}(a) = (w_1(a),\dots,w_k(a))$ for a in the compact real interval I. Let S_i be the projection of S on the t_i-axis; by assumption $\mathcal{L}_1(S_i) = 0$. Let $A_i \supset S_i$ be a Borel set of zero measure and set $B_i = w_i^{-1}(A_i)$. Then B_i is a measurable subset of I. Clearly $S \cap C \subset \underset{\sim}{w}(B)$ where $B = \bigcap_1^k B_i$. By Lemma 1.1 $\dot{w}_i(a) = 0$ a.e. in B_i, $i = 1,\dots,k$, and hence $\dot{\underset{\sim}{w}}(a) = \underset{\sim}{0}$ a.e. in B. It follows from Lemma 1.3 that $\dot{s}(a) = 0$ a.e. in B, where s is a length function for $\underset{\sim}{w}$. Since s is absolutely continuous we have that $\mathcal{L}_1(s(B)) = 0$ [4,p. 227] and hence by Lemma 1.3 that $\mathcal{H}_1(\underset{\sim}{w}(B)) = 0$, as claimed.

Remark: A set S satisfying the hypothesis of Lemma 1.6 is in particular an $\mathcal{L}_k$-null set. However the null intersection property can hold even for non-measurable sets.

§2. The Lipschitz Case,

Hereafter we will consider mappings G of the form:

$$(2.1) \qquad G(u_1,\ldots,u_m)(x) = g(x,u_1(x),\ldots,u_m(x)) \quad x \in Q,$$

where $g(x,\underset{\sim}{t})$ is defined for x in the domain Q of R_n and $\underset{\sim}{t}$ in R_m, and the $\{u_j\}$ are measurable functions on Q. We shall denote by S_g the set of points in $Q \times R_m$ where g does not possess a total differential. If g is continuous then S_g is a Borel set ([1],p.211). In particular, S_g is an ℓ_{n+m}-null set if g is locally Lipschitz in $Q \times R_m$, while S_g may equal all of $Q \times R_m$ if g is merely absolutely continuous in Tonelli's sense [4,p.300].

In the present section we shall explore conditions under which a locally Lipschitz g corresponds to a G which maps one Sobolev space $W_{1,p}(Q)$ into another. In Section 4 we reexamine this question for functions g which need not be locally Lipschitz.

Lemma 2.1. Suppose that $g(x,\underset{\sim}{t})$ is defined for $x \in Q$, $\underset{\sim}{t} \in R_m$ and that the following conditions hold:

(i) g is locally Lipschitz on $Q \times R_m$;

(ii) there exists a null subset N of Q such that the projection of $S_g - N \times R_m$ on R_m (to be denoted by S_g^T) has the null intersection property in R_m.

Then for every $\underset{\sim}{u} = (u_1,\ldots,u_m) \in A(Q)^m$ the function $v = G(\underset{\sim}{u})$ is in $A(Q)$ and the chain rule holds, i.e., (2.2) $$\frac{\partial v}{\partial x_i} = \frac{\partial g}{\partial x_i} + \sum_{j=1}^{m} \frac{\partial g}{\partial t_j}(x,u)\frac{\partial u_j}{\partial x_i} \quad \text{a.e. in } \Omega, \quad i = 1,\ldots,n,$$

where the products on the right side are to be interpreted as zero whenever their second factor is zero, irrespective of whether $\frac{\partial g}{\partial t_j}$ is defined.

Remarks 3.1. 1. The chain rule is not valid under significantly weaker hypotheses on S_g, as is clear from the following example for $n = 1$. Let $g(t_1,\ldots,t_m) = \max(t_1,\ldots,t_m)$ and $u_1(x) = \ldots = u_m(x) = x$, $x \in (0,1)$. Then the right side of (2.2) is nowhere defined on $\Omega = (0,1)$, while the left side is identically unity.

2. Note that when $m = 1$ and $g(x,t) = g(t)$, then condition (i), or even the weaker requirement that g be locally absolutely continuous on R_1, already implies condition (ii). This comes about because in one-dimension the existence of a derivative is equivalent to the existence of a total differential.

Proof; The assertion that $v \in A(\Omega)$ is a consequence of the fact, easily established by direct calculation, that a Lipschitz function of an absolutely continuous R_m-valued function on an interval is itself absolutely continuous on that interval.

By elementary arguments, (2.2) is valid for $\frac{\partial v}{\partial x_i}$ at each point $x^0 \in \Omega$ such that the derivatives $\frac{\partial u_j}{\partial x_i}$, $j = 1,\ldots,m$, exist at x^0 and the total differential of g exists at $(x^0, u(x^0))$.

Now let Γ be a line parallel to the x_i-axis in R_n such that $u_1,\dots,u_m$ and hence also v are locally absolutely continuous on $\Gamma \cap \Omega$, and such that $\Gamma \cap N$ is an $\mathcal{L}_1$-null set. Then $C_o = \underset{\sim}{u}(\Gamma\cap\Omega)$ is a countable union of tracks of absolutely continuous curves, and hence $C_o \cap S'$ is an $\mathcal{H}_1$-null set. Let $M = \underset{\sim}{u}^{-1}(C_o \cap S') \cap \Gamma$. By the preceding paragraph the chain rule for $\frac{\partial v}{\partial x_i}$ holds $\mathcal{L}_1$-a.e. on $\Gamma \cap \Omega - M$. Moreover $u_j(M)$ is an $\mathcal{L}_1$-null set, $j = 1,\dots,m$. This follows from the fact that the projection of the $\mathcal{H}_1$-null set $C_o \cap S'$ on any coordinate axis in R_m is $\mathcal{L}_1$-null. Hence by Lemma 1.1 $\frac{\partial u_j}{\partial x_i} = 0$ $\mathcal{L}_1$-a.e. in M, $j = 1,\dots,m$. Let $x^o \in \Gamma \cap \Omega$ be a point where $\frac{\partial \underset{\sim}{u}}{\partial x_i} = \underset{\sim}{0}$ and $\frac{\partial v}{\partial x_i}$ exists. Then we claim that

$$(2.3) \qquad \frac{\partial v}{\partial x_i}(x^o) = \frac{\partial g}{\partial x_i}(x^o, \underset{\sim}{u}(x^o)).$$

Indeed, setting $x_h = (x_1^o,\dots,x_i^o+h,\dots,x_n^o)$, we have:

$$g(x_h,\underset{\sim}{u}(x_h)) = g(x_h,\underset{\sim}{u}(x^o)+o(1)h) = g(x_h,\underset{\sim}{u}(x^o)) + o(1)h,$$

where $o(1)$ tends to zero with h. Here, the fact that $\frac{\partial \underset{\sim}{u}}{\partial x_i}(x^o) = \underset{\sim}{0}$ and that g is locally Lipschitz has been used. It now follows that

$$\frac{v(x_h) - v(x^o)}{h} = \frac{g(x_h,\underset{\sim}{u}(x^o)) - g(x^o,\underset{\sim}{u}(x^o))}{h} + o(1).$$

Letting h tend to zero we obtain (2.3). But this shows that the chain rule for $\frac{\partial v}{\partial x_i}$ also holds $\mathcal{L}_1$-a.e. in M, and hence $\mathcal{L}_1$-a.e.

on $T \cap \Omega$. Since the assumptions on the line T hold for almost every line parallel to the x_i-axis and since the choice of i was arbitrary the proof is complete.

Condition (ii) of the above lemma can be weakened in a special but rather important case. This case is introduced next.

Lemma 2.2. Suppose that $g(x,t) = g(x,t_0,t_1,\ldots,t_n)$ satisfies the following conditions:

(i) g is locally Lipschitz in $\Omega \times R_{n+1}$;

(ii') there exists a null subset N of Ω such that with $S' = S_g - N \times R_{n+1}$ and with T the track of any absolutely continuous curve in R_{n+1}, the projection of $S' \cap T$ on the t_j-axis is $\mathcal{L}_1$-null.

Then for a function $u_0 \in A(\Omega)$ which is such that $\frac{\partial u_0}{\partial x_i}$ coincides $\mathcal{L}_n$-a.e. in Ω with a function $u_i \in A(\Omega)$, $i = 1,\ldots,n$, the function $v = G(u_0,u_1,\ldots,u_n) = G(u)$ is in $A(\Omega)$ and satisfies

$$\frac{\partial v}{\partial x_i} = \frac{\partial g}{\partial x_i} + \sum_{j=0}^{n} \frac{\partial g}{\partial t_j}(x,u)\frac{\partial u_j}{\partial x_i} \quad \mathcal{L}_n\text{-a.e. in } \Omega,\ i = 1,\ldots,n, \tag{2.4}$$

where the products on the right side are to be interpreted as zero wherever their second factor is zero, irrespective of whether $\frac{\partial g}{\partial t_j}$ is defined.

Proof; As noted in Lemma 2.1, the fact that $v \in A(\Omega)$ follows from condition (i). Thus we need only prove (2.4).

We may assume that the set N in (ii') is a Borel null set, since it is in any event contained in such a set. Since S_g is also a Borel set ([1],p.211) it follows that S' is Borel, and hence that the set $M = u^{-1}(S') \cap \Omega$ is a measurable set.

Let Γ be any line parallel to the x_i-axis of R_n chosen so that u and hence also v is locally absolutely continuous on $\Gamma \cap \Omega$. Since $u(M\cap\Gamma) \subset u(M) \cap u(\Gamma\cap\Omega) = S' \cap u(\Gamma\cap\Omega)$, it follows from (ii') that the range of u_0 on $M \cap \Gamma$ is an λ_1-null set. Hence by Lemma 1.1 $\frac{\partial u_0}{\partial x_i} = 0$ $\mathcal{L}_1$-a.e. in $M \cap \Gamma$. It follows that $\frac{\partial u_0}{\partial x_i} = 0$ $\mathcal{L}_n$-a.e. in M, with $i = 1,\dots,n$ since the choice of i above was arbitrary. Therefore $u_{0,i} = 0$ $\mathcal{L}_n$-a.e. in M, $i = 1,\dots,n$. Denote by M'_j that $\mathcal{L}_n$-null subset of M where $(u_{1,j},\dots,u_{n,j}) \neq (0,\dots,0)$.

Next let Γ be any line parallel to the x_j-axis chosen so that the conditions of the preceding paragraph are met and, in addition, so that $(N \cup M'_j) \cap \Gamma$ is an $\mathcal{L}_1$-null set. Then the absolute continuity of u on $\Gamma \cap \Omega$ implies that $u_j(M\cap\Gamma)$ is an λ_1-null set, $j = 1,\dots,n$, since this set differs by at most the point 0 from $u_j((N\cup M'_j)\cap\Gamma)$. Hence by Lemma 1.1 we also have $\frac{\partial u_j}{\partial x_j} = 0$ $\mathcal{L}_1$-a.e. in $M \cap \Gamma$, $j = 1,\dots,n$.

Now for the set $\Gamma \cap \Omega - M$ we again deduce by elementary arguments that equation (2.4) for $\frac{\partial v}{\partial x_j}$ holds $\mathcal{L}_1$-a.e. . On the other hand, on $\Gamma \cap M$ we find as in the previous proof that (2.3) holds at every point x^0 at which $\frac{\partial v}{\partial x_j}$ exists and $\frac{\partial u}{\partial x_j} = (0,\dots,0)$.

Hence the chain rule for $\frac{\partial v}{\partial x_i}$ holds $\mathcal{L}_1$-a.e. on $T \cap O$, and thereby $\mathcal{L}^n$-a.e. in Ω. Since the choice of i was arbitrary this completes the proof.

