# THE SELF-CIRCUMFERENCES OF POLAR CONVEX DISKS

by

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## 1. Introduction

Let $E$ be an oriented real euclidean two-dimensional vector space, and let $K$ be a convex disk (compact convex set with non-empty interior) in $E$, with $0$ in its interior. The guage function of $K$,

$$\|x\|_K = \min\{\lambda : x \in \lambda K\}, \quad x \in E \tag{1}$$

determines a (not necessarily symmetric) "Minkowski metric". For each rectifiable curve in $E$ (rectifiability in terms of the euclidean metric of $E$) it is then possible to define, in the usual way, its $K$-length determined by this "metric".

In [2], Gołąb discussed, apparently for the first time, the self-circumferences of $K$, i.e., the $K$-lengths of the two simple closed curves (one the opposite of the other) that describe the boundary of $K$. If $\partial K$ is this boundary, let $\partial_+ K$ and $\partial_- K$ be the simple closed convex (hence rectifiable) curves whose image is $\partial K$; for definiteness, $\partial_+ K$ leaves $K$ on its left according to the given orientation of $E$. We let $\sigma_+(K)$, $\sigma_-(K)$ be the $K$-length of $\partial_+ K$, $\partial_- K$, respectively; these are the self-

circumferences of K (with respect to 0). Further work on these parameters appears in [3] and [1], and in papers there cited.

Let $K^!$ be the convex disk polar to K with respect to the inner product of E. The purpose of this paper is to show that $cr_+(K) = a_(K^!)$, $a_(K) = o^(K<)$.

If instead of a single euclidean plane we had considered a pair of oriented two-dimensional vector spaces in duality, we could have stated our result without involving any Euclidean structure. We shall not deprive the reader of the pleasure of carrying out this reformulation in the general case. For disks symmetric with respect to 0 such a reformulation is, however, related to some geometric problems that will be mentioned at the end of the paper.

## 2. The Main Result

We denote the inner product of u,veE with respect to the euclidean structure of E by u-v. An oriented line *I* in E is an oriented line of support of the convex disk K if it contains a point of K and if K lies entirely in the left closed half-plane determined by *I* and the orientation of E. If *l,V* are oriented lines in E, the ordered pair *{1₉V )* forms ≤a right angle if the (euclidean) unit vector of *V* is obtained from the unit vector of *I* by a rotation of $-T\frac{1}{2}T$

according to the orientation of E.

It follows at once from (1) and from basic facts about convex sets that

(2) $\|x\|_K = \max\{\varphi(x) : \varphi \text{ a linear functional on } E,\ \varphi(K) \subset (-\infty, 1]\}$
$= \max\{x \cdot y : y \in K'\}, \quad x \in E.$

1. Theorem. Let K be a convex disk in the oriented real euclidean two-dimensional vector space E, with O in its interior. Let K' be the polar disk. Then $\sigma_+(K) = \sigma_-(K')$, $\sigma_-(K) = \sigma_+(K')$.

Proof. 1. It is clearly sufficient to prove the first equality: the second follows on reversing the orientation of E. On account of the existence of an obvious approximation procedure, it is sufficient to prove this first equality for polygonal K; indeed, more specifically, for K satisfying

(3) K is a polygonal disk, and the parallel through O to each side of K contains no vertex of K.

The polar disk of a polygonal disk is polygonal (with the same number of sides); Condition (3) is easily seen to be equivalent to

(4) K is a polygonal disk, and no side of K is perpendicular to a side of the polygonal disk K'.

2. We now assume that $K$ satisfies (4) and let $n$ be the number of its sides. We consider an ordered pair $(\ell,\ell')$ of oriented lines in $E$ forming a right angle, and such that $\ell$ is an oriented line of support of $K$ and $\ell'$ an oriented line of support of $K'$. We let this pair rotate counterclockwise; as it does so, the pair of points where $\ell$ supports $K$ and $\ell'$ supports $K'$ describes a cyclic sequence $((u_j,v_j))$, $j\in Z_{2n}$, such that

(5) $u_j \neq u_{j-1}$ or $v_j \neq v_{j-1}$, but not both, for each $j\in Z_{2n}$.

This property is a consequence of (4). The fact that the sequence has $2n$ terms follows from (5) and from the fact that each one of the $n$ vertices of $K$ and the $n$ vertices of $K'$ appears in the sequence, each term introducing one of them for the first time.

