L-ORTHOGONALLY SCATTERED MEASURES

by

K. Sundaresan[1] and W. A. Woyczynski[2]

Research Report 71-38

June, 1971

[1] The research work of the first author was in part supported by a Scaife Faculty Grant administered by Carnegie-Mellon University.

[2] The research of the second author was in part supported by NSF Grant U-9056 while on leave from Wroclaw University and the Institute of Mathematics of the Polish Academy of Sciences, Poland.

L-Orthogonally Scattered Measures

K. Sundaresan[1] and W. A. Woyczynski[2]

1. Introduction

In spite of many questions yet to be answered the theory of vector measures (i.e. measures taking values in a Banach space) seems to be a well developed discipline today, Dincleanu [1]. In particular when the range of the measure is a Hilbert space and the measure is orthogonally scattered (i.e. has orthogonal values on disjoint sets) the theory is especially deep and elegant and its origins go as far back as the early Wiener's paper [13]. As highlights of this theory we mention the description of the space of real functions that are integrable with respect to such a measure and the applications of harmonic analysis of such measures to stationary Stochastic processes in the Wiener-Kolmogorov prediction theory.

Unlike his predecessors, recently Masani [6] wrote an expository paper on the subject without using any probabilistic terminology. Generalizations of this theory were pursued in several directions by a number of authors. All of them, to the best of our knowledge, were trying to replace the Hilbert space in

---

[1] The research work of the first author was in part supported by a Scaife Faculty Grant administered by Carnegie-Mellon University.

[2] The research of the second author was in part supported by NSF Grant GU-9056 while on leave from Wroclaw University and the Institute of Mathematics of the Polish Academy of Sciences, Poland.

the range of the measure by a more general space (not necessarily even locally convex) while retaining the property of "independent" scatteredness of values of measure on disjoint sets. For example the "independent" scatteredness means the orthogonality of operators when the values of the measure are hermitian operators in a Hilbert space [7], and it means stochastic independence in the case of general random measures considered in Urbanik, [12]. When the range space is a topological vector lattice we have a natural concept of orthogonality namely the lattice theoretic one. In the present paper we are making an attempt to study vector valued measures taking values in certain topological vector lattices interpreting "independent" scatteredness in terms of lattice theoretic orthogonality (ℓ-orthogonally scattered measures). The elegance of the Wiener-Kolmogorov theory stems from the fact that the positive measure associated with the Hilbert-space valued orthogonally scattered measure is not only mutually absolutely continuous with respect to the vector measure in question but it is also algebraically closely connected with it. In the case under consideration we have a similar advantage.

It might be mentioned that there are several orthogonality concepts available in arbitrary Banach spaces. For an extensive discussion of such concepts we refer to James [4]. As pointed out in [4] the most interesting of these concepts is the following. (D). Let B be a Banach space. If $x, y \in B$, x is said to be orthogonal to y if $\|x+\lambda y\| \geq \|x\|$ for all real numbers A. It has been shown recently in Sundaresan [11] that if $\dim B \geq 2$ and if F is a continuous function on $B \to R$ such that $F(x+y) = F(x) + F(y)$ whenever $x \perp y$ then F is of the form $c\|x\|^2 + \ell(x)$ $(\ell(x))$ where $\ell \in B^*$ if B is a Hilbert space (if B is not isometric with a Hilbert space). It is for this reason that we have not considered measures orthogonally scattered where orthogonality is interpreted following the definition in (D).

In this paper we discuss the following three problems concerning *--orthogonally scattered measures:

1) Hahn extension of these measures (Section 3)

2) representation theorem for such measures (Section 4)

3) Radon-Nikodym theorem for these measures (Section 5)

Finally we indicate some applications to random measures in the concluding Section 6. The problem in Section 6 is the essential motivation for the results discussed in Sections 3-5.

2. In this section we state the notation, few definitions and elementary facts which are required in the subsequent sections.

$\mathfrak{X}$ denotes a vector lattice in which $x \perp y$, $x,y\in\mathfrak{X}$, means $|x|\wedge|y| = 0$. In what follows $\mathfrak{X}$ is frequently a space of equivalence classes of measurable functions on a σ-finite measure space (Ω,Σ,μ), the ideal of μ-null sets in Σ being denoted by Δ. The functional $F\ \mathfrak{X}\quad R$ is said to be additive if $F(x+y) = F(x) + F(y)$ whenever $x \perp y$. If $\mathfrak{X}$ is a topological vector lattice we require F to be continuous. An additive functional ρ on a vector lattice $\mathfrak{X}$ is said to be a modular if it satisfies the following conditions: 1) $x = 0 \Leftrightarrow \rho(x) = 0$ 2) $|x| \leq |y| \quad \rho(x) \leq \rho(y)$, $x,y\in\mathfrak{X}$ 3) $0 \leq x_n \uparrow x \Rightarrow \rho(x) = \sup \rho(x_n)$. $\|x\|_\rho \overset{df}{=} \inf\{\alpha : \rho(\frac{x}{\alpha}) \leq 1\}$ is an F-norm on $\mathfrak{X}$ rendering $\mathfrak{X}$ a complete metric space, Matuszewska and Orlicz [8].