Theorem 2.1. Let g be as in Lemma 2.1. Suppose that $u_1, \dots, u_m \in W^{loc}_{1,1}(\Omega)$ and set $v = G(u_1, \dots, u_m) = Gu$. Then v is in $W^{loc}_{1,1}(\Omega)$ if and only if the functions

$$(2.5) \qquad v_i = \frac{\partial g}{\partial x_i}(x,u) + \sum_{j=1}^{m} \frac{\partial g}{\partial t_j}(x,u) S_i u_j, \quad i = 1,\dots,n,$$

belong to $L_1^{loc}(\Omega)$, where S_i denotes the distribution derivative and where the products are to be interpreted according to the convention of Lemma 2.1. Moreover we then have

$$(2.6) \qquad v_i = S_i v \quad \mathcal{L}_n\text{-a.e. in } \Omega, \quad i = 1,\dots,n.$$

Proof: For each $j = 1,\dots,m$ let $\tilde{u}_j$ be a function in $A(\Omega)$ which coincides $\mathcal{L}_n$-a.e. in Ω with u_j and is such that

$\frac{\partial \tilde{u}_j}{\partial x_i} = S_i u_j$ $\mathcal{L}_n$-a.e. in Ω, $i = 1,\dots,n$. Existence of such functions is ensured by Lemma 1.5. Let $\tilde{v} = g(x,\tilde{u})$, $\tilde{u} = (\tilde{u}_1, \dots, \tilde{u}_m)$. By Lemma 2.1 $\tilde{v}$ is in $A(\Omega)$ and

$$\frac{\partial \tilde{v}}{\partial x_i} = \frac{\partial g}{\partial x_i}(x,\tilde{u}) + \sum_{j=1}^{m} \frac{\partial g}{\partial t_j}(x,\tilde{u}) \frac{\partial \tilde{u}_j}{\partial x_i} \quad \mathcal{L}_n\text{-a.e. in } \Omega, \ i = 1,\dots,n,$$

with the usual convention regarding products. Obviously $\tilde{v} = v$ $\mathcal{L}_n$-a.e. in Ω and $\frac{\partial \tilde{v}}{\partial x_i} = v_i$ $\mathcal{L}_n$-a.e. in Ω, $i = 1,\dots,n$. Therefore if $v_i \in L_1^{loc}(\Omega)$ then by Lemma 1.5 $v \in W_{1,1}^{loc}(\Omega)$. If, on the other hand, it is assumed that $v \in W_{1,1}^{loc}(\Omega)$ then by Remark 1.1 $\partial_i v$ coincides $\mathcal{L}_n$-a.e. in Ω with $\frac{\partial \tilde{v}}{\partial x_i}$, and since $\frac{\partial \tilde{v}}{\partial x_i} = v_i$ $\mathcal{L}_n$-a.e. in Ω, it follows that $v_i \in L_1^{loc}(\Omega)$. This completes the argument.

By making use of the techniques of the above theorem it is sometimes possible to make stronger statements about the Sobolev space to which v belongs. For example if a given $\underset{\sim}{u} = (u_1, \ldots, u_m) \in W_{1,1}^{loc}(\Omega)^m$ is such that for some $p \in (1,\infty)$,

$$\frac{\partial g}{\partial x_i}(x, \underset{\sim}{u}) \in L_p^{loc}(\Omega), \quad i = 1, \ldots, n, \text{ and}$$

$$\frac{\partial g}{\partial t_j}(x, \underset{\sim}{u})\, \partial_i u_j \in L_p^{loc}(\Omega), \quad j = 1, \ldots, m, \quad i = 1, \ldots, n,$$

then the methods used above suffice to prove that

$v \in W_{1,p}^{loc}(\Omega) \subset W_{1,1}^{loc}(\Omega)$.

We proceed next to describe a set of circumstances in which $v = G\underset{\sim}{u}$ is in $W_{1,p}(\Omega)$ for <u>all</u> $\underset{\sim}{u} \in W_{1,q}(\Omega)^m$. That is, we give conditions under which the mapping G which corresponds to a function g satisfying (i), (ii) of Lemma 2.1 is a mapping from a space $W_{1,q}(\Omega)^m$ into $W_{1,p}(\Omega)$.

<u>Theorem 2.2. Let Ω be a bounded domain in</u> R_n satisfying the cone property. Suppose that the function g defined on $\Omega \times R_m$ satisfies conditions (i) and (ii) of Lemma 2.1 and, in addition,

(iii) for every $(x,\underset{\sim}{t}) \in \Omega \times R_m$ where the derivative mentioned below exists,

(2.7) $$\left|\frac{\partial g}{\partial x_i}(x,\underset{\sim}{t})\right| \leq a_1(x) + b_1|\underset{\sim}{t}|^{\nu} \qquad i = 1,\ldots,n$$

(2.8) $$\left|\frac{\partial g}{\partial t_j}(x,\underset{\sim}{t})\right| \leq a_2(x) + b_2|\underset{\sim}{t}|^{\nu-1} \qquad j = 1,\ldots,m,$$

where $\nu \geq 1$ is a fixed number; a_1 is in $L_p(\Omega)$ for some $1 \leq p < n$; a_2 is in $L_r(\Omega)$ with $r = \frac{\nu}{\nu-1}\cdot\frac{np}{n-p}$ [$r=\infty$ for $\nu=1$]; and $|\underset{\sim}{t}| = |t_1| + \ldots + |t_m|$.

Then

(2.9) $$G : W_{1,q}(\Omega)^m \longrightarrow W_{1,p}(\Omega) \quad \text{with} \quad q = \nu p\cdot\frac{n}{n+(\nu-1)p},$$

and

(2.10) $$\|G(\underset{\sim}{u})\|_{W_{1,p}(\Omega)} \leq \text{const}(1+\|\underset{\sim}{u}\|^{\nu}_{W_{1,q}(\Omega)^m}) \qquad \forall \underset{\sim}{u} \in W_{1,q}(\Omega)^m,$$

where the constant depends on Ω, a_1, a_2, b_1, b_2 and $g(x,0) = f(x)$, but not on $\underset{\sim}{u}$.

<u>Proof</u>; Since the local Lipschitz property for g implies local absolute continuity along lines in R_m, we have by (i) and (iii),

(2.11) $$\begin{aligned} |g(x,\underset{\sim}{t})-g(x,\underset{\sim}{0})| &\leq \left|\int_0^{t_1}\frac{\partial g}{\partial t_1}(x,\sigma,0,\ldots,0)d\sigma\right| + \ldots \\ &\quad + \left|\int_0^{t_m}\frac{\partial g}{\partial t_m}(x,t_1,\ldots,t_{m-1},\sigma)\,d\sigma\right| \\ &\leq a_2(x)|\underset{\sim}{t}| + b_2|\underset{\sim}{t}|^{\nu}. \end{aligned}$$

Now by Lemma 2.1 $g(\cdot,\underset{\sim}{t})$ is in $A(\Omega)$ for each $\underset{\sim}{t} \in R_m$. Hence (2.7) and Lemma 1.5 yield that $g(\cdot,\underset{\sim}{t})$ is in $W_{1,p}(\Omega)$. In particular, $f(x) = g(x,\underset{\sim}{0}) \in W_{1,p}(\Omega)$.

Note that since $1 < p < n$ we have $1 < q < n$. Moreover, $u_j \in W_{1,q}(\Omega)$ implies by the Sobolev imbedding theorem (Sobolev [7], Gagliardo [2]) that $u_j \in L_{q^*}(\Omega)$, where $q^* = \frac{nq}{n-q}$, and

$$\|u_j\|_{L_{q^*}(\Omega)} \leq c_0 \|u_j\|_{W_{1,q}(\Omega)}, \tag{2.12}$$

where the constant c_0 depends on Ω. (It is here that we need the assumption that Ω satisfies the cone property.) Expressing q^* in terms of ν and p we obtain: $q^* = \nu p \cdot \frac{n}{n-p}$. Note too that for $f_1 \in L_r(\Omega)$, $f_2 \in L_q(\Omega)$, Holder's inequality implies that

$$\|f_1 f_2\|_{L_p(\Omega)} \leq \|f_1\|_{L_r(\Omega)} \|f_2\|_{L_q(\Omega)}.$$

Now let $u \in W_{1,q}(\Omega)^m$ and set $v = Gu$. By the proof of Theorem 2.1, v coincides $\mathcal{L}_n$-a.e. in Ω with the function $\tilde{v} \in A(\Omega)$ defined by $\tilde{v} = G\tilde{u}$, where $\tilde{u}_j \in A(\Omega)$ coincides $\mathcal{L}_n$-a.e. in Ω with u_j, $j = 1,\dots,m$. Applying (2.7) and (2.8) to (2.5) we have

$$\begin{aligned}
\left\|\frac{\partial \tilde{v}}{\partial x_i}\right\|_{L_p(\Omega)} &\leq \|a_1 + b_1 |\tilde{u}|^{\nu}\|_{L_p(\Omega)} + \sum_{j=1}^{m} \left\|[a_2 + b_2|\tilde{u}|^{\nu-1}]\frac{\partial \tilde{u}_j}{\partial x_i}\right\|_{L_p(\Omega)} \\
&\leq \|a_1\|_{L_p(\Omega)} + b_1 \||\tilde{u}|\|^{\nu}_{L(\Omega)} \\
&\quad + \sum_{j=1}^{m} \left[\|a_2\|_{L_r(\Omega)} \left\|\frac{\partial \tilde{u}_j}{\partial x_i}\right\|_{L_q(\Omega)} + b_2 \||\tilde{u}|\|_{L_{q^*}(\Omega)}^{\frac{\nu-1}{\nu}\frac{n-p}{n}} \left\|\frac{\partial \tilde{u}_j}{\partial x_i}\right\|_{L_q(\Omega)}^{\frac{n+(\nu-1)p}{n\nu}} \right. \\
&\leq \mathrm{const}(1 + \|\tilde{u}\|^{\nu}_{W_{1,q}(\Omega)^m}),
\end{aligned} \tag{2.13}$$

where the last inequality utilizes (2.12). By Lemma 1.5 we conclude that v is in $W_{1,p}^n(\Omega)$.

Next, we observe that by (2.11) and (2.13) we have the estimate

$$\|\tilde{v}-\tilde{f}\|_{L_p(\Omega)} = \|g(x,\tilde{u})-g(x,0)\|_{L_p(\Omega)} \leq \|a_{\tilde{}}|\tilde{u}|+b_{\tilde{}}|\tilde{u}|^{\nu}\|_{L_p(\Omega)}$$

(2.14)
$$\leq \|a\|_{L_r(\Omega)}\||\tilde{u}|\|_{L_q(\Omega)} + b_2\||\tilde{u}|\|^{\nu}_{L_{p\nu}(\Omega)}$$

$$\leq \mathrm{const}(1+\|\tilde{u}\|^{\nu}_{W_{1,q}(\Omega)^m}) .$$

Combining (2.13) and (2.14) we obtain (2.10), which completes the proof.

It is clear that the above ideas can readily be extended to situations in which the functions u_j belong to distinct Sobolev spaces $W_{1,q_j}(\Omega)$, $j = 1,\ldots,m$. This will require that the $\frac{\partial g}{\partial t_j}$, $j = 1,\ldots,m$, possess different rates of growth. However we content ourselves with giving only Theorem 2.2 here, since in a subsequent paper we shall treat these matters from a more general viewpoint, including an analysis of the continuity properties of G.

Our next result concerns a theorem on mappings G from $W_{2,1}^{loc}(\Omega)$ to $W_{1,1}^{loc}(\Omega)$.

