NONOSCILLATION AND DISCONJUGACY OF SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

by

Zeev Nehari

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Abstract

The differential equations under consideration are of the form

$$\frac{dx}{dt} = A(t)x, \tag{1}$$

where $A(t)$ is a piecewise continuous real $n\times n$-matrix on a real interval α, and the vector $x = (x_1,\ldots,x_n)$ is continuous on α. The equation is said to be nonoscillatory on α if every non-trivial real solution vector x has at least one component x_k which does not vanish on α.

The principal concern of this paper is the derivation of conditions, expressed in terms of various norms of A, which guarantee the nonoscillation of (1) in a given interval.

NONOSCILLATION AND DISCONJUGACY OF SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

by

Zeev Nehari*

In the present paper we shall discuss various oscillatory properties of differential equations of the form

$$\frac{dx}{dt} = A(t)x, \tag{1.1}$$

where $A(t)$ is a real $n \times n$ matrix whose elements are defined on a real interval α, and $x = (x_1,\ldots,x_n)$ is an n-vector. While the case of principal interest is that in which A is continuous on α, it soon becomes apparent that this assumption is too restrictive and that, even for an adequate discussion of the continuous case, it is necessary to consider coefficient matrices A which may have a finite number of discontinuities at interior points of α. Accordingly, we shall assume that $A(t)$ is continuous on α, with the possible exception of a finite number of interior points at which both the left and the right limit of $A(t)$ exist. The value of $A(t)$ at a discontinuity t_0 will be defined as $\lim_{t \to t_0^+} A(t)$; this enables us to define a unique (and

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continuous) continuation of a solution $x(t)$ as the point t_0 is passed from left to right. A vector $x(t)$ will be called a solution of (1) on a if it is continuous on a and satisfies equation (1) in all subintervals of a which do not contain discontinuities of $A(t)$.

A nontrivial solution $x(t)$ of (1.1) will be said to be <u>oscillatory</u> on a if each of its components $x_1, \ldots, x_n$ vanishes at some point of a. Equation (1.1) will be said to be <u>nonoscillatory</u> on a if none of its solutions are oscillatory on a, i.e., if every nontrivial solution has at least one component which does not vanish on a.

An alternative description of the oscillatory behavior of equation (1.1) may be based on the concept of the <u>conjugate point</u>, which generalizes a similar notion employed in the study of scalar n-th order equations [3,4,5,6,11]. If $[a,b] \in a$, b will be said to be a conjugate point of a with respect to equation (1.1) if there exists a nontrivial solution vector x of (1.1) such that each component of x vanishes at either a or b and if, moreover, b is the smallest number for which this is the case. If no point of a possesses a conjugate point, the equation is said to be <u>disconjugate</u> on a. Evidently, an equation which is nonoscillatory on a is also disconjugate on this interval. The converse is in general not true. While there exist classes of equation (1.1) for which the concepts of nonoscillation and

disconjugacy coincide (e.g., equations which are equivalent to scalar equations of the form $y^{(n)} + p(t)y = 0$ [6]), an equation (1.1) may be disconjugate on an interval without being nonoscillatory. As an example, we consider the equation

$$(1.2) \qquad y_k' = 2(t-a_k)\ [\sum_{\nu=1}^{n} (t-a_\nu)^2]^{-1} \sum_{\nu=1}^{n} y_\nu, \quad k = 1,\ldots,n$$

(a_k real). Clearly, (1.2) is solved by the vector $y^{(1)} = [(t-a_1)^2,\ldots,(t-a_n)^2]$, and also by the constant vectors $y^{(k)}$ $(k=2,\ldots,n)$ whose only non-zero components are $y_{k-1}^{(k)} = 1$ and $y_n^{(k)} = -1$. If Y is the solution matrix formed by the column vectors $y^{(1)},\ldots,y^{(n)}$, it is easily seen that the determinant of Y has the value $(t-a_1)^2+\ldots+(t-a_n)^2$ and this shows that, unless all the a_k coincide, Y is a fundamental solution matrix on $(-\infty,\infty)$. On an interval α containing the points $a_1,\ldots,a_n$ each component of $y^{(1)}$ has a zero, and the equation thus is oscillatory on α. Nevertheless (except in the case in which the set $a_1,\ldots,a_n$ contains only two different numbers), no point in $(-\infty,\infty)$ has a conjugate point. In fact, as the following argument shows, all real solutions of (1.2) which are not constant multiples of $y^{(1)}$ are nonoscillatory on $(-\infty,\infty)$. Since Y is a fundamental matrix, all real solutions are of the form Yc, where $c = (\alpha_1,\ldots,\alpha_n)$ is a constant vector. If the n components of Yc are to vanish at points $t_1,\ldots,t_n$, respectively, it follows from the special form of Y that $\alpha_1(t_k-a_k)^2 + \alpha_{k+1} = 0$ $(k=1,\ldots,n-1)$, $\alpha_1(t_n-a_n)^2 - \sum_{\nu=2}^{n} \alpha_\nu = 0$.

Adding these equations, we obtain $\alpha_1 \sum_{k=1}^{n} (t_k - a_k)^2 = 0$. Since $\alpha_1 = 0$ would lead to the trivial solution, this implies $t_k = a_k$, $k = 1,\ldots,n$, and thus $\alpha_2 = \ldots = \alpha_n = 0$, i.e., the only oscillatory solutions of (1.2) are $\alpha_1 y^{(1)}$, $\alpha_1 \neq 0$. A pair of conjugate points a,b $(a<b)$ is thus possible only if all the numbers $a_1,\ldots,a_n$ coincide with either a or b. In this case, a indeed possesses a conjugate point, but no other point does.

Finally, we consider a property of equations of the type (1.1) which is closely related to both nonoscillation and disconjugacy, and which has the merit that it can be defined without reference to the components of a solution vector. The equation is said to be <u>suborthogonal</