$\mathfrak{B}_o$ stands for a pre-ring of subsets of a set T i.e. 1) $A,B\in\mathfrak{B}_o \quad A \cap B\in\mathfrak{B}_o$, 2) $A,B\in\mathfrak{B}_o$, $A \subset B$ there is a finite increasing sub-family $C_1,\ldots,C_n\in\mathfrak{B}_o$ such that $A = C_1$, $B = C_n$ and $C_i \sim C_{i-1}\in\mathfrak{B}_o$, $i = \quad,\ldots,n$. We denote the σ-ring generated by $\mathfrak{B}_o$ by $\mathfrak{B}$.

<u>Definition</u>: a) A mapping $\xi : \mathfrak{B}_o \longrightarrow \mathfrak{X}$ is said to be a ℓ-orthogonally scattered (l-o.s.) measure if $\xi(A \cup B) = \xi(A) + \xi(B)$, and $\xi(A) \perp \xi(B)$ whenever $A \cap B = \emptyset$, $A,B\in\mathfrak{B}_o$.
b) If $\mathfrak{X}$ is a topological vector lattice then $\xi : \mathfrak{B}_o \longrightarrow \mathfrak{X}$ is said to be a countably additive ℓ-orthogonally scattered (c.a.l-o.s.) measure if ξ is a l.-o.s. measure and in addition

$\xi(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \xi(A_i)$ whenever $\{A_i\}$ is a pairwise disjoint sequence from $\mathcal{B}_o$ such that $\bigcup_{i=1}^{\infty} A_i \in \mathcal{B}_o$.

Example. (Standard c.a.l.-o.s. measure). Let $T = \Omega$, $\mathcal{B} = \Sigma$ and $f \in \mathfrak{X}$ where $\mathfrak{X}$ is normal (i.e. if $g \in \mathfrak{X}$, g' is a non-negative measurable function such that $0 \leq g' \leq g$ then $g' \in \mathfrak{X}$). Then ξ^f defined by the formula $\xi^f(A) = f \cdot \chi_A$ is a c.a.l.-o.s. measure on Σ $\mathfrak{X}$.

If ξ is a c.a.l.-o.s. measure on $\mathcal{B}_o$ with values in a topological vector lattice $\mathfrak{X}$ and F is an additive functional on $\mathfrak{X}$ then, $\xi_F = F \circ \xi$ is a countably additive real measure on $\mathcal{B}_o$. We collect here few elementary facts concerning ξ and ξ_F. Since the results are immediate consequences of the definitions, the proofs are not supplied.

(1) If $A, B \in \mathcal{B}_o$ and $B \subset A$ then $F(\xi(A) - \xi(B)) = F(\xi(A \sim B)) + F(-\xi(B \sim A))$.

(2) If $A, B \in \mathcal{B}_o$ and $B \subset A$ then $F(\xi(A) - \xi(B)) = F(\xi(A)) - F(\xi(B))$.

(3) If $A, B \in \mathcal{B}_o$ and $A \sim B \in \mathcal{B}_o$ then $\xi(A \sim B) = \xi(A) - \xi(A \cap B)$

(4) If $A, B \in \mathcal{B}_o$, $A \sim B$, $B \sim A \in \mathcal{B}_o$ then

$\xi(A \triangle B) = \xi(A) + \xi(B) - 2\xi(A \cap B)$ if $A \triangle B \in \mathcal{B}_o$.

Further we have the following elementary lemma.

<u>Lemma 1</u>. Suppose $\xi : \mathcal{B}_o \longrightarrow \mathfrak{X}$ is a c.a.l.-o.s. measure and $F : \mathfrak{X} \longrightarrow R$ is an additive functional. Then the following two statements are equivalent.

(a) For all $A, B \in \mathcal{B}_o$ such that $B \subseteq A$

$$F(\xi(A) - \xi(B)) = \nu(A) - \nu(B)$$

where $\nu : \mathcal{B}_o$ R is a c.a. measure.