Theorem 2.3. Let g be as in Lemma 2.2. Given $u_0 \in W_{2,1}^{loc}(\Omega)$ set $u_i = S_i u_0$, $i = 1,\ldots,n$. Let $v = G(u_0,u_1,\ldots,u_n) = G\tilde{u}$. Then v is in $W_{1,1}^{loc}(\Omega)$ if and only if the functions

(2.15) $$v_i = \frac{\partial g}{\partial x_i}(x,u) + \frac{\partial g}{\partial t_0}(x,u)\,\partial_i u_0 + \sum_{j=1}^{n} \frac{\partial g}{\partial t_j}(x,u)\,\partial_i \partial_j u_0, \quad i = 1,\dots,n$$

belong to $L_1^{loc}(\Omega)$, where the products are to be interpreted as zero wherever the second factor is zero, irrespective of whether the first factor is defined. Moreover we then have

(2.16) $$v_i = \partial_i v \quad \mathcal{L}_n\text{-a.e. in } \Omega, \; i = 1,\dots,n.$$

Proof: By Lemma 1.5 there exists for each $j = 0,1,\dots,n$ a function $\tilde{u}_j \in A(\Omega)$ such that $\tilde{u}_j = u_j$ $\mathcal{L}_n$-a.e. in Ω and such that $\tilde{u}_j = \frac{\partial \tilde{u}_0}{\partial x_j}$ $\mathcal{L}_n$-a.e. in Ω. Moreover we have by Remark 1.1 that $\frac{\partial \tilde{u}_0}{\partial x_j} = \partial_j u_0$ $\mathcal{L}_n$-a.e. in Ω and $\frac{\partial \tilde{u}_j}{\partial x_i} = \partial_i \partial_j u_0$ $\mathcal{L}_n$-a.e. in Ω. Set $\tilde{v} = g(x,\tilde{u})$. By Lemma 2.2 and the preceding observations, $\tilde{v} \in A(\Omega)$ and,

(2.17) $$\frac{\partial \tilde{v}}{\partial x_i} = v_i \quad \mathcal{L}_n\text{-a.e. in } \Omega, \; i=1,\dots,n.$$

Now suppose that $v \in W_{1,1}^{loc}(\Omega)$. Since $\tilde{v} = v$ $\mathcal{L}_n$-a.e. in Ω it follows by Remark 1.1 that

(2.18) $$\partial_i v = \frac{\partial \tilde{v}}{\partial x_i} \quad \mathcal{L}_n\text{-a.e. in } \Omega, \; i = 1,\dots,n.$$

Thus (2.15) and (2.16) follow from (2.17), (2.18). On the other hand, given that $v_i \in L_1^{loc}(\Omega)$, $i = 1,\dots,n$, it follows from Lemma 1.5 and (2.17) that $v \in W_{1,1}^{loc}(\Omega)$, which completes the argument.

Here again it is possible to prescribe growth conditions on $\frac{\partial g}{\partial x_i}$ and $\frac{\partial g}{\partial t_j}$, $i = 1,\ldots,n$, $j = 1,\ldots,m$, so as to ensure that G is a mapping from a Sobolev space $W_{2,q}(\Omega)$ into a Sobolev space $W_{1,p}(\Omega)$. We give a result of this type below.

<u>Theorem 2.4</u>. Let Ω be a bounded domain in R^n satisfying the cone property. Suppose that the function g defined on $\Omega \times R_{n+1}$ satisfies conditions (i) and (ii') of Lemma 2.2 and, in addition,

(iii') for every $(x, t_0, \underset{\sim}{t}') \in \Omega \times R_{n+1}$ where the derivative mentioned below exists,

$$(2.19) \qquad \left|\frac{\partial g}{\partial x_i}\right| \le a(x) + b|t_0|^{\nu} + c|\underset{\sim}{t}'|^{\omega}, \quad i = 1,\ldots,n,$$

$$(2.20) \qquad \left|\frac{\partial g}{\partial t_0}\right| \le a_0(x) + b_0|t_0|^{\nu_0} + c_0|\underset{\sim}{t}'|^{\omega_0},$$

$$(2.21) \qquad \left|\frac{\partial g}{\partial t_j}\right| \le a_1(x) + b_1|t_0|^{\nu_1} + c_1|\underset{\sim}{t}'|^{\omega_1}, \quad j = 1,\ldots,n,$$

where $\nu, \nu_0, \nu_1, \omega, \omega_0, \omega_1 \ge 1$ are fixed numbers; a is in $L_p(\Omega)$ for some $1 < p < n$; a_0 and a_1 are respectively in $L_{r_0}(\Omega)$ and $L_{r_1}(\Omega)$ for $r_0 = \frac{np\alpha}{(n-p)\alpha - 1}$ and $r_1 = \frac{np\alpha}{n\alpha - 1}$ with

$$\alpha = \max\left\{\frac{\omega}{n+\omega p}, \frac{\nu}{n+2\nu p}, \frac{\omega_0+1}{n+(\omega_0+1)p}, \frac{\nu_0+1}{n+(2\nu_0+1)p}, \frac{\omega_1+1}{n+\omega_1 p}, \frac{\nu_1+1}{n+2\nu_1 p}\right\};$$

and $|\underset{\sim}{t}'| = |t_1| + \ldots + |t_n|$.

Then

$$(2.22) \qquad G : W_{2,q}(\Omega) \longrightarrow W_{1,p}(\Omega) \quad \text{where} \quad q = np\alpha,$$

and

$$(2.23)\qquad \|G(u)\|_{W_{1,p}} \le const(1+\|u_0\|^{\mu+1}_{W_{2,q}(\Omega)}) \qquad \forall u_0 \in \overset{0}{W}_{2,q}(\Omega),$$

where $\mu = \max(\nu, \nu_0, \nu_1, \omega, \omega_0, \omega_1)$ and the constant depends on Ω, a, a_0, a_1, b, b_0, b_1, c, c_0, c_1 and $g(x,0) \equiv f(x)$, but not on u_0.

The proof, which follows from the chain rule of Lemma 2.2 by the same pattern utilized in proving Theorem 2.2, will be omitted.

§3. Absolute Continuity on a Track in R_m.

In the next two sections we examine the mapping G for functions g which are not necessarily locally Lipschitz. Much of our analysis concerns the case $g(x,\underset{\sim}{t}) = g(\underset{\sim}{t})$. The crucial idea occurs already when $n = 1$, and involves the use of a chain rule for $\frac{d}{dx} G(\underset{\sim}{u})$ when $\underset{\sim}{u} : J \to R_m$ is an absolutely continuous curve and g is merely "absolutely continuous on the track of u". This latter notion, which is introduced in the present section, is apparently new. However its properties are analogous to those of the usual notion of absolute continuity for functions defined on a real interval, and it reduces to that notion in the case of a track which is a real interval. The earliest prototype of our chain rule is Lemma 1.2, due to Vallée Poussin for the case $n = m = 1$.

We restrict attention for the present to dimension $n = 1$. Thus we begin by recalling certain notions in the theory of curves which will be needed below (see also [3], and [1,p.235]).

Definition 3.1. Let T be a closed subset of R_m. Given a point $\underset{\sim}{y} \in T$, a unit vector $\underset{\sim}{\theta}$ is called a (bilateral) tangent to T at $\underset{\sim}{y}$ provided that there exist sequences $\{\underset{\sim}{y}_i\}$, $\{\underset{\sim}{y}'_i\}$ in T such that

$$\underset{\sim}{y}_i \to \underset{\sim}{y}, \quad \underset{\sim}{y}'_i \to \underset{\sim}{y}, \tag{3.1}$$

and,

$$\frac{\underset{\sim}{y}_i - \underset{\sim}{y}}{|\underset{\sim}{y}_i - \underset{\sim}{y}|} \to \underset{\sim}{\theta}, \quad \frac{\underset{\sim}{y}'_i - \underset{\sim}{y}}{|\underset{\sim}{y}'_i - \underset{\sim}{y}|} \to -\underset{\sim}{\theta}. \tag{3.2}$$

The (possibly empty) set of tangents to T at y is denoted by $\Theta_T(y)$ [sometimes, $\Theta(y)$].

Note that by this definition θ is in $\Theta_T(y)$ if and only if $-\theta$ is in this set: $-\Theta_T(y) = \Theta_T(y)$. It can be shown that $\Theta_T(y)$ is a closed set of unit vectors [F,p.233].

Definition 3.2. Let T be a closed subset of R, and let f be a real-valued function on T. Then f is said to have the θ-derivative $Df(\theta,y)$ at a point $y \in T$ provided that $\theta \in \Theta(y)$ and for every pair of sequences $\{y_i\}$, $\{y_i'\}$ satisfying (3.1), (3.2) one has

$$(3.3) \qquad \frac{f(y_i) - f(y)}{|y_i - y|} \to Df(\theta,y), \qquad \frac{f(y_i') - f(y)}{|y_i' - y|} \to -Df(\theta,y).$$

If $Df(\theta,y)$ exists and has the same magnitude for all $\theta \in \Theta(y)$:

$$|Df(\theta,y)| = D_T f(y) \geq 0 \qquad \forall \theta \in \Theta(y),$$

then the quantity $D_T f(y)$ is called the tangential derivative of f at y.

Note that by (3.3) $Df(\theta,y)$ exists if and only if $Df(-\theta,y)$ exists, and then $Df(-\theta,y) = -Df(\theta,y)$.

Lemma 3.1. Let $T \subset R_m$ be the track of an absolutely continuous curve v. Then W_1-a.e. in T:

(i) Θ (•%) consists of a single pair of opposing unit vectors;

(ii) v^{-1}(£) is a finite set;

(3.4) 111) (v(t) ^ 0; tev ¹ (y)};

(iv) 0(£) = {± v(t) / $|\dot{v}(t)|$: t∈v^{-1}(y)}.

Proof: By hypothesis v : I --* R , with I a compact interval, is absolutely continuous. Thus $\dot{v}$ is defined £$_1$ - a.e. on I. Let A_1 c I be the null set where $\dot{v}$ is not defined and let A_0 c: i be the set where $\dot{v}$ = 0. If s is an arc length function for v then s is absolutely continuous and $\dot{s}$ = $|\dot{v}|$ $_1$ - a.e• on I (Lemma 1.3). Hence both s(A_1) and s(A_2) are t^-null sets [4, pp. 225, 227]. It follows by Lemma 1.3 that v(A.$_1$ U A«) c: T is an M^null set, which implies (iii) .

We show next that (iv) holds for all yeT - v(A_n U A,J . On the one hand, for v(t) = jr, the quantity **yv(t + h)** / |v(t + h) - y| has the opposing limits + $\dot{v}$(t) / |**$\dot{v}$(t)**| as h -> 0+ and h → 0-, so that the right side of (iv) is a subset of the left. On the other hand, if [%$_i$} in T is such that

$$y_i \quad y, \qquad \overline{yy_i}\Big/|y_i - y| \to \theta\in\Theta(y),$$

then there exists a sequence t.el such that y. = v(t.). We may select a convergent subsequence {t$'_i$} which converges to its limit one sidedly, say t$'_i$ → t_0+. By continuity v(t_0) = y and, by the choice of y, $\dot{v}$(t^) ,£ 0 exists. We then find

$$9 = \lim \overline{yy'_i}\Big/|y'_i - y| = \lim[v(t'_i) - v(t_0)]\Big/|v(t'_i) - v(t_Q)| = \frac{\dot{v}(t_0)}{|\dot{v}(t_0)|}$$

(if $t'_i \to t_o$ ~ this argument gives $\theta = -\dot{v}(t_o) / |\dot{v}(t_o)|$).

Finally let B_1 denote the set of £eT for which the cardinality of v^{-1}(^) is infinite. By a result of Federer [1, p. 245], (see also [4, p. 278]), B_1 is an W_1-null set, so (ii) is proved. Since (iv) ensures that for all £ in $T^* = T - \hat{}(v(A_1 U A_2) U B_1)$, ©(£) is a finite set, we may apply a result of Roger [3] to conclude that ©(jO consists of a unique opposing pair of vectors H_1 - a.e. on T . This yields (i) and thus completes the proof.

