LINEAR PROCESSES GENERATED BY INDEPENDENT RANDOM VARIABLES

by

W. A. Woyczynski

Report 71-35

July, 1971

This research was supported in part by NSF Grant GU-2056

-CS-

LINEAR PROCESSES GENERATED BY INDEPENDENT RANDOM VARIABLES

by

W. A. Woyczynski

Abstract

In a recent paper R. Dudley gave a characterization of those sequences of independent and identically distributed random variables which are ℓ_p-compatible for $p \geq 1$. In the present note we extend his result into $p \in (0,1]$ and provide some conditions (necessary or sufficient) for ℓ_φ-compatibility of a sequence of independent random variables not necessarily identically distributed.

LINEAR PROCESSES GENERATED BY INDEPENDENT RANDOM VARIABLES

by

W. A. Woyczynski[1]

Let $(X_n\}$ ^^((f^tfjP) be a sequence of random variables. If for every element (a_n) of a complete metric linear space X of real sequences in which co-ordinate functionals $(a_n) —*\sim a_m$, m = 1,2,..., are continuous the series $La_n X_n$ converges in probability (we shall say in this case that $\{X_n\}$ is X-compatible) then the mapping

$$X : Xs(a_n) \longrightarrow \Sigma a_n X_n \in \mathfrak{M}(\Omega, \mathfrak{F}, P)$$

defines a linear continuous stochastic process on X (for basic definitions the reader is referred to [1], [4] and references given therein) . *Tfhe* linearity is clear and the continuity follows from the classical Banach's theorem which says that every Borel homomorphism of a complete metric group into a metric is continuous and from a remark that X is a pointwise limit of continuous (because finite-dimensional) , hence Borel functions $S_{h=l} a_n X_n$. R. Dudley ([2]) gave a characterization of those sequences of independent and identically distributed random variables which are $£_p$-compatible for p ^ 1.

AMS 1970 Subject Classification: Primary 6OBO5, 6OB99.

Key Words and Phrases: linear stochastic process, $£_{TM}$ -compatible, weakly 9-finite.

[1] The author acknowledges support received under NSF Grant GU-2056, administered by Carnegie-Mellon University, while on leave from Wroclaw University and the Mathematical Institute of the Polish Academy of Sciences.

In this note we extend his result onto pe(O,l] and provide a necessary condition for the sequence of independent random variables (not necessarily identically distributed) to be K_{φ} -compatible and the sufficient condition for the same sequence to be $£_p$-compatible for $p > 0$. However it seems that they are rather far from the conditions which are necessary and sufficient simultaneously.

Now let $\{X_i\}$ be a sequence of independent random variables with distribution functions F_n and let cp,0 :3R$^+$ —> TR$^+$ be non-decreasing continuous functions vanishing only at the origin. Call the sequence f$^x{}_n$} (°r ^$^F{}_n$}) lower [upper] weakly cp-finite if for some $C > 0$

$$\liminf_n [\sup_n] \; P\{|X_n| \geq M\} \leq \frac{C}{\varphi(M)}, \qquad 0 \leq M < \infty .$$

We shall say that cp satisfies A_2 condition $(\varphi \in \Delta_2)$ **if for** some positive constant K, cp(2t) £ Kcp(t)$_3$ 0 £ t < oo . If $\varphi \in \Delta_2$ then the sequence space ;

$$\ell_{\varphi} \equiv \{(a_n) \in \mathbb{R}^{\infty} : \textstyle\sum \varphi(|a_n|) < \infty\}$$

is a linear space which can be equipped with a non-homogeneous norm (e.g. $\|(a_n)\|_{\varphi} = \inf\{c : c > 0, \sum_{\varphi} (|a_n/c|) \leq c\}$ which makes ℓ_{φ} a complete metric linear space with continuous co-ordinate functionals (see e.g. [3], Theorems 1.4, 1.62 and 1.82).

Given below propositions explain what is the relationship between lower [upper] finiteness and the existence of moments for a sequence $\{X_n\}$.

PROPOSITION 1. <u>If for some subsequence</u> $\{n_k\}$, $k = 1,2,\cdots$, $E\varphi(|X_{n_k}|)$ are finite and $\liminf_k E\varphi(|X_{n_k}|) = L < \infty$, <u>then the sequence</u> $\{X_n\}$ <u>is lower weakly</u> φ<u>-finite</u>.