(b) $\xi_F = \nu$

3. Here we proceed to discuss the Hahn extension of c.a.l.-o.s. measure defined on a pre-ring $\mathfrak{B}_o$. In this connection we recall the classical Hahn extension theorem on p.54 in Halmos [2]. For convenience if ν is any real countable additive measure on $\mathfrak{B}_o$ then its Hahn extension to $\mathfrak{B}$ is denoted by $\tilde{\nu}$.

<u>Theorem 1</u>. Let $\xi : \mathfrak{B}_o \quad \mathfrak{X}$ be a c.a.l.-o.s. measure where $(\mathfrak{X},\rho)$ is a modular space and $\tilde{\xi}_\rho$ be the Hahn extension of ξ_ρ to $\mathfrak{B}$. Let $\mathfrak{F} = \{A | A\in\mathfrak{B}, \xi_\rho(A)<\infty\}$. Then there exists a unique extension $\tilde{\xi}$ of ξ to $\mathfrak{F}$ such that $(\tilde{\xi})_\rho = \tilde{\xi}_\rho|_{\mathfrak{F}}$.

<u>Proof</u>. Let $\mathfrak{R}$ be the ring generated by $\mathfrak{B}_o$. Let ξ be an extension of ξ to $\mathfrak{R}$. The existence of such an extension is verified as follows. If $A\in\mathfrak{R}$ chose an arbitrary family of pairwise disjoint sets $A_1,\dots,A_n\in\mathfrak{B}_o$ such that $A = \bigcup_{i=1}^{n} A_i$ and let $\hat{\xi}(A) = \sum_{i=1}^{n} \xi(A_i)$. It is verified that $\hat{\xi}$ is a c.a.l.-o.s. measure on $\mathfrak{R}$ into $\mathfrak{X}$ and an extension of ξ. By the theorem D, p.56 of [2], it follows that if $A\in\mathfrak{F}$, $\epsilon > 0$, there exists $B\in\mathfrak{R}$ such that $\tilde{\xi}_\rho(A\triangle B) < \epsilon$. Thus there exists a sequence $\{B_n\} \subset \mathfrak{R}$ such that $\tilde{\xi}_\rho(A\triangle B_n) \to 0$ as $n\to\infty$. Further it is verified that $\tilde{\xi}_\rho(B_m\triangle B_n) \to 0$ as $n\to\infty$. Since

$$\tilde{\xi}_\rho(B_m\triangle B_n) = \rho(\xi(B_m \sim B_n) + \xi(B_n \sim B_m)) =$$

$$= \rho(\xi(B_m \sim B_n)) + \rho(\xi(B_n \sim B_m)),$$

it follows that

$$\rho(\xi(B_m\sim B_n)) \to 0 \quad \text{and} \quad \rho(\xi(B_n\sim B_m)) \to 0 .$$

Hence, from (1) in Section 2

$$\rho(\xi(B_m) - \xi(B_n)) \longrightarrow 0$$

Since p is absolutely continuous

$$I(B_m) - \xi(B_n) I \to 0 .$$

Since X is complete there exists an xeX such that £ (B_n) — ^x. If now $\{c_n\}$ c 3 is another sequence such that 4_ρ (AAC_n)—^0 then § (C_n) — ^ y for some yeX. Thus p(x-y) = 0 then x = y. Let us now define for AGJ?, £ (A) = x if there exists a sequence $B_n$eil such that $£_\rho$ (AAB^)—>0 and ^ (B_n)—^x. From the preceding it follows that £ : J5—^X is a function. We proceed to verify that £; is a ca.l.-o.s. measure on the ring 5. Let AjBeJJ, B c A and let ^ , F-^e R, k = 1,2,..., be such that $£_\rho$ (H^AB)—^O and $?_p(t_kA(A\sim B))$—>0. Thus $4_p((E_kUF_k)AA)$~^O. Let $G_k = E_k \cup F_k$. Note that G.Gfc. Since lim p (£ (G_v) - 4(F>)> = 4^ (A)-$?_n$ (B) it follows that p (I (A) - g (B)) = g (A) - § (B) . Now from the Lemma 1 we infer that $(|)_\rho = §_\rho$ |Jf. We proceed to exhibit a counterexample to show that in general a c.a.l.-o.s. measure does not admit a Hahn extension with respect to arbitrary additive functionals on X.

<u>Example</u>; Let ft be the ring of all finite unions of disjoint bounded left-closed right-open intervals in the real line R. Let X = L_1(/i) where \x is the Lebesgue measure.

Let <p : R R be a continuous function such that <p has support in [-j,l], <P ^ ° and range <p c [o,l]. Let F : L.,^)—>R be the additive functional defined by F(x) = j cp(x)d|a.
R

Let £ :&—>L (H) be a c.a.l.-o.s. measure defined by 4 (A) = S^ x_{Af1l_n} , where l_n = [n,n+l).