Corollary 3.1. With T as above suppose that $v = v^* : J \to R_m$ is an absolutely continuous curve parametered by its arc length s. Then (i) and (ii) hold and (iii) and (iv) can be replaced by the assertions that W_1 - a.e. one has

(3.4*)
$$C\{|v(s)| = 1 : s \in v^{*-1}(y)\},$$
$$®(y) = \{\pm x(s) : s \in v^{*-1}(y)\} = \{\pm \theta_y\}.$$

Proof: Clearly $A_1 U A_2 c A_3$, the latter being the subset at which $\dot{v}$ is undefined or else $|\dot{v}| \neq 1$. However in the present situation Lemma 1.3 implies that $A^{\wedge} c J$ is an $\mathcal{L}_1$-null set and hence $v(A_o)$ is an M_1-null subset of T. The remainder of the argument is as given above.

Lemma 3.2. Let $T c R_m$ be the track of an absolutely continuous curve and let f be a real valued function on T having a tangential derivative $D_T f$ $\mathcal{H}_1$ - a.e. Then $D_T f$ is an $\mathcal{H}_1$-measurable function.

Proof: By assumption T is the track of an absolutely continuous curve v. Hence T is also the track of the Lipschitz curve $v^*: J \to \mathbb{R}^n$ which is obtained from v by reparametrizing by its arc length [1, p. 110], and $|\dot{v}^*(s)| = 1$ $\mathcal{X}_1$ - a.e. on J (Lemma 1.3). By Corollary 3.1 there exists a subset $T' \subset T$ of full $\mathcal{H}^1$-measure such that (3.4) holds, i.e.

$$\Theta(y) = \{\pm \dot{v}^*(s) : s \in v^{*-1}(y)\} = \{\pm \theta_y\},$$

and

$$\{|\dot{v}^*(s)| = 1 : s \in v^{*-1}(y)\}, \quad \forall y \in T'.$$

We proceed to show that the existence of $D_T f(\xi)$ for a point $\xi \in T'$ implies the existence of $\frac{d}{ds} f(v^*(s))$ for all $s \in v^{*-1}(y)$, and in addition implies that

(3.5) $$D_T f(\xi) = \left|\frac{d}{ds} f(v^*(s))\right|, \quad \xi \in T' \cap \text{Domain } D_T f, \; s \in v^{*-1}(y).$$

Let $s \in v^{*-1}(\xi)$. Then for any sequence $h_i \to 0$,

(3.6) $$\frac{f(v^*(s+h_i)) - f(v^*(s))}{h_i} =$$

$$= \frac{f(v^*(s+h_i)) - f(v^*(s))}{h_i} \cdot \frac{|v^*(s+h_i) - v^*(s)|}{h_i},$$

the manipulation being justified by (3.4_2^*). Moreover in order to compare with (3.2) we note that

$$\frac{v^{*}(s+h_i)-v^{*}(s)}{|v^*(s+h_i)-v^*(s)|} = \frac{v^{*}(s+h_i)-v^{*}(s)}{h_i} \cdot \frac{h_i}{|v^*(s+h_i)-v^*(s)|} \to \dot{v}^*(s)\varepsilon,$$

with $\varepsilon = 1$ for $h_i > 0$ and $\varepsilon = -1$ for $h_i < 0$. Hence the expression in (3.6) converges to the limit

$$Df(\dot{v}^*(s), y) \cdot 1 \quad \text{when } h_i > 0,$$

$$Df(-\dot{v}^*(s), y) \cdot (-1) \quad \text{when } h_i < 0.$$

The equality of these expressions implies existence of $\frac{d}{ds} f(v^*(s))$, and in addition,

$$\text{(3.7)} \qquad \frac{d}{ds} f(v^*(s)) = Df(\dot{v}^*(s), \varepsilon) = \pm D_T f(x).$$

This last equation, which follows from the definition of $D_T f$, yields (3.5).

To complete the proof of the lemma it suffices to show that for every Borel set $B \subset R_1$, the set $(D_T f)^{-1}(B) \cap T'$ is H_1-measurable, i.e. is the union of a Borel set and an H_1-null set. Now by a theorem of Banach [4, p. 113] the function $|\frac{d}{ds}(f \circ v^*)| = h$ is defined on an $\mathcal{L}_1$-measurable subset of J and is $\mathcal{L}_1$-measurable there. Moreover, by (3.5)

$$(D_T f)^{-1}(B) \cap T' = v^*(h^{-1}(B)) \cap \text{Domain}(D_T f) \cap T'.$$

However, if M is any $\mathcal{L}_1$-measurable subset of J then $M = N \cup M'$ where M' is a countable union of compact sets and N is an $\mathcal{L}_1$-null set. Hence $v^*(M')$ is a Borel set while $v^*(N)$ is an H_1-null set (by Lemma 1.3). Therefore $v^*[h^{-1}(B)]$ is an H_1-measurable set and the proof is complete.

We are now able to define the notion of "absolute continuity on a track" referred to earlier.

Definition 3.13. Let $T \subset R_m$ be the track of an absolutely continuous curve and let f be a real valued function on T for which $D_T f$ is defined $\mathcal{H}_1$ - a.e. Then f is said to be pre-absolutely continuous on T provided that for each $y_1, y_2 \in T$

$$(3.8) \qquad |f(y_2)-f(y_1)| \leq \int_U D_T f(y)\, d\mathcal{H}_1 y \leq \infty ,$$

whenever U is a closed connected subset of T containing y_1 and y_2. The function f is said to be absolutely continuous on T if in addition to the above, $D_T f$ is $\mathcal{H}_1$-summable.

Remarks 3.14. (1) When $T \subset R_1$ is a real interval we show that the above definition of absolute continuity coincides with the usual one. On the one hand, one sees from (3.8) that $\Sigma|f(y_{i+1})-f(y_i)|$ can be made arbitrarily small by requiring the total length $\Sigma|y_{i+1}-y_i|$ of the disjoint intervals $\{[y_i, y_{i+1}]\}$ to be sufficiently small. On the other hand, absolute continuity of a real function f implies, since $D_T f(y) = |\frac{df}{dy}(y)|$, that

$$|f(y_2)-f(y_1)| = |\int_{(y_1,y_2)} (\frac{d}{dy}f)dy|$$

$$\leq \int_U |\frac{df}{dy}|dy < \infty .$$

(2) If $y : I \to R_m$ is an absolutely continuous curve and $f : T_y \to R_1$ is absolutely continuous on T_y, then the composite function $w = f \circ y$ is continuous. This follows from the

observation that whenever $\{I_n\}$ is a nested family of closed intervals contracting to a point $t_o \in I$ then $\{U_n = \underset{\sim}{v}(I_n)\}$ is a nested family of closed connected subsets of T contracting to $\underset{\sim}{v}(t_o) \in T_{\underset{\sim}{v}}$ and hence by (3.8) the real intervals $w(I_n) = f(U_n)$ have length approaching zero.

We proceed next to show that with the above definition a Lipschitz function on R_m is necessarily absolutely continuous on the track of every absolutely continuous curve.

<u>Lemma</u> 3.3. Let $T \subset R_m$ be the track of an absolutely continuous curve and let $f : R_m \to R_1$ be a Lipschitz function. Then $f|_T$ is absolutely continuous on T. Moreover H_1 - a.e. $D_T f|_T$ coincides with the directional derivative of f along a tangent ray in R_m:

$$(3.9) \qquad \lim_{h \to 0+} \frac{f(\underset{\sim}{x}+h\underset{\sim}{\theta})-f(\underset{\sim}{x})}{h} = Df(\underset{\sim}{\theta},\underset{\sim}{x}) = \pm D_T f(\underset{\sim}{x}), \quad \underset{\sim}{\theta} \in \Theta(\underset{\sim}{x}).$$

Proof: Let $v : I \to R$ be any absolutely continuous curve whose track is T. As in Lemma 3.2 we utilize the Lipschitz curve $\underset{\sim}{v} : J \to R_m$ which is obtained when $\underset{\sim}{v}$ is reparametrized by its arc length s. It is to be proved that $D_T f$ is defined H_1 - a.e. on T and that (3.8) and (3.9) hold.

Consider the composite function $w = f \circ \underset{\sim}{v}^* : J \to R_1$. Since f and $\underset{\sim}{v}^*$ are Lipschitz so is w. We treat w as a curve in R_1. Now by Corollary 3.1 there is a subset $T' \subset T$ of full H_1-measure on which (3.4) holds. Moreover since w is absolutely continuous, the set $A_4 \subset J$ of points where w is non-differentiable is an $\mathcal{L}_1$-null set. Therefore $\underset{\sim}{v}^*(A_4) \subset T$

is W_1-null (Lemma 1.3).

We proceed to show that $D_T f$ exists everywhere on $T'' = T^n - \underset{\sim}{v}(A_4)$. Given $\underset{\sim}{y} \in T''$, hereafter kept fixed, we may partition $\underset{\sim}{v}^{*-1}(\underset{\sim}{y})$ into two subsets

$$C_+ = \{s \in \underset{\sim}{v}^{*-1}(\underset{\sim}{y}) : \underset{\sim}{v}^*(s) = \underset{\sim}{\theta}_{\underset{\sim}{y}}\}, \quad C_- = \{s \in \underset{\sim}{v}^{*-1}(\underset{\sim}{y}) : \underset{\sim}{v}^*(s) = -\underset{\sim}{\theta}_{\underset{\sim}{y}}\}.$$

Now since $\underset{\sim}{y} \in T''$, it follows that the limits below exist for all $s \in \underset{\sim}{v}^{*-1}(\underset{\sim}{y})$:

$$(3.10) \quad \lim_{h \to 0\pm} \frac{f(\underset{\sim}{v}^*(s+h)) - f(\underset{\sim}{v}^*(s))}{|\underset{\sim}{v}^*(s+h) - \underset{\sim}{v}^*(s)|} =$$

$$= \lim_{h \to 0\pm} \frac{f(\underset{\sim}{v}^*(s+h)) - f(\underset{\sim}{v}^*(s))}{h} \cdot \frac{h}{|\underset{\sim}{v}^*(s+h) - \underset{\sim}{v}^*(s)|} = \varepsilon \frac{d}{ds} f(\underset{\sim}{v}^*(s)),$$

where $\varepsilon = 1$ or -1 according as $h \to 0+$ or $h \to 0-$. Moreover the limit obtained is the same for all $s \in C_+$ [respectively, for all $s \in C_-$]. This is a consequence of the relation

$$\lim_{h \to 0\pm} \frac{f(\underset{\sim}{v}^*(s+h)) - f(\underset{\sim}{v}^*(s))}{|\underset{\sim}{v}^*(s+h) - \underset{\sim}{v}^*(s)|} = \lim_{h \to 0\pm} \frac{f(\underset{\sim}{v}^*(s) + h\dot{\underset{\sim}{v}}^*(s) + o(h)) - f(\underset{\sim}{v}^*(s))}{|h\dot{\underset{\sim}{v}}^*(s) + o(h)|}$$

$$(3.11) \quad = \begin{cases} \lim_{h \to 0\pm} \dfrac{f(\underset{\sim}{y} + h\underset{\sim}{\theta}_{\underset{\sim}{y}}) - f(\underset{\sim}{y})}{|h|} & s \in C_+ \\[2ex] \lim_{h \to 0\pm} \dfrac{f(\underset{\sim}{y} - h\underset{\sim}{\theta}_{\underset{\sim}{y}}) - f(\underset{\sim}{y})}{|h|} & s \in C_-, \end{cases}$$

where we have utilized the fact that f is Lipschitz. Note that we may also deduce from this relation that in (3.10) the limits obtained for points $s \in C_+$ are equal in magnitude but opposite in sign to the corresponding limits for points $s \in C_-$.