<u>Proof</u>.

$$\liminf_n P\{|X_n| \geq M\} \cdot \varphi(M) \leq$$

$$\leq \liminf_k P\{|X_{n_k}| \geq M\} \cdot \varphi(M) \leq$$

$$\leq \liminf_k \int_{|X_{n_k}| \geq M} \varphi(|X_{n_k}|)\,dP \leq \liminf_k E\varphi(|X_{n_k}|) = L < \infty .$$

PROPOSITION 2. <u>Let</u> $\varphi \in A_2$, $\theta \geq 0$ <u>be differentiable with monotone</u> φ', <u>and let</u> $\theta(t)\varphi'^{-1}(t) \to 0$ as $t \to +\infty$ <u>in such a way that</u> $\theta(t)\varphi'^{-1}(t)\,t^{-1} \in L^1(a,\infty)$ <u>for some</u> $a > 0$. <u>In those conditions if</u> $\varphi(M) \limsup_n P\{|X_n| \geq M\} \leq C$, $0 \leq M < \infty$ <u>(even more so if</u> $\{X_n\}$ <u>is upper weakly</u> φ<u>-finite) then</u> $\limsup_n E\theta(|X_n|) < \infty$.

<u>Proof</u>, Note that

$$\limsup_n E\theta(|X_n|) =$$

$$\limsup_n \int_{-\infty}^{-a} \theta(|t|)\,dF_n(t) + \limsup_n \int_{|x| \leq a} \theta(|X_n|)\,dP + \limsup_n \int_a^{\infty} \theta(|t|)\,d(F_n(t)-1).$$

The second term being bounded by $\theta(a)$ it is sufficient to evaluate the first and the third ones. We restrict ourselves to the evaluation of the third term (the procedure for the first one is analogous). Integrating by parts we get that

$$\limsup_n \int_a^\infty \theta(t)\,d(F_n(t)-1) =$$

$$= \limsup_n \left[\frac{\theta(t)}{\varphi(t)}\,\varphi(t)\,(F_n(t)-1)\Big|_a^\infty + \int_a^\infty \theta'(t)\,(1-F_n(t))\,dt \right]$$

By our assumptions $\theta(t)\cdot\varphi^{-1}(t) \longrightarrow 0$ as $t \longrightarrow \infty$ and $\limsup_n \varphi(t)(F_n(t)-1)$ remains bounded. Hence the first summand is bounded by $\theta(a)$. As to the second one we have that

$$\limsup_n \int_a^\infty \theta'(t)\,(1-F_n(t))\,dt \le c\int_a^\infty \frac{\theta'(t)}{\varphi(t)}\,dt \le C\int_a^\infty \frac{\theta(t)}{t\,\varphi(t)}\,dt$$

We shall prove that $\varphi'(t)t/\varphi(t)$ is bounded so that the last integral is finite. Indeed, when φ' is increasing then (because of the A_2-condition)

$$K\varphi(t) \ge \varphi(2t) = \int_0^{2t} \varphi'(\tau)\,d\tau \ge \int_t^{2t} \varphi'(\tau)\,d\tau \ge t\varphi'(t)$$

and in the case φ' is decreasing

$$\varphi(t) = \int_0^t \varphi'(\tau)\,d\tau \ge \int_{t/2}^t \varphi'(\tau)\,d\tau \ge \tfrac{1}{2}t\varphi'(t)$$

what was to be proved.

Remarks. From the Proposition 2 and its proof it follows that

a) if $\varphi, \theta \in A_2$ have both monotone derivatives, $\theta(t)\varphi^{-1}(t) \to 0$

as $t \to +\infty$ in such a way that $\Phi'(t)/\varphi(t) \in L'(a, \infty)$ for some $a > 0$, and $\{X_n\}$ is upper weakly φ-finite then $\limsup_n E\,\Phi(|X_n|) < \infty$.

b) if $\{X_n\}$ is upper weakly φ-finite with $\varphi(u) = u^p$, $p > 0$, then $\limsup_n E|X_n|^q < \infty$ for every q, $0 < q < p$.