It is verified that if £ is the extension of £„ to <J (bl) then $i_F(R)$ < oo, indeed $4_p(R)$ = lira $4_p[0,n)$ = $4_p[0,2)$. If 4 has an extension ^ to 3 = {A | A c R, 4_F(A)<00}, then since £T-,(R) < oo, ^ (R) is defined. However since 4 is c.a.l.-o.s. measure

$$\tilde{\xi}(R^+) = \sum_{n\geq 1} \tilde{4}(I_n) = \sum \frac{1}{n} X_{I_n}$$

which is not in L.(JLX), a contradiction.

4. In the present section we deal with certain structure theorems for c.a.l.-o.s. measures.

Proposition 1, Let I be an F-space of equivalence classes of measurable functions supported by a measure space (^,£,1*0, and let £ : (S—>% be a c.a.l.-o.s. measure on a a-ring <S (c:2^T). Then there exists a function fel and a cr-homomorphism h^A : B—»£/A such that

$$£(A) = fX_{h^A(A)}, \quad Aefi .$$

Proof, We note that if G c B such that A ^ A ^ G implies $A_{\pm}$ n A_2 = <p and AeG implies § (A) ^ *O* then card G ^ K_Q. For if card G > ^_o consider

{||4 (A) || |A€G}. There exists G ^ G card $G_Q = K_Q$ and a positive number a such that AeG_Q implies ||£(A)|| > a. in such a case, since £ is an l.-o.s. measure it is at once verified that [Z £(A.)|A.GG } is not a Cauchy sequence. Hence it does not converge to £(UA^), a contradiction.

Now applying Zorn's lemma it follows that there exists a maximal family G_o c B such that G_o is pair-wise disjoint and AeG_o implies 4(A) / 0. Let $T_Q = \bigcup_{A \in G_o} A$ and $£(T_Q)$ = *fel.* Let h^{Δ} : B L/A be the mapping defined by $h^{\Delta}(A)$ = supp £(A), AGB. The mapping h^{Δ} is a cr-homomorphism on B into I/A. Indeed

(1) h^A(AUB) = support(^(A~B) + *£*(AOB) + £(B~A)) =

= support(^(A-B) + ^(AfiB)) U support(£(AnB) + 4(B-A))

= h^A(A) U h^A(B)

and (2) if A <= B then

h^A(&~A) = h^A(B) - h^A(A)

since h^A(B) = h^A(B-A) U h^A(A) and h^A(A) and h^A(B-A) are disjoint. The fact that h^{Δ} is a a-homomorphism is implied by the a-additivity and £-orthogonal scatteredness of *£.* Now if j\ cz T_o (the case A ⊄ T_o may be reduced to the latter because 4(A~T_o) =0) then T_o = A U (T_o~A) and

$$f = £(T_0) = \wedge(A) + 4(T_Q - A).$$

Hence

$$f = f X_{h^A(A)} + f X_{h^{\Delta}(T_o \sim A)} .$$

Since $\xi(A) \perp \xi(T_o \sim A)$ and support $f\chi_{h^{\triangle}(A)} = h^{\triangle}(A) =$ support $\xi(A)$ we obtain $\xi(A) = f\chi_{h^{\triangle}(A)}$, completing the proof of the proposition.

We recall that a σ-algebra of sets is said to be σ-perfect if every σ-filter in it is determined by some point. Further we note that every σ-algebra of sets is isomorphic to a σ-perfect σ-algebra of sets, Sikorski [10]. We proceed to show that the proposition 1 concerning the structure of ξ could be improved in certain special cases. We adopt the following notation: $\mathcal{L}_{\infty}(\Omega,\Sigma,\mu)$ is the space of all bounded measurable functions $\tilde{f}$ on (Ω,Σ,μ) and $L_{\infty}(\Omega,\Sigma,\mu)$ is the space of all equivalence classes of $\mathcal{L}_{\infty}(\Omega,\Sigma,\mu)$. In the next proposition we choose for $\mathfrak{X} = L_{\infty}(\Omega,\Sigma,\mu)$ with $\mathcal{L}_{\infty} \ni \tilde{f} \longrightarrow (\tilde{f})^{\triangle} \in L_{\infty}$ as the canonical mapping.