The sides of $K$, taken as oriented segments leaving $K$ on the left, are exactly the oriented segments $u_{j-1}u_j$ for those $j\in Z_{2n}$ for which $u_j \neq u_{j-1}$. By the construction of the sequence and by (2) we have

$$\|u_j-u_{j-1}\|_K = (u_j-u_{j-1})\cdot v_j = u_j\cdot v_j - u_{j-1}\cdot v_{j-1} \qquad j\in Z_{2n},\ u_j \neq u_{j-1},$$

where (5) was used for the last equality. Thus

(6) $$\sigma_+(K) = \sum\{u_j\cdot v_j - u_{j-1}\cdot v_{j-1} : j\in Z_{2n},\ u_j \neq u_{j-1}\}.$$

Using (5) to reshuffle (6), we find

$$(7) \qquad \mathrm{CF}_{+}(K) = \sum_{j r Z_{2n}} \rho_j u_j \cdot v_j$$

where

$$(8) \qquad Pj = \begin{cases} 1 & \text{if } u_3 \wedge u_{3-1} \text{ and } v_3 \wedge v_{3+1} \\ -1 & \text{if } v_3 \wedge v_{3-1} \text{ and } u_3 \wedge u_{3+1} \\ 0 & \text{otherwise.} \end{cases} \qquad j \wedge Z_{2n}$$

3. in order to compute $a_{-}(K^{!})$, we apply the preceding line of argument to $K^{!}$ instead of K, with the orientation of E reversed. Since $K^{?T} = K$, $K^{T}$ satisfies the analogue of (4); the analogue of the construction at the beginning of Part 2 of the proof yields the same lines with the opposite orientation, paired in the opposite order, and rotating "backwards", i.e., in the <u>new</u> counterclockwise sense. The pair of points at which $K^{1}$, K are supported then describes the sequence $((v_{-j}, u_{-j}))$, $J \epsilon Z_{2n}$, and we conclude that

$$(9) \qquad a_{-}(KM = \sum_{j \epsilon Z_{2n}} \rho'_j v_{-j} \cdot u_{-j},$$

where

$$(10)\qquad p_j = \begin{cases} 1 & \text{if } v_{-j} = v_{-j+1} \text{ and } u_{-j} \neq u_{-j-1} \\ -1 & \text{if } u_{-j} = u_{-j+1} \text{ and } v_{-j} \neq v_{-j-1} \\ 0 & \text{otherwise.} \end{cases} \qquad j \in \mathbb{Z}_{2n}$$

Inspection of (8) and (10) shows that $p_j^2 = p_j$, $j \in \mathbb{Z}_{2n}$, hence comparison of (7) and (9) yields $a_+(K) = a_-(K)$, as desired.

## 3. Normed Spaces and the Girth of Spheres

If $X$ is a real normed space, we let $\mathcal{L}(X)$ denote its unit ball. If $\dim X = 2$, we denote by $L(X)$ one-half the length (in terms of the norm of $X$) of the simple closed curve describing the boundary of the symmetric convex disk $\mathcal{L}(X)$ (see [4]). Thus, if $E$ is any oriented euclidean space coinciding algebraically with $X$, we have $L(X) = a_+(\mathcal{Z}(x)) = v_2(E(x))$. Now the same vector space, with the polar disk $(\mathcal{L}(X))^f$ as unit ball, is congruent to $X^*$, the dual space of $X$. We obtain the following consequence of Theorem 1.

2. *Corollary.* *If $X$ is a real normed space with $\dim X = 2$, and $X^*$ is its dual space, then $L(X^*) = L(X)$.*

More generally, if $x$ is a normed space with $\dim X \geq 2$, one can define $m(X)$, the infinum of the lengths of rectifiable curves with antipodal endpoints and lying entirely in the boundary of $\mathcal{Z}(X)$; $2m(X)$ is the *girth* of $\mathcal{L}(x)$. We refer to [4] for a

detailed discussion of this definition. Corollary 2 then gives an affirmative answer for dim X = 2 to the following conjecture, which has been verified in a few isolated additional cases.

3. Conjecture. If X is a real normed space with dim X $\geq$ 2, and $X^*$ is its dual space, then $m(X^*) = m(X)$.

For the reader acquainted with [4] we remark that if we replace the parameter m in the conjecture by either M or D, the statement, while still true for dim X = 2 by Corollary 2, becomes false for every other dimension.

## REFERENCES

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