In what follows we shall need the following condition for the function φ

$$(+) \qquad 0 < \text{const} \leq \varphi(t)/\varphi(t^{-1}) \leq \text{CONST} < \infty .$$

As examples of such a function we may give $\varphi(t) = t^p$, $p > 0$ and

$$\varphi(t) = \begin{cases} \int_0^t \frac{|\sin u|}{u}\,du & \text{for } 0 < t \leq \pi \\ \varphi(IT) + \int_{\pi}^{t} |\sin u|\,du & \text{for } \pi < t. \end{cases}$$

THEOREM 1. Let φ satisfies condition (+). Then this case if $\{X_n\}$ is $\mathcal{L}_\varphi$-compatible then it is lower weakly φ-finite. If in addition φ' exists and is monotone then $\mathcal{L}_\varphi$-compatibility of (X_n) implies that $\varphi'(t) \liminf_n \int_{-t}^{t} s\,dF_n(s)$ is bounded for $0 \leq t < \infty$.

Proof. Assume that $\{F_n\}$ is not lower weakly φ-finite. Then for every $k = 1, 2, \ldots,$ we can find M_k such that

$$\min(\varphi(M_k), 1) \geq k^2$$

and

$$\varphi(M_k) \liminf_n P(|X_n| \geq M_k) > k$$

It follows that for every $k = 1,2,\ldots,$ there exists an integer N_k such that

$$(*) \qquad P(|X_j| \geq M_k) > \frac{k}{\varphi(M_k)}, \qquad j \geq N_k.$$

Let now $a_j = \frac{1}{M_k}$ for $m_k \leq j < m_k + r_k$, where $m_1 = N_1$, $m_{k+1} = \max(m_k + r_k, N_{k+1})$ and where r_k is an integer such that

$$(**) \qquad 1 \leq \frac{k^2 r_k}{\varphi(M_k)} \leq 2,$$

and $a_j = 0$ for j's not listed above (if any). We shall check that $(a_j) \in t_\varphi$. Indeed

$$\sum_j \varphi(|a_j|) = \sum_k r_k \varphi\left(\frac{1}{M_k}\right) \leq \text{CONST} \sum_k \frac{r_k}{\varphi(M_k)} \leq \text{CONST} \sum_k \frac{2}{k^2} < \infty$$

because of the condition (+). However (*) and (**) cause that the series

$$\sum_j P(|a_j X_j| \geq 1) = \sum_{k=1}^{\infty} \sum_{j=m_k}^{m_k+r_k-1} P(|X_j| \geq M_k) \geq$$

$$\geq \sum_{k=1}^{\infty} \frac{r_k k}{\varphi(M_k)} \geq \sum_{k=1}^{\infty} \frac{1}{k}$$

is divergent causing the divergence of $\sum_j a_j X_j$ in probability by the Kolmogorov three series theorem.

Now assume φ' is monotone and $\varphi'(t) \liminf_n \int_{-t}^{t} s\,dF_n(s)$ is unbounded on the positive half-line. Then, as in the preceding part of the proof, for every $k = 1,2,\ldots,$ we can find $M_k \ge N_k$ such that $\min(\varphi(M_k), 1) \ge k^2$ and

(***) $\quad I_{jk} = \varphi'(M_k) \int_{|t| \le M_k} s\,dF_j(s) > k, \quad j \ge N_k$

(or $-I_{jk} > k$; in this case we proceed similarly). Having chosen (a_j) as previously in $\mathcal{L}_\varphi$, we see in view of (***) that

$$\sum_j E(a_j X_j)^{1*} \geq \sum_{k=1}^{\infty} \frac{1}{M_k} \sum_{j=m_k}^{m_k+r_k-1} \int^{M_k} s\,dF_j(s) >$$

$$> \sum_{k=1}^{\infty} \frac{r_k k}{M_k \varphi'(M_k)} \ge \frac{1}{\max(2,K)} \sum_{k=1}^{\infty} \frac{r_k k}{\varphi(M_k)} \ge \frac{1}{\max(2,K)} \sum_{k=1}^{\infty} \frac{1}{k}$$

the penultimate inequality being motivated as in the proof of the Proposition 2 (p.) by Δ_2-condition and monotonicity of φ'. Thus again by Kolmogorov three series theorem we conclude that $\{X_j\}$ is not $\mathcal{L}_\varphi$-compatible what ends the proof.