<u>Proposition 2</u>. If μ is σ-finite, $\xi : \mathcal{B} \quad L_{\infty}(\Omega,\Sigma,\mu)$ is an c.a.l.-o.s. measure on a σ-perfect σ-algebra $\mathcal{B}$ of subsets of T then the homomorphism $h^{\triangle}$ of the preceding proposition is induced by a point wise mapping i.e. there exists a mapping $\tau : \Omega \longrightarrow T$ and there exists an $f \in \mathcal{L}_{\infty}(\Omega,\Sigma,\mu)$ such that

$$(*) \qquad \xi(A) = (\widetilde{\widetilde{f\chi}}_{\tau^{-1}(A)})^{\triangle}, \quad A \in \mathcal{B}.$$

<u>Proof</u>: From the hypothesis on μ (causing its strict localizability) there exists a lifting on L_{∞} i.e. a multiplicative linear mapping $\lambda : L_{\infty} \longrightarrow \mathcal{L}_{\infty}$ such that a) $(\lambda(f))^{\triangle} = f$, b) $\lambda(1) = 1$, c) $f \geq 0 \quad \lambda(f) \geq 0$. (Chapter 4, Ionescu-Tulcea [3]).

It follows that λ is continuous and it is a vector lattice homomorphism. Hence $\lambda\,\xi : \mathcal{B} \longrightarrow \mathcal{L}_\infty$ is a c.a.l.-o.s. measure. Now we proceed as in the proof of the previous proposition. Let us define $h : \mathcal{B} \longrightarrow \Sigma$ by setting $h(A) = \text{support}\ \lambda(\xi(A))$ and as before it is verified that h is a σ-homomorphism of $\mathcal{B}$ into Σ. Since $\mathcal{B}$ is σ-perfect every σ-homomorphism of $\mathcal{B}$ into any σ-algebra of sets is induced by some point-wise mapping [10]. Hence there is $\tau : \Omega \longrightarrow T$ such that $h(A) = \tau^{-1}(A)$, $A \in \mathcal{B}$. As in Proposition 1 it is verified that $\lambda(\xi(A)) = \tilde{\tilde{f}}\chi_{\tau^{-1}(A)}$, $A \in \mathcal{B}$, $\tilde{f}$ being by definition equal to $\lambda(\xi(T))$ from which $(*)$ follows.

Since for $1 \leq p < \infty$ there exists no positive linear lifting on L_p the proof of Proposition 2 cannot be carried over for L_p-spaces. Hence in general we cannot claim that every c.a.l.-o.s. measure is a composition of standard c.a.l.-o.s. measure and some rearrangement (τ) of the underlying set. However if we restrict the class of σ-algebras $\mathcal{B}$ on which ξ is defined we can prove the similar result. For completeness sake let us recall that the σ-algebra $\mathcal{B}$ is said to be an solute Borel σ-algebra if it is of the Hilbert cube [10] For instance every σ-algebra of Borel subsets of a separable complete metric space is absolutely Borel.

<u>Proposition 3</u>. Let $\mathfrak{X}$ be an F-space of equivalence classes of measurable functions supported by a measure space (Ω, Σ, μ) and let $\xi : \mathcal{B} \quad \mathfrak{X}$ be a c.a.l.-o.s. measure on the σ-algebra $\mathcal{B}$ of subsets of T where it is assumed that $\mathcal{B}$ is absolutely Borel. Then the σ-homomorphism $h^{\triangle}$ of Proposition 1 is induced by a point-wise mapping i.e. there exists a function $f \in \mathfrak{X}$ and a mapping $\tau : \Omega \quad T$ such that

$$\xi(A) = f \cdot \chi^{\triangle}_{\tau^{-1}(A)}, \quad A \in \mathcal{B}.$$

$$\xi(A) = f\cdot\chi^{\triangle}_{\tau^{-1}(A)}, \quad A\in\mathcal{B}.$$

The proof of this proposition is very similar in details to the preceding one after noting Theorem 3.2.5, p.139 [10]. Therefore the details are omitted.

5. The real measure ξ_F as defined in section 2 is absolutely continuous with respect to the c.a.l.-o.s. measure ξ, and in this section we discuss the existence of Radon-Nikodym derivative of ξ with respect to ξ_F.

<u>Theorem 2</u>. Let $\xi : \mathcal{B} \quad L_p(\Omega,\Sigma,\mu)$, $1 \leq p \leq \infty$, be an c.a.l.-o.s. measure. Let F be a fixed non-negative additive functional on $L_p(\Omega,\Sigma,\mu)$. If there exists $A\in\mathcal{B}$ such that (1) $0 < \xi_F(A) < \infty$ (2) $(A,\mathcal{B}_A,|\xi|)$ is a non-atomic measure space,where $|\xi|$ is a variation of ξ and $\mathcal{B}_A$ is the trace of $\mathcal{B}$ on A then ξ does not admit a Radon-Nikodym derivative with respect to ξ_F.