Now let $\{y_i\}$, $\{y_i'\}$ be sequences in T satisfying (3.1), (3.2) relative to y, i.e.

(3.12) $$y_i \to y, \quad \overline{yy_i} \big/ |y_i - y| \to e_y,$$

$$y_i' \to y, \quad \overline{yy_i'} \big/ |y_i' - y| \to -e_y.$$

We must show that the following quotients

(3.13) $$[f(y_i) - f(y)] \big/ |y_i - y|, \quad [f(y_i') - f(y)] \big/ |y_i' - y_i|,$$

converge respectively to limits $D f(y, e_y)$, $D f(y, -e_y)$ of equal magnitude but opposite sign, which are the same for ~~all~~ sequences $\{y_i\}$ and $\{y_i'\}$ satisfying (3.12). Consider the quotient in (3.13$_1$). Since $\{y_i\} \in T$ there exists a sequence $\{s_i\} \in J$ such that $y_i = v^*(s_i)$. To show convergence of this quotient it suffices to show that one obtains one and the same limit for all subsequences $\{f_i\}$ which correspond to one-sidedly convergent subsequences of $\{s_i\}$. Let $s_j' \to s_0'+$ and $s_k'' \to s_0''-$ be two such one-sidedly convergent subsequences. Then by (3.12$_1$) and (3.4)

$v^*(s_0') = v^*(s_0'') = y$ and

$$\dot{v}^*(s_0') = \lim \frac{v^*(s_j') - v^*(s_0')}{s_j' - s_0'} = \lim \frac{v^*(s_j') - v^*(s_0')}{|v^*(s_j') - v^*(s_0')|} \cdot \frac{|v^*(s_j') - v^*(s_0')|}{s_j' - s_0'}$$

$$= e_y$$

$$\dot{\underset{\sim}{v}}^*(s''_0) = \lim \frac{\underset{\sim}{v}^*(s''_k)-\underset{\sim}{v}^*(s'_0)}{s'_k-s''_0} = \lim \frac{\underset{\sim}{v}^*(s''_k)-\underset{\sim}{v}^*(s''_0)}{|\underset{\sim}{v}^*(s''_k)-\underset{\sim}{v}^*(s''_0)|} \cdot \frac{|v^*(s''_k)-v^*(s''_0)|}{s_k-s''_0}$$

$$= -\underset{\sim}{e}_{\underset{\sim}{y}}$$

so that S'LGC,, s"eC_. Consequently (3.11) implies existence of these limits with

$$\lim \frac{f(\underset{\sim}{y}_j)-f(\underset{\sim}{y})}{|\underset{\sim}{y}_j-\underset{\sim}{y}|} = \lim_{h\to 0+} \frac{f(\underset{\sim}{y}+h\underset{\sim}{\theta}_{\underset{\sim}{y}})-f(\underset{\sim}{y})}{h},$$

$$\lim \frac{f(\underset{\sim}{y}_k)-f(\underset{\sim}{y})}{|\underset{\sim}{y}_k-\underset{\sim}{y}|} = \lim_{h\to 0-} \frac{f(\underset{\sim}{y}-h\underset{\sim}{\theta}_{\underset{\sim}{y}})-f(\underset{\sim}{y})}{-h}.$$

Here the right sides are clearly equal and independent of the particular subsequences $\{s'_j\}$, $\{s''_k\}$ chosen, so that convergence of the quotient in (3.13-1) has been proved. Moreover, the same argument shows that the quotient in (3.13₉) converges to a limit of equal magnitude but opposite sign. It now follows by Definition 3.2 that $D_T f(\underset{\sim}{y})$ exists and that

$$D_T f(z) = \left|\lim_{h\to 0+} \frac{f(\underset{\sim}{y}\pm h\underset{\sim}{\theta})-f(\underset{\sim}{y})}{h}\right|,$$

which yields (3.9).

It remains to prove absolute continuity. By Lemma 1.3 one has for any measurable subset $E \subset j_9$

$$\mathcal{H}_1(w(E)) \leq \mathcal{L}_1(\rho(E)),$$

where ρ is an arc length function for w. Now consider the closed subset $\underset{\sim}{v}^{*-1}(U) \subset j$. By a theorem of Banach [4, p. 282] there exists a measurable subset $E \subset v^{*-1}(U)$ such that $v^*(E) = U$ and v^* is injective on E. Thus the set

$$w(E) = w(v^{*-1}(U)) = f(U) \subset R_1$$

is a connected set containing the numbers $f(y_1)$, $f(y_2)$. Since on R_1, $\mathcal{H}_1$-measure coincides with $\mathcal{L}_1$-measure we have

$$|f(y_2)-f(y_1)| \le \mathcal{H}_1(f(U)) = \mathcal{H}_1(w(E)).$$

On the other hand since by Lemma 1.3 ρ is absolutely continuous on J, it follows from [4, p. 227] that

$$\mathcal{L}_1(\rho(E)) \le \int_E \dot{\rho}(s)\,ds.$$

Combining these inequalities, we obtain

$$|f(y_2)-f(y_1)| \le \int_E \dot{\rho}(s)\,ds = \int_E |\dot{w}(s)|\,ds,$$

the last equation following by Lemma 1.3. Now for $s \in v^{*-1}(T'')$ it follows from (3.9) that

$$|\dot{w}(s)| = \left|\frac{d}{ds} f(v^*(s))\right| = D_T f(v^*(s)).$$

Thus we may write

(3.14) $$|f(y_2)-f(y_1)| \le \int_{E \cap v^{*-1}(T'')} D_T f(v^*(s))\,ds + \int_{E'} |\dot{w}(s)|\,ds,$$

where $E' = E - E \cap v^{*-1}(T'')$. Moreover, since $v^*(E') = v^*(E) - T''$ is $\mathcal{H}_1$-null it is not difficult to show using the Lipschitz property of f and the definition of $\wedge_1$.-measure that the set

$$w(E') = f(\underset{\sim}{v}^* (E'))$$

is IL_1-null. It then follows by Lemma 1.1 that the second integral in (3.14) is zero.

Finally, using the fact that $D_T f$ is W_1-measurable (Lemma 3.2) and that $|\dot{v}^*(s)| = 1$ on $E \cap v^{*-1}(T'')$, we may employ a result of Federer [1, p. 245] to write the first integral in (3.14) as follows:

$$\int_{E \cap \underset{\sim}{v}^{*-1}(T'')} D_T f(\underset{\sim}{v}^*(s))ds = \int_{\underset{\sim}{v}^*(E) \cap T''} D_T f(\underset{\sim}{y}) \cdot 1 d\mathcal{H}_1 \underset{\sim}{y} = \int_U D_T f(\underset{\sim}{y}) d\mathcal{H}_1 \underset{\sim}{y},$$

where the factor 1 in the second integral is a consequence of our choice of E. This completes the proof.

Our next result yields some of the conclusions of Lemma 3.3 for functions with the null intersection property.

<u>Lemma 3.4</u>. Let $T \subset R_m$ be the track of an absolutely continuous curve and let $f : R_m \to R_n$ be such that Sp_1 has the null intersection property. Then $f|_1$ has an W_1-measurable tangential derivative defined W_1 - a.e., and except on an W_1-null set, $D_T f|_T$ coincides with the directional derivative of f along a tangent ray:

$$(3.15) \qquad Df(\theta, y) = \lim_{h \to 0+} \frac{f(\underset{\sim}{y}+h\underset{\sim}{\theta}) - f(\underset{\sim}{y})}{h} = \nabla f(\underset{\sim}{y}) \cdot \underset{\sim}{\theta}, \quad \underset{\sim}{\theta} \in \Theta(\underset{\sim}{y}).$$

<u>Proof</u>: As noted before there exists a Lipschitz curve

$\underset{\sim}{v}^*$: $J \to R_m$ whose track is T and which is parametrized by arc length. By Corollary 1.1 there exists a subset $T' \subset T$ of full $\mathcal{H}_1$-measure such that (3.4*) is valid for all $\underset{\sim}{y} \in T'$. We proceed to show that $D_T f(\underset{\sim}{y})$ exists and satisfies (3.15) for every $\underset{\sim}{y} \in T = T' - S_f$. Let $\{y_i\}$, $\{y_i'\}$ be sequences in T satisfying (3.12) relative to $\underset{\sim}{y}$. We claim that the following quotients

$$f(\underset{\sim}{y}_i)-f(\underset{\sim}{y}) \Big/ |\underset{\sim}{y}_i-\underset{\sim}{y}| , \quad f(\underset{\sim}{y}_i')-f(\underset{\sim}{y}) \Big/ |\underset{\sim}{y}_i'-\underset{\sim}{y}| ,$$

converge respectively to limits $Df(\theta_{\underset{\sim}{y}},\underset{\sim}{y})$, $Df(-\theta_{\underset{\sim}{y}},\underset{\sim}{y})$ of equal magnitude and opposite sign, which are the same for all sequences $\{y_i\}$, $\{y_i'\}$ satisfying (3.12). In fact, since $\underset{\sim}{y} \notin S_f$ we have

$$\frac{f(\underset{\sim}{y}_i)-f(\underset{\sim}{y})}{|\underset{\sim}{y}_i-\underset{\sim}{y}|} = \frac{\nabla f(\underset{\sim}{y})\cdot(\underset{\sim}{y}_i-\underset{\sim}{y})+o(|\underset{\sim}{y}_i-\underset{\sim}{y}|)}{|\underset{\sim}{y}_i-\underset{\sim}{y}|} \to \nabla f(\underset{\sim}{y})\cdot\theta_{\underset{\sim}{y}} = Df(\theta_{\underset{\sim}{y}},\underset{\sim}{y})$$

$$\frac{f(\underset{\sim}{y}_i')-f(\underset{\sim}{y})}{|\underset{\sim}{y}_i'-\underset{\sim}{y}|} = \frac{\nabla f(\underset{\sim}{y})\cdot(\underset{\sim}{y}_i'-\underset{\sim}{y})+o(|\underset{\sim}{y}_i'-\underset{\sim}{y}|)}{|\underset{\sim}{y}_i'-\underset{\sim}{y}|} \to \nabla f(\underset{\sim}{y})\cdot(-\theta_{\underset{\sim}{y}}) = Df(-\theta_{\underset{\sim}{y}},\underset{\sim}{y}) .$$

Thus $D_T f(\underset{\sim}{y})$ exists and (3.15) holds. Measurability of $D_T f\big|_T$ follows by Lemma 3.2.

It can be seen from the above proof that we actually have:

<u>Corollary 3.2</u>. Let T be as above and let $f : R_m \to R_1$ be a function such that $S_f \cap T$ is $\mathcal{H}_1$-measurable. Then $f\big|_T$ has a tangential derivative $\mathcal{H}_1$ - a.e. on $T - S_f$, and except for an $\mathcal{H}_1$-null set $D_T f\big|_T$ coincides wherever it exists with the tangential derivative of f along a tangent ray in the sense of (3.15).

One of the important properties of an absolutely continuous function f on a real interval I is that it carries null sets into null sets:

$$A \subset I,\ \mathcal{L}_1(A) = 0 \Rightarrow \mathcal{L}_1(f(A)) = 0.$$

This property of real valued functions f was introduced by Lusin, who called it the (N) condition [4, p. 224]. We end this section with a proof that a function f which is absolutely continuous in our sense on a track T, satisfies the exact analogue of the (N) condition.

Theorem 3.1. Let T be the track of an absolutely continuous curve and let $f : T \to R_1$ be a function which is absolutely continuous on T. Then f satisfies the following condition:

(N_T) $$B \subset T,\ \mathcal{H}_1(B) = 0 \Rightarrow \mathcal{L}_1(f(B)) = 0.$$

Remark. It should be noted that the (N) condition does not characterize absolutely continuous functions on real intervals, and thus the (N_T) condition does not characterize absolutely continuous functions on T.