Corollary. If $\sup_t \varphi'(t) = \infty$ (e.g. if $\varphi(t) = t^p$, $p > 1$) and $\{X_n\}$ is $\mathcal{L}_\varphi$-compatible then $\liminf_n EX_n^1 = 0$.

THEOREM 2. If $\{X_n\}$ is uniformly weakly φ-finite, where $\varphi(t) = t^p$, $0 < p < 2$, and $t^{p-1} \sup_n \left| \int_{-t}^{t} s\,dF_n(s) \right| \le C' < \infty$, $0 \le t < \infty$, then X_n is $\mathcal{L}_\varphi$-compatible.

Proof. We know that $M^p \sup_n P(|X_n| > M) < C$ and we shall

---

*) $X^1(\omega)$ is equal to $X(\omega)$ if $|X(\omega)| \le 1$ and 0 otherwise.

check the convergence of all three series in Kolmogorov theorem, (a_i) always is supposed to belong to l_p.

$$\text{(I)}\quad \sum_i P(|a_iX_i|>1) \le \sum_i P\left(|X_i|>\frac{1}{|a_i|}\right) \le \sum_i \sup_n P\left(|X_n|>\frac{1}{|a_i|}\right) \le$$

$$\le C\sum_i |a_i|^p < \infty .$$

$$\text{(II)}\quad \sum_i \left|E(a_iX_i)^1\right| = \sum_i |a_i| \left|\int_{|t|<1/|a_i|} s\,dF_i(s)\right| \le \sum_i |a_i| \sup_n \left|\int_{|t|<1/|a_i|} t\,dF_n\right| \le$$

$$\le C' \sum_i |a_i|\,|a_i|^{p-1} < \infty ,$$

$$\text{(III)}\quad \sum_i E((a_iX_i)^1)^2 = \sum_i a_i^2 \int_{|t|<1/|a_i|} t^2\,dF_i(dt).$$

It is sufficient to evaluate the series

$$\sum_i a_i^2 \int_0^{1/|a_i|} t^2\,dF_i(t) \le \sum_i a_i^2\left(-\frac{1}{a_i^2} + 2\int_0^{1/|a_i|} tP_i(t)\,dt\right) =$$

$$= \sum_i a_i^2\, 2\int_0^{1/|a_i|} t(1-F_i(t))\,dt \le 2\sum_i a_i^2\left(1+C\int_1^{1/|a_i|} t^{1-p}\,dt\right) =$$

$$= 2\sum_i \left(1-\frac{c}{2-p}\right)a_i^2 + \frac{2c}{2-p}\sum_i |a_i|^p < \infty .$$

<u>Remarks</u>, 1) If the random variables are $\{X_n\}$ are identically distributed with distribution function F then the Theorems 1 and 2 give the following corollary which is

the extension of R. Dudley's theorem 7.2 of [2]: if $0 < p < 2$ then X_n is ℓ_p-compatible if and only if F is weakly t^p-finite (i.e. $M^p P(|X_1| \geq M)$ is bounded) and $t^{p-1} \int_{-t}^{t} s\,dF(s)$ is bounded.

2) For $p > 1$ assumptions of theorem 2 cause, clearly, that $EX_n = 0$.

References

[1] Badrikian, A., "Seminaire sur les mesures cylindriques et les fonctions lineaires aleatoires", Springer Lecture Notes 139, Berlin-Heidelberg-New York, 1970.

[2] Dudley, R., "Random linear functionals", Trans. Amer. Math. Soc. 136. (1969), 1-24.

[3] Mazur, S. and W. Orlicz, "On some classes of linear spaces", Studia Math. 17 (1958), 97-119.

[4] Woyczynski, W. A., "Representation of additive functionals and invariant characteristic functionals of linear processes with independent pieces", Bull. Acad. Polon. Sci., 19 (1971), 221-230.

Carnegie-Mellon University
Pittsburgh, Pennsylvania 15213

/cs