<u>Proof</u>: We can assume that ξ is absolutely continuous with respect to ξ_F, for otherwise ξ does not admit a Radon-Nikodym derivative. Further let f be the function in $L_p(\Omega,\Sigma,\mu)$ determined by ξ so that $\xi(A) = f\cdot\chi_{h^{\triangle}(A)}$, $h^{\triangle}$ being the map defined in proposition 1, section 4. For the sake of simplicity and without loss of generality we assume that $f \geq 0$. Because of the representation of ξ and since $\xi << \xi_F$ it is verified that $|\xi| << \xi_F$. Indeed if $B\in\mathcal{B}$ and $\xi(B) = 0$ and $|\xi|(B) = \delta > 0$

there exists $B_1 \in \mathcal{B}$ such that $\|\xi(B_1)\| > 0$. Hence $\xi_F(B_1) \neq 0$. Since F is additive $\xi(B_1) \neq 0$. Since $\xi(B) = 0$ and $f \geq 0$, $f\chi_{h^\triangle(B_1)} = \xi(B_1) = 0$. Further we note the following: a) there exist $\alpha > 0$, and $A_1 \in \mathcal{B}$, $A_1 \subset A$ such that $\xi(A_1) \neq 0$ and $f\chi_{h^\triangle(A_1)} \geq \alpha\chi_{h^\triangle(A_1)}$, b) the measure $\xi_F|_{\Sigma_{A_1}} << \mu h^\triangle|_{\Sigma_{A_1}}$.

For if (a) were to be false then for each $\alpha > 0$ and for every set $B \subset A$, $B \in \mathcal{B}$ such that $\xi(B) \neq 0$ there exists a set $B_1, B_1 \in \mathcal{B}$, $B_1 \subset B$ with $f\chi_{h^\triangle(B_1} < \alpha$. Pick $A_1 \subset A$ such that $\xi(A_1) \neq 0$ and $f\chi_{h^\triangle(A_1)} < \alpha$. Then either $\xi(A \sim A_1) = 0$ or else repeating the above procedure we obtain

$\xi(A_2) \neq 0$ and $f\chi_{h^\triangle(A_2)} < \alpha$. Thus we obtain a finite family of pairwise disjoint $\mathcal{B}$-measurable sets $\{A_1, A_2, \ldots, A_n\}$ such that $\xi(A \sim \bigcup_{i=1}^{n} A_i) = 0$ and $f\chi_{h^\triangle(\cup A_i)} < \alpha$ or else there exists an infinite sequence of pair-wise disjoint $\mathcal{B}$-measurable sets $\{A_i\}_{i\geq 1}$, $\xi(A_i) \neq 0$ such that $f\chi_{h^\triangle(\cup A_i)} < \alpha$. In the first case $f\chi_{h^\triangle(A)} < \alpha$. We shall show in the second case also the same holds. Let $\Gamma = \{\{A_i\}_{i\geq 1} | A_i \cap A_j = \emptyset,\ i \neq j,\ A_i \in \mathcal{B}, A_1 \subset A$ and $f\chi_{h^\triangle(A_i)} < \alpha\}$. Partially order Γ by inclusion. If C is a chain in Γ and if $A, B \in \cup C$ then $A \cap B = \emptyset$. Since $A \in \cup C$ implies $\xi(A) \neq 0$, $\cup C$ is countable. Further $A \in \cup C$ implies $f\chi_{h^\triangle(A)} < \alpha$. Thus $\cup C$ is a member of Γ. Hence by Zorn's Lemma there exists a maximal sequence $G_o = (A_i)_{i\geq 1}$ ^ r . Now if $|(A - \bigcup_{i\geq 1} A_i) \wedge 0$ it is verified that the maximality of G_o is contradicted. Thus % (A) = § (U A.) =fx $_{h^A(UA_i)}$ < a i.e. i^l

fX A $_{h^A(A)}$ < a. Thus if (a) is false for every a > 0 fx $_{h^A(A)}$ < a. Since f]> 0 this implies that § (A) = fx $_{h^A(A)}$ = 0, contradicting $\S_F(A) > 0$. This completes the proof of a) and there exists A_1 e 8, A_1 c A, $Z(A_1)$ ji 0 such that fx . $_{h^A(A_1)}$ 1 a $\chi_{h^A(A_1)}$ for some positive number a.

b) follows directly from the representation of § stated in proposition 1.