Proof: As noted earlier there exists a Lipschitz curve $v : J \to R$ whose track is T and which is parametrized by arc length. If $B \subset T$ is $\mathcal{H}_1$-null then by Lemma 1.1 $v' = 0$ $\mathcal{L}_1$-a.e. on $A = v^{-1}(B)$. Since $|v'| = 1$ $\mathcal{L}_1$-a.e. it follows that $A \subset J$ is an $\mathcal{L}_1$-null set. Given $\varepsilon > 0$ let $\{J_n\}_{n \ge 1} \subset J$ be a countable collection of disjoint closed intervals covering A such that $\sum_n \mathcal{L}_1(J_n) < \varepsilon$. Then

$\mathcal{B} = \{B_n = \underset{\sim}{v}^*(J_n)\}_{n\geq 1}$, is a cover of B_5 which consists of closed connected subsets of T\ We observe next that $C = \bigcup_{n=1}^{\infty} B_n \Rightarrow B$ can be decomposed as a union $C = \bigcup_i C_i$ of disjoint connected sets C_i which are "connected chains" in the following sense: each C_i is the union of countably many sets $\{B_\alpha^{(i)}\}_{\alpha\geq 1} \in \mathcal{B}$ such that for all $k \geq 1$, $\bigcup_{\alpha=1}^{k} B_\alpha^{(i)} \subset C_i$ is connected as well as closed. We describe the construction of C_1. Set $B_1^{(1)} = B_1$. If $B_n \cap B_1 = \emptyset$ for all $n > 1$, then $C_1 = B_1^{(1)}$. Otherwise let $k_2 > 1$ be the smallest index such that $B_{k_2} \cap B_1^{(1)} \neq \emptyset$, and set $B_2^{(1)} = B_{k_2}$. If $B_1^{(1)} \cup B_2^{(1)}$ is disjoint from the other sets in $\mathcal{B}$ then $C_1 = B_1^{(1)} \cup B_2^{(1)}$. Otherwise let $k_3 \neq 1$ or k_2 be the smallest index such that B_{k_3} meets $B_1^{(1)} \cup B_2^{(1)}$, and set $B_3^{(1)} = B_{k_3}$. Proceeding in this manner we either terminate after finitely many steps or obtain a sequence $\{B_\alpha^{(1)}\}_{\alpha\geq 1}$. In either case the union C_1, of the resulting set $\{B_\alpha^{(1)}\}$ is a "connected chain", which is disjoint from all elements of $\mathcal{B}_{(2)} = \mathcal{B} - \{B_\alpha^{(1)}\}_{\alpha\geq 1}$. If $C_1 = \bigcup_1^{\infty} B_n$ we are finished. Otherwise we may construct in the same manner as above a connected chain C_2 from the elements of $\mathcal{B}_{(2)}$ beginning with the $B_n \in \mathcal{B}_{(2)}$ of lowest index. Obviously $\mathcal{B}$ will be exhausted after at most countably many steps. Thus

$$\bigcup_{i=1}^{\infty} C_i = \bigcup_{n=1}^{\infty} B_n \supset B.$$

We now utilize the absolute continuity of f to show that for any connected chain C_i, the length of the interval $f(C_i) \subset R_1$ satisfies

(3.16) $$\mathcal{L}_1(f(C_i)) \leq \int_{C_i} D_T f(y)\, d\mathcal{H}_1 y.$$

This follows from the fact that whenever $y_1, y_2 \in C_i$ then by the "connected chain" property there exists a finite index k such that

$$y_1, y_2 \in \bigcup_{a=1}^{k} A^{(i)} = U_{i,k},$$

and thus since $U_{i,k}$ is both closed and connected we have

$$|f(y_1) - f(y_2)| \leq \int_{U_{i,k}} VW^{d} \wedge \mathcal{L} \int_{I} V(y)\, d\mathcal{H}_1 y.$$

To complete the proof observe that by Lemma 1.3

$$\mathcal{H}_1(v^*(J_n)) \leq \mathcal{L}_1(J_n).$$

Hence, denoting by $J_a^{(i)}$ an interval J_n for which $v^*(J_n) = B_J^{(i)}$, we have

$$\mathcal{H}_1(C_i) \leq \sum_{a=1}^{\infty} \cdots \sum_{a=1}^{\infty} \quad)).$$

Thus by (3.16) the total length of the family of intervals $\{f(C_i)\}$ satisfies

$$\sum_i \mathcal{L}_1(f(C_i)) \leq \sum_i \sum_a \mathcal{L}_1(J^{(i)}) \leq \sum_n \mathcal{L}_1(J_n) < \varepsilon.$$

Since

$$f(B) \subset f(C) = \bigcup_{i=1}^{\infty} f(C_i),$$

it follows that f(B) is covered by families of intervals of arbitrarily small total length. This completes the proof.

§4. The Chain Rule for Tracks; Applications.

We begin this section by giving our version in R_m of Lemma 1.2, the chain rule obtained by Vallée Poussin for $m = 1$. The analogues of the direct and converse portions of Lemma 1.2 are stated separately.

Theorem 4.1. Let $\underset{\sim}{v} : I \to R_m$ be an absolutely continuous curve and let $g : R_m \to R_1$ be such that S_g has the null intersection property. If the following conditions hold

(*) $\nabla g(\underset{\sim}{v}) \cdot \dot{\underset{\sim}{v}}$ is $\mathcal{L}_1$-summable on I (with the product interpreted as zero wherever $\dot{\underset{\sim}{v}} = \underset{\sim}{0}$),

(ac_1) $g|_{T_{\underset{\sim}{v}}}$ is pre-absolutely continuous on $T_{\underset{\sim}{v}}$

then $g|_{T_{\underset{\sim}{v}}}$ is actually absolutely continuous on $T_{\underset{\sim}{v}}$. In addition $w = g \circ \underset{\sim}{v}$ is absolutely continuous on I, and the chain rule holds, i.e.

(4.1) $$\dot{w} = \nabla g(\underset{\sim}{v}) \cdot \dot{\underset{\sim}{v}} \qquad \mathcal{L}_1\text{-a.e. on } I,$$

with the above interpretation for the product term when $\dot{\underset{\sim}{v}} = \underset{\sim}{0}$.

Remarks 4.1. 1. The function g is assumed to be defined on all of R_m merely for convenience. What is actually needed for the proof of (4.1) is that $g|_{T_v}$ be extendable to a function possessing a total differential $\mathcal{H}_1$-a.e. on $T_{\underset{\sim}{v}}$ and satisfying (*).

2. It follows from the null intersection property that $D_T g$ exists $\mathcal{H}_1$-a.e. and is $\mathcal{H}_1$-measurable on $T_{\underset{\sim}{v}}$ (Lemma 3.4). Moreover

it will be seen from the proof that even without (ac_1), condition (*) implies that D_g is H.-summable on $\underset{\sim}{T}$.

Proof; Along with $\underset{\sim}{v}$ we again examine the Lipschitz curve $\underset{\sim}{v}^* : J \to R_m$ which is obtained from $\underset{\sim}{v}$ by reparametrizing by its arclength s. We proceed to show that the functions $g \circ \underset{\sim}{v}$ and $g \circ \underset{\sim}{v}^*$ are absolutely continuous on their respective intervals. Given $a_1 < a_2 \in I$ set $s_1 = s(a_1)$, $s_2 = s(a_2)$ and $\underset{\sim}{y}_1 = \underset{\sim}{v}(a_1) = \underset{\sim}{v}^*(s_1)$, $\underset{\sim}{y}_2 = \underset{\sim}{v}(a_2) = \underset{\sim}{v}^*(s_2)$. Then (ac_1) gives,

$$|g(y_2)-g(y_1)| \le \int_U D\_g(y)\,dH_1(y) \le \int_U D\_g(y)N(v,[a_1,a_2],y)\,dH_1 y =$$

(4.2)

$$= \int_U D\_g(y)N(v^*,[s_1,s_2],y)\,dH_1 y,$$

where $U = \underset{\sim}{v}([a_1,a_2]) = \underset{\sim}{v}^*([s_1,s_2])$, and $N(\underset{\sim}{v},[a_1,a_2],y)$ [respectively, $N(\underset{\sim}{v}^*,[s_1,s_2],y)$] denotes the cardinality of $\underset{\sim}{v}^{-1}(y) \cap [a_1,a_2]$ [respectively, of $\underset{\sim}{v}^{*-1}(\underset{\sim}{y}) \cap [s_1,s_2]$]. The H_1-measurability of both functions N follows by results of Federer [1,p.177], while the equality of the last two integrals can be seen as follows. The sets $\underset{\sim}{v}^{-1}(\underset{\sim}{y}) \cap [a_1,a_2]$ and $\underset{\sim}{v}^{*-1}(\underset{\sim}{y}) \cap [s_1,s_2]$ have precisely the same cardinality unless $\underset{\sim}{y}$ is such that for some $a' < a'' \in I$, $\underset{\sim}{v}(a') = \underset{\sim}{v}(a'') = \underset{\sim}{y}$ and $s(a') = s(a'')$. Since the monotone function s is then constant on $[a',a'']$ it follows that $\underset{\sim}{y} \in \underset{\sim}{v}(A)$ where $A = \{a : \dot{s}(a) = 0\} \subset I$. However by absolute continuity of s, $s(A)$ is an $\mathcal{L}_1$-null set [4,p. 227]. Therefore by Lemma 1.3 $\underset{\sim}{v}(A)$ is an $\mathcal{H}_1$-null set, so the integrands in the last two integrals of (4.2) are equal $\mathcal{H}_1$-a.e.

Now by Lemma 3.1, Corollary 1.1 and (3.15) there is a subset $S \subset U$ of full $\mathcal{H}_1$-measure such that

$$D_T g(y) = |\nabla g(y) \cdot \dot{v}(\sigma)/|\dot{v}(\sigma)||$$

$$= |\nabla g(y) \cdot \dot{v}^*(s)| \qquad \forall y \in S,\ \sigma \in v^{-1}(y),\ s \in v^{*-1}(y).$$

By use of this relation and a result of Federer [1,p.245] we deduce from (4.2) the relations

$$|g(y_2)-g(y_1)| \le \int_U |\nabla g(y) \cdot \dot{v}/|\dot{v}|| \cdot N(v,[\sigma_1,\sigma_2],y)\, d\mathcal{H}_1 y = \int_{\sigma_1}^{\sigma_2} |\nabla g(v(\sigma)) \cdot \dot{v}(\sigma)|\, d\sigma$$

(4.3)

$$|g(y_2)-g(y_1)| \le \int_U |\nabla g(y) \cdot \dot{v}^*| \cdot N(v^*,[s_1,s_2],y)\, d\mathcal{H}_1 y = \int_{s_1}^{s_2} |\nabla g(v^*(s)) \cdot \dot{v}^*(s)|\, ds.$$

(Note that the values of the integrand over the $\mathcal{H}_1$-null set $U - S$ are irrelevant since by Lemma 1.1 $\dot{v} = 0$ $\mathcal{L}_1$-a.e. on $v^{-1}(U-S)$ and $\dot{v}^* = 0$ $\mathcal{L}_1$-a.e. on $v^{*-1}(U-S)$.) By appeal to (*) we see that the right side of (4.3_1) and hence, by equality of the left hand integrals above (see(4.2)), also the right side of (4.3_2) is finite. In other words, we have

$$|g(v(\sigma_2))-g(v(\sigma_1))| \le \int_{\sigma_1}^{\sigma_2} |\nabla g(v(\sigma)) \cdot \dot{v}(\sigma)|\, d\sigma < \infty,$$

$$|g(v^*(s_2))-g(v^*(s_1))| \le \int_{s_1}^{s_2} |\nabla g(v^*(s)) \cdot \dot{v}^*(s)|\, ds < \infty.$$

These relations yield the absolute continuity of $g \circ v$ and $g \circ v^*$ by direct calculation.