Since |k(C)||^P = $\int_\Delta f^P \chi_{h^A(c)} d\mu$ < oo for all CeH_1 C c & $_1$

it is verified that uOh $|L^{A_1}$ is a finite measure. Since

$\S_p|E^{A_1}$ is a finite positive measure we can chose a j3 > 0 such that $\%_F(A_1)$ - $P\backslash J.Ol\backslash^A(Aj)$ > 0. If for every B c A^ BG» §F (B) - £uOh^A(B) ;> 0 then $\S_p(B)$ = 0 implies |J h^A(B) = 0 i.e. (*) |aoh^A| $L7_{A_1}$ « 5«| $£_{A_1}$. Thus either (**) there exists a

set A_2 c A_1 such that O ^ v = $\S_F|L_{A_2}$ - j3f-iOh^A| $_{\Sigma_{A_2}}$ ^ 0 and $v(A_2)$ <0 or else (*) holds. Next suppose (*) holds. Since $|(A_1)$ / 0

from the representation of 5 it is verified that |ar$^<$h^A(A_1) ^ 0. Hence from (*) it follows that | $_F$(A^) ^ 0. Thus there is a real number *fi* > 0 such that (/3^oh^A|$£_{A_1}$ -$\S_F$)(A_1) > 0. Hence from Jordan decomposition theorem it follows that there is a B—measurable set $P_{2} \subset A_1$

satisfying the property (-**) . Thus from the preceding remarks we can choose the set A itself to have the properties

1) there exists a > 0 such that $f\chi_{h^A(A)} \geq \alpha\chi_{h^A(A)}$

2) there exists a $\gamma > 0$ such that for all $B \in \mathcal{B}$, $B \subset A$, $g_p(B) \geq \gamma\, \mu \circ h^A(B)$.

Now choose a pairwise disjoint sequence of S-measurable sets $\{A_i\}_{i \geq 1}$, $A_i \subset A$, $g(A_i) \neq 0$. Since $\xi \ll g_p$, $g_p(A_i) > 0$. We complete now the proof of non-existence of Radon-Nikodym derivative of § with respect to ξ_F by discussing the two cases $1 \leq p < \infty$, $p = \infty$ separately. If $1 \leq p < \infty$ with the sequence $\{A_i\}_{i \geq 1}$ as chosen above consider the inequality, for $i \neq j$,

$$\left\| \frac{f\chi_{h^\Delta(A_i)}}{\xi_F(A_i)} - \frac{f\chi_{h^\Delta(A_j)}}{\xi_F(A_j)} \right\|^p = \left\| \frac{\xi_F(A_j) f\chi_{h^\Delta(A_i)} - \xi_F(A_i) f\chi_{h^\Delta(A_j)}}{\xi_F(A_i)\,\xi_F(A_j)} \right\|^p$$

$$\geq \frac{\int |f\chi_{h^\Delta(A_i)}|^p \, d\mu}{\xi_F(A_i)} \geq \frac{\alpha \, \mu \circ h^\Delta(A_i)}{\gamma \, \mu \circ h^\Delta(A_i)} = \frac{\alpha}{\gamma} \, .$$

Thus the sequence $\left\{ \frac{f\chi_{h^\Delta(A_i)}}{\xi_F(A_i)} \right\}_{i \geq 1}$ does not admit a convergent subsequence. Hence it follows from Theorem 1, Rieffel [9] that ξ_p does not admit a Radon-Nikodym derivative with respect to ξ. If $p = \infty$ it is verified that

$$\left\| \frac{f\chi_{h^\Delta(A_i)}}{\xi_F(A_i)} - \frac{f\chi_{h^\Delta(A_j)}}{\xi_F(A_j)} \right\| \geq \left\| \frac{f\chi_{h^\Delta(A_i)}}{\xi_F(A_j)} \right\| \geq \frac{\alpha}{\xi_F(A)} \, .$$

Once again applying Rieffel's theorem the proof of the theorem is completed for the case $p = \infty$.

Remark 1. We note that from the preceeding theorem and theorem 1 in [8] that ξ admits a Radon-Nikodym derivative with respect ξ_F if and only if
(1) $\xi << \xi_F$, (2) $|\xi|$, the variation of ξ, is a finite measure and (3) $(T, \mathcal{B}, |\xi|)$ is purely atomic.

Remark 2. In §4, Masani [6] obtained sufficient conditions for the non-existence of Radon-Nikodym derivative of a c.a.o.s. measure ξ taking values in a Hilbert space $\mathfrak{X}$ with respect to the measure $\xi_F(\cdot) = \|\xi(\cdot)\|^2$ and $F(x) = \|x\|^2$. The following analogue of Masani's theorem for c.a.l.-o.s. measures ξ is an immediate consequence of the preceeding theorem.