Now since the functions $g \circ v^*$, s and $w = g \circ v = (g \circ v^*) \circ s$ are all absolutely continuous on their respective intervals it follows by Lemma 1.2 that

(4.4) $$\dot{w} = \frac{d}{ds}(g \circ v^*) \cdot \dot{s} \qquad \mathcal{L}_1\text{-a.e. on } I,$$

with the convention that the right side is zero wherever $\dot{s} = 0$. Moreover for a full subset $T_1 \subset T$ we have

(4.5) $dg(y)$ exists, $\dot{v}(a)/|\dot{v}(a)| = \dot{v}^*(s(a))$, $\dot{s}(a) = |\dot{v}(a)|$ $\forall y \in T_1$, $a \in v^{-1}(y)$.

Hence by Lemma 3.4,

$$\frac{d}{ds}(g \circ v^*) \dot{s} = Dg\left(\frac{\dot{v}^*}{|\dot{v}^*|}, v\right)|\dot{v}| = \nabla g(v) \cdot \dot{v}, \text{ for } v = v^*(s) = y \in T_1.$$

On the other hand on $A = v^{-1}(T-T_1)$, the functions $\dot{v}$ and $\dot{s} = |\dot{v}|$ are zero $\mathcal{L}_1$-a.e. by Lemma 1.1. Moreover by Theorem 3.1

$$w(A) = g(T-T_1)$$

is an $\mathcal{L}_1$-null set, so that, again by Lemma 1.1, $\dot{w} = 0$ $\mathcal{L}_1$-a.e. on A. Together these facts yield (4.1).

Theorem 4.2. Let $v : I \to R_m$ and $g : R_m \to R_1$ be as in Theorem 4.1. Suppose in addition that the following conditions hold,

(ac_1) $g|_{T_v}$ is absolutely continuous on T_v,

(ac_2) $w = g \circ v$ is absolutely continuous on I.

Then the chain rule (4.1) holds, i.e.

$$\dot{w} = \nabla g(\underset{\sim}{v}) \cdot \dot{\underset{\sim}{v}}, \qquad £_1\text{-a.e. on } I$$

(with the same interpretation for the product as before), and hence (*) is valid.

Proof: Let us introduce $v^* : J \to R_m$ as before and let $T_1 \subset T$ again be a subset of full W_1-measure such that (4.5) is valid. Since $T - T_1$ is W_1-null we deduce as before that

$$\dot{\underset{\sim}{v}} = 0, \ \dot{w} = 0 \quad £_1\text{-a.e. on } A = \underset{\sim}{v}^{-1}(T-T_1).$$

Moreover for all $\sigma \in I - A = \underset{\sim}{v}^{-1}(T_1)$ we have, using (4.5)

$$\frac{\underset{\sim}{w}(a+h)-\underset{\sim}{w}(a)}{h} = \frac{g(\underset{\sim}{v}(a+h))-g(\underset{\sim}{v}(a))}{h} = \frac{\nabla g(\underset{\sim}{v}(a))\cdot(\underset{\sim}{v}(a+h)-\underset{\sim}{v}(a))+o(|\underset{\sim}{v}(a+h)-\underset{\sim}{v}(a)|)}{h}$$

$$= \nabla g(\underset{\sim}{v}(a))\cdot(\dot{\underset{\sim}{v}}(\sigma)+o(1))+o(1)\cdot\frac{|\underset{\sim}{v}(a+h)-\underset{\sim}{v}(a)|}{h}$$

$$\to \nabla g(\underset{\sim}{v}(a))\cdot\dot{\underset{\sim}{v}}(o).$$

Together these results yield (4.1) and thereby (*), since absolute continuity of w implies $£_1$-summability of $\dot{w}$.

We now wish to employ Theorems 4.1 and 4.2 in obtaining generalized versions of the results obtained in section 2 for locally Lipschitz functions. It will be necessary to introduce the following definition.

<u>Definition 4.1.</u> Let Q be a domain in R_n and let $\underset{\sim}{u} = (u_1,\ldots,u_m)$ be in $A(Q)^m$. Suppose that $g : R_m \to R_1$ is a real-valued function on R_m. We shall say that g is <u>locally</u> $\underset{\sim}{u}$-<u>absolutely continuous</u> provided that g is Borel measurable and, for almost all lines r parallel to any one of the axes in R_n, g is absolutely continuous on every track of the form $T_{u,I} = \underset{\sim}{u}(I)$ where the interval I c r fl Q, is compact.

If $\underset{\sim}{u} = (u_1,\ldots,u_m)$ where the u_l are $£_n$-measurable functions on Cl which are equal a.e. to functions $\tilde{u}_1 \in A(Q)$, i = l,...,m, then g is said to be <u>locally</u> $\underset{\sim}{u}$-<u>absolutely continuous</u> provided that it is locally $\underset{\sim}{\tilde{u}}$-absolutely continuous, where $\underset{\sim}{\tilde{u}} = (\tilde{u}_1,\ldots,\tilde{u}_m)$.

Our next result is an analogue of Theorem 2.1.

<u>Theorem 4.3.</u> Let Q be a domain in R_n and let $u = (u_1,\ldots,u_m)$, where $u_1,\ldots,u_m \in W_{1,1}^{loc}(Cl)$. Suppose that $g : R_m \to R_l$ is locally $\underset{\sim}{u}$-absolutely continuous and S_g has the null intersection property. Set $v = G(u_1,\ldots,u_m) = G\underset{\sim}{u}$. Then v is in $W_{1,1}^{loc}(Q)$ if and only if the functions

$$(4.6) \qquad v_l = \sum_{j=1}^{m} \frac{\partial g}{\partial t_j}(\underset{\sim}{u})\partial_i u_j \qquad i = 1,\ldots,n,$$

belong to $L_1^{loc}(Cl)$, where d_i denotes a distribution derivative and where the products in (4.6) are to be interpreted as zero wherever their second factor is zero. Moreover we then have

$$(4.7) \qquad v_l = d_i v \qquad £_n\text{-a.e. in } £2, \quad i=l,\ldots,n.$$

Remark; It will be seen from the proof that the " if" portion also follows under the assumption that g is only "$\underset{\sim}{u}$-pre-absolutely continuous"9 where the definition of this concept is obvious.

Proof; By Lemma 1.5, for each j = l,...,m, u_D coincides £n-a.e. in *Cl* with a function $u_j^{\sim}$ *e A(Cl)* fl L_1^{loc}*(Q)* such that

$$\frac{d\tilde{u}_j}{ox_i} = B_1 u_j \quad £_n\text{-a.e. in } Q, \ i = l,...,n.$$

Since the function $\tilde{v} = G(\tilde{u}_1\hat{}...{}_{J?}\tilde{u}_m) = G\underset{\sim}{\tilde{u}}$ coincides with v $£_n$-a.e. in *Q,,* we will have $v e W_{1,1}^{loc}(Q)$ if and only if $v \hat{G} W_{1,1}^{loc}$ *(Cl)* .

Suppose first that (4.6) holds. Then also the functions

(4.8) $$\tilde{v}_1 = \sum_{j=1}^{m} \frac{\hat{f}\Lambda}{ot_j}(\underset{\wedge}{u}) \frac{Q\tilde{u}_{\Lambda}}{ox_i} \qquad i = 1,...,n,$$

are in t_1^{loc}(O) . Now let *r* be a line parallel to the x_i-axis such that (1) $u_1, . . . ,u_m$ are locally absolutely continuous on r Pi *Q,,* (2) g is absolutely continuous on every track $T_{\underset{\sim}{u},I}$ for compact intervals I c r fl fl, and (3) $\tilde{v}_i$ is locally summable on r fl fl. It follows by Theorem 4.1 that $\tilde{v}$ is locally absolutely continuous on r D Q and that

$$\frac{S\tilde{v}}{ox_I} = \tilde{v}_I e G L_1^{loc}(rflQ) \qquad £_1\text{-a.e. on } r.$$

Since almost all lines r parallel to the x_i-axis satisfy con-ditions (1), (2), and (3), it follows from this that $\tilde{v}$eA(fl) and that $\frac{\Re v}{ox_1}$ e L_1^{loc}(Q) , i = l....n. This shows by Lemma 1.5 that

$\tilde{v} \in W_{1,1}(\Omega')$ for each bounded subdomain $\Omega' \subset \Omega$ which satisfies the cone condition, and hence that $v \in W^{loc}_{1,1}(\Omega)$.

Conversely, suppose that v is in $W^{loc}_{1,1}(\Omega)$. It follows that $\tilde{v}$ is in $W^{loc}_{1,1}(\Omega)$ and hence coincides $\mathcal{L}_n$-a.e. on C with a function $v^{\#} \in W^{loc}_{1,1}(\Omega) \cap A(\Omega)$. Let Γ be a line parallel to the x_i-axis satisfying conditions (1) and (2) above as well as: (3$'$) $v^{\#}$ is locally absolutely continuous on $\Gamma \cap \Omega$, and (4$'$) $\tilde{v}$ coincides with $v^{\#}$ $\mathcal{L}_1$-a.e. on $\Gamma \cap \Omega$. It follows by the continuity of $\tilde{v}$ on $\Gamma \cap \Omega$, (Remark 3.1$_2$) that actually $\tilde{v}$ is itself locally absolutely continuous on $\Gamma \cap \Omega$. Since almost all lines Γ parallel to the x_i-axis satisfy conditions (1), (2), (3$'$), (4$'$) ($i=1,\ldots,n$) it follows that $\tilde{v} \in A(\Omega) \cap W^{loc}_{1,1}(\Omega)$. Moreover, by Theorem 4.2 we have $\frac{\partial \tilde{v}}{\partial x_i} = v_i$ $\mathcal{L}_1$-a.e. on $\Gamma \cap \Omega$, for all such Γ. Conclusions (4.6) and (4.7) now follow from Lemma 1.5 and the relations:

$$\delta_i v = \frac{\partial \tilde{v}}{\partial x_i} \ \mathcal{L}_n\text{-a.e. in } \Omega, \quad v_i = v_i \ \mathcal{L}_n\text{-a.e. in } \Omega.$$

In order to give analogues of Theorems 2.2 and 2.3 we first introduce the following definition and prove an important lemma.

<u>Definition 4.2. A function $g : R_n \to R_1$ is said to be fully absolutely continuous</u> provided that it is Borel measurable and is absolutely continuous on Γ for <u>every</u> Γ which is the track of an absolutely continuous curve. The class of all such g is denoted by $\mathfrak{F}$.

Remark; By Lemma 3.3 the class $\mathfrak{F}$ includes all locally Lipschitz functions on R_m. It would be very interesting to have a good characterization of $\mathfrak{F}$.

Lemma 4.1. Let $g : R_m \to R_1$ be a Borel function. Then for any $j = 1,\ldots,m$ the domain of $\frac{\partial g}{\partial t_j}$ is a Borel set and $\frac{\partial g}{\partial t_j}$ is a Borel function.

Proof: For j fixed let e_j be the unit vector in the direction of the t_j-axis. Since g is Borel measurable, the function on $R_m \times R_1' \times R_1$ ($R_1' = R_1 - \{0\}$) defined by

$$Q(t,h,\ell) = \frac{g(t+he_j)-g(t)}{h} - \ell$$

is Borel measurable. Consequently for any $\varepsilon > 0$ the following subset of $R_m \times R_1' \times R_1$ is a Borel set

$$\hat{C}_\varepsilon = \{(t,h,\ell) : |Q(t,h,\ell)| \leq \varepsilon\}.$$

It follows that for each pair of integers $i,k > 0$, the set

$$C_{i,k} = \{(t,\ell) : |Q(t,h,\ell)| \leq \tfrac{1}{i}, \text{ when } h \in [-\tfrac{1}{k},\tfrac{1}{k}]'\}$$

is a Borel set in $R_m \times R_1$, where $[-\frac{1}{k},\frac{1}{k}]' = [-\frac{1}{k},\frac{1}{k}] - \{0\}$. In fact

$$C_{i,k} = R_m \times R_1 - \pi(R_m \times [-\tfrac{1}{k},\tfrac{1}{k}]' \times (R_1 - \hat{C}_{\frac{1}{i}})),$$

where π is the canonical projection of $R_m \times R_1' \times R_1$ onto $R_m \times R_1$. However the set

$$G = \bigcap_{l=1}^{\infty} \bigcup_{k=1}^{\infty} C_{l,k} \subset R_m \times R_1$$

is the graph of the relation $\frac{\partial g}{\partial t_k}$, which completes the argument.

<u>Theorem 4.4</u>. Let Ω be a bounded domain in R_n possessing the cone property. Let g be a function on R_m satisfying the hypotheses of Theorem 4.3 and denote $h_j = \frac{\partial g}{\partial t_j}$, $j = 1,\dots,n$. Then each h_j is necessarily Borel measurable and defined $\mathcal{L}_n$-a.e. Given p, $1 \le p \le n$, suppose that for some q, $p < q < n$, the functions h_j determine, via composition, mappings which satisfy:

$$h_j : L_{q^*}(\Omega)^m \to L_{q'}(\Omega) \quad j = 1,\dots,n, \text{ with } q^* = \frac{nq}{n-q}, \quad q' = \frac{qp}{q-p}.$$

Then g yields, via composition, a mapping which satisfies

$$g : W_{1,q}(\Omega)^m \to W_{1,p}(\Omega).$$

Moreover, with $v = g(u_1,\dots,u_m)$ one has, for $u = (u_1,\dots,u_m) \in W_{1,q}(\Omega)^m$,

$$\partial_i v = \sum_{j=1}^{m} \frac{\partial g}{\partial t_j}(u)\,\partial_i u_j \qquad i = 1,\dots,n,$$

the products being interpreted as zero wherever the second factor is zero.

<u>Remark</u>; The hypotheses of the theorem as regards the h_j, $1 \le j \le n$, are met, in particular, if the h_j satisfy growth conditions of the type in (2.8). However the theorem is not limited to such cases.

Since h_j is defined at all points of $R_m - S_g$, the Borel measurability of the functions h_j ensures that they are defined $\mathcal{L}_n$-a.e. Thereafter the proof utilizes Gagliardo's characterization (and Theorem 4.3) in almost exactly the same way as was done in Theorem 2.2.

Theorem 4.5. Let Ω be a domain in R_n. Given $u_0 \in W^{loc}_{2,1}(\Omega)$ denote $u_i = \partial_i u_0$, $i = 1,\dots,n$, and set $\underset{\sim}{u} = (u_0, u_1, \dots, u_n)$. Suppose that $g : R_{n+1} \to R_1$ is locally $\underset{\sim}{u}$-absolutely continuous and that for T the track of any absolutely continuous curve in R_{n+1}, the projection of $S_g \cap T$ on the t_0-axis is $\mathcal{L}_1$-null. Set $v = G(u_0, u_1, \dots, u_n) = G\underset{\sim}{u}$. Then v is in $W^{loc}_{1,1}(\Omega)$ if and only if the functions

$$(4.9) \qquad v_i = \frac{\partial g}{\partial t_0}(\underset{\sim}{u})\,\partial_i u_0 + \sum_{j=1}^{n} \frac{\partial g}{\partial t_j}(\underset{\sim}{u})\,\partial_i \partial_j u_0 \qquad i = 1,\dots,n,$$

belong to $L^{loc}_1(\Omega)$, where the products are to be interpreted as zero wherever their second factor is zero. Moreover we then have

$$(4.10) \qquad v_i = \partial_i v \quad \mathcal{L}_n\text{-a.e. in } \Omega, \quad i = 1,\dots,n.$$

Proof; By Lemma 1.5 there exists for each $j = 0,1,\dots,n$ a function $\tilde{u}_j \in A(\Omega)$ such that $\tilde{u}_j = u_j$ $\mathcal{L}_n$-a.e. in Ω and such that $\tilde{u}_j = \frac{\partial \tilde{u}_0}{\partial x_j}$ $\mathcal{L}_n$-a.e. in Ω. Moreover we have by Remark 1.1 that $\frac{\partial \tilde{u}_0}{\partial x_j} = \partial_j u_0$ $\mathcal{L}_n$-a.e. in Ω and $\frac{\partial \tilde{u}_j}{\partial x_i} = \partial_i \partial_j u_0$ $\mathcal{L}_n$-a.e. in Ω. Set $\tilde{v} = G\tilde{\underset{\sim}{u}}$. Clearly $\tilde{v}$ coincides with v $\mathcal{L}_n$-a.e. in Ω, so that $\tilde{v} \in W^{loc}_{1,1}(\Omega)$ if and only if $v \in W^{loc}_{1,1}(\Omega)$.

Suppose first that (4.9) holds. Then also the functions

$$\tilde{v}_i = \frac{\partial g}{\partial t_0}(\tilde{\underset{\sim}{u}})\frac{\partial \tilde{u}_0}{\partial x_i} + \sum_{j=1}^{n} \frac{\partial g}{\partial t_j}(\tilde{\underset{\sim}{u}})\frac{\partial \tilde{u}_j}{\partial x_i}$$

belong to $L^{loc}_1(\Omega)$. Set $M = \tilde{\underset{\sim}{u}}^{-1}(S_g)$. Now let T be a line parallel to the x_i-axis such that

(1) $\tilde{u}_0, \dots, \tilde{u}_n$ are locally absolutely continuous on $\Gamma \cap \Omega$,

(2) g is absolutely continuous on every track $T_{\tilde{u},I}$ for compact intervals $I \subset \Gamma \cap \Omega$. Since $\tilde{u}(M \cap \Gamma) \subset \tilde{U}(M) \cap \tilde{u}(\Gamma \cap \Omega) \subset S_y \cap u(\Gamma \cap \Omega)$, it follows that the range of $\tilde{u}_0$ on $M \cap \Gamma$ is an $\mathcal{L}_1$-null set. Hence by Lemma 1.1 $\frac{\partial \tilde{u}_0}{\partial x_i} = 0$ $\mathcal{L}_1$-a.e. on $M \cap \Gamma$. It follows that $\frac{\partial \tilde{u}_0}{\partial x_i} = 0$ $\mathcal{L}_n$-a.e. in M, with $i = 1, \dots, n$ since the choice of i above was arbitrary. Thus $\tilde{u}_{0x_i} = 0$ $\mathcal{L}_n$-a.e. in M, $i = 1, \dots, n$. Let $M' \subset M$ be the $\mathcal{L}_n$-null set where $(\tilde{u}_1, \dots, \tilde{u}_n) \neq (0, \dots, 0)$.

We proceed to show that $\frac{\partial \tilde{u}_j}{\partial x_i} = 0$ $\mathcal{L}_n$-a.e. in M. For this let Γ be a line parallel to the x_i-axis which in addition to (1) and (2) above satisfies (3) $M' \cap \Gamma$ is an $\mathcal{L}_1$-null set, and (4) $\tilde{v}_i$ is locally summable on Γ. Then $\tilde{u}(M \cap \Gamma) \subset \tilde{u}(M' \cap \Gamma) \cup \{(0, \dots, 0)\}$. Hence by the local absolute continuity of $\tilde{u}$ on $\Gamma \cap \Omega$ and the fact that $M' \cap \Gamma$ is a null set we conclude (Lemma 1.3) that $\tilde{u}(M \cap \Gamma)$ is $\mathcal{L}_1$-null. Hence Lemma 1.1 implies that

(4.11) $$\frac{\partial \tilde{u}_j}{\partial x_i} = 0 \quad \mathcal{L}_1\text{-a.e. in } M \cap \Gamma, \quad j = 0, 1, \dots, n.$$

From this it follows that $\frac{\partial \tilde{u}_j}{\partial x_i} = 0$ $\mathcal{L}_n$-a.e. in M, with $i, j = 1, \dots, n$, since the conditions (1), (2), (3), (4) hold for almost all Γ parallel to the x_i-axis, and the choice of i was arbitrary. In addition we point out that S meets each of the tracks $T_{\tilde{u},I}$ for I a compact subinterval of $\Gamma \cap \Omega$, in a set which is actually W_1-null, not merely one whose projection on the t_o-axis is $\mathcal{L}_1$-null. This follows from (4.14) and Lemma 1.3, since $\dot{s} = |\tilde{u}|_{\Gamma\cap U} = 0$ $\mathcal{L}_1$-a.e. in $\Gamma \cap M$ implies that $s(\Gamma\cap M)$ is an $\mathcal{L}^1$-null set [4,p.227]. Therefore we may conclude by Theorem 4.1 (and Remark 4.1.1), that $\tilde{v}$ is locally absolutely continuous on $\Gamma \cap \Omega$ and $\frac{\partial \tilde{v}}{\partial x_i}$ coincides with $\tilde{v}_i$ $\mathcal{L}_1$-a.e. on $\Gamma \cap \Omega$, whenever Γ is an interval satisfying (1), (2), (3), (4). Since these conditions hold for almost all lines and the choice of i was arbitrary, we deduce by Lemma 1.5 that $\tilde{v} \in A(\Omega) \cap W^{loc}_{1,1}(\Omega)$.

Conversely, if v is in $W^{loc}_{1,1}(\Omega)$ then $\tilde{v}$ is in $W^{loc}_{1,1}(\Omega)$ and coincides almost everywhere with a function $v^* \in A(\Omega) \cap W^{loc}_{1,1}(\Omega)$. On almost all lines Γ parallel to any axis the absolute continuity of g on $T_{\tilde{u},I}$ ensures continuity of $\tilde{v}$, as in the proof of Theorem 4.3. Hence $\tilde{v}$ coincides with v^* on $\Gamma \cap \Omega$ for almost all lines Γ, and it follows that $\tilde{v} \in A(\Omega) \cap W^{loc}_{1,1}(\Omega)$. We can then show as above that every $S_g \cap T_{\tilde{u},I}$ is $\mathcal{L}_1$-null for almost all choices of Γ, and hence we obtain (4.12), (4.13) by use of Theorem 4.2. This completes the proof.

Clearly one could now give a direct analogue of Theorem 2.4 as well. We omit the obvious formulation and proof.

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Bibliography

[1] Federer, H., Geometric Measure Theory, (Grundlehren d. Math. Wiss., Vol.153), New York, Springer, 1969.

[2] Gagliardo, E., "Proprietà di alcune classi di funzioni in più variabili", Richerche di Mat., 2(1958)> 102-137.

[3] Roger, F., "Sur la relation entre les propriétés tangentielles et metriques des ensembles cartésiens", C. R. Acad. Sci. Paris 201(1935), 871-872.

[4] Saks, S., Theory of the Integral, Second ed. (Monografje Math., Vol.7) Warsaw, 1937. Reprinted Stechert-Hafner Pub. Co., New York. New York, Dover, 1964.

[5] Serrin, J., Personal communication, 1971.

[6] Serrin, J. and D. E. Varberg, "A General Chain Rule for Derivatives and the Change of Variables Formula for the Lebesgue Integral", Amer. Math. Monthly 76.(1969), 514-520.

[7] Sobolev, S. L., Applications of Functional Analysis in Mathematical Physics, (Transl. of Math. Monographs, Vol.7), Providence, Amer. Math. Soc., 1963.

[8] Valle'e Poussin, Ch. J. de la, "Sur l'integrale de Lebesgue", Trans. Amer. Math. Soc. 16(1915), 435-501.