Corollary. Let $\xi : \mathcal{B} \quad \mathcal{H}$ be a c.a.l.-o.s. measure, where $\mathcal{H}$ is the Hilbert space $L_2(\Omega, \Sigma, \mu)$. Let $F : \mathcal{H} \longrightarrow R$ be the additive functional defined by $F(x) = \|x\|^2$. Then ξ admits a Radon-Nikodym derivative w.r.t. ξ_F if and only if (1) $|\xi|$ is a finite measure and (2) $(T, \mathcal{B}, |\xi|)$ is purely atomic.

6. In this section we apply the results obtained in the preceeding sections to a probabilistic problem concerning independently scattered random measures.

Let $\mathfrak{M}(\pi,\mathcal{P},P)$ be a complete metric linear space (topology determined by convergence in probability) of all random variables on a probability space $\mathfrak{M}(\pi,\mathcal{P},P)$. An independently scattered random measure on the Borel subsets $\mathcal{B}$ on the unit interval T is a mapping $M : \mathcal{B} \longrightarrow \mathfrak{M}$ enjoying the following properties.

(+) for every sequence $\{E_i\}$ of pairwise disjoint Borel sets

$$M(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} M(E_i)$$

where the series converges with probability 1.

(++) for every sequence $E_1,\ldots,E_n$ of disjoint Borel sets the random variables $M(E_1),\ldots,M(E_n)$ are independent For the theory of such measures the reader is referred to [12] and references given therein. The measure M is said to be non-atomic if $M(\{a\}) = 0$ p.a.e. for every one point set $\{a\}$. Let $[M]$ denote the closed subspace spanned in $\mathfrak{M}$ by range of M. The definition and the properties of the integral of real functions on T with respect to M may be found in [12]. $L_1(M)$ denotes the space of M-integrable real-valued functions on T.

<u>Theorem 3</u>. Let M be a non-atomic and non-gaussian independently scattered $\mathfrak{M}$-valued random measure. If $N : \mathcal{B} \longrightarrow [M]$ is an independently scattered random measure then there exists a Borel measurable mapping $\tau : T \longrightarrow T$ and a function $f \in L_1(M)$ such that the measure N has the following representation

$$N(A) = \int_{\tau^{-1}(A)} f(t)M(dt), \quad A \in \mathcal{B}.$$

<u>Proof</u>: For every Borel set A, $N(A) \in [M]$. Hence from the representation theorem in Section 2 of [12], there exists an $f_A \in L_1(M)$ such that $N(A) = \int_T f_A(t) M(dt)$. Since the mapping $N(A) \to f_A$ is a continuous linear mapping the set function $\mathcal{B} \ni A \to f_A$ is a measure with values in the complete metric space $L_1(M)$. If A and B are disjoint Borel sets then $N(A)$ and $N(B)$ are independent and from theorem 2.1 in [12] f_A and f_B are orthogonal in the sense of section 1. Thus the set function $\mathcal{B} \ni A \to f_A$ is a c.a.l.-o.s. $L_1(M)$ valued measure. It follows from Proposition 3 in section 4 that

$$f_A = f \chi_{\tau^{-1}(A)}$$

for some $f \in L_1(M)$ and a Borel measurable mapping $\tau : T \to T$. This completes the proof of the theorem.

References

[1] N. Dinculeanu, Vector Measures, Pergamon Press 1967

[2] P. R. Halmos, Measure Theory, Van Nostrand 1950.

[3] A. and C. Ionescu-Tulcea, Topics in the Theory of Lifting Springer-Verlag, 1969.

[4] R. C. James, "Orthogonality and Linear Functionals in Normed Linear Spaces", Trans. Amer. Math. Soc. 61 (1947) 261-292.

[5] A. N. Kolmogorov, "Stationary sequences in Hilbert spaces" (in Russian), Bulletin of M.G.U. 2 (1941), 1-40.

[6] P. Masani, "Orthogonally scattered measures", Advances in Math. 2 (1968), 61-117.

[7] _________, "Quasi-isometric measures and their applications", Bull. Amer. Math. Soc. 76 (1970), 427-528.

[8] W. Matuszewska and W. Orlicz, "Note on modular spaces VI, Bull. Polon. Ac. Sci. 11 (1963), 449-454.

[9] M. A. Rieffel, "The Radon-Nikodym theorem for the Bochner integral", Trans. Amer. Math. Soc. 131 (1968), 466-487.

[10] R. Sikorski, Boolean Algebras, Springer-Verlag 1964.

[11] K. Sundaresan, "Orthogonality and Nonlinear Functionals on Banach Spaces, Report 71-22, Carnegie-Mellon University.

[12] K. Urbanik, "Random measures and harmonizable sequences", Studia Math. 31 (1968), 61-88.

[13] N. Wiener, "Differential space", Journal of Math. and Phys. 2 (1923), 131-174.

Carnegie-Mellon University
Pittsburgh, Pennsylvania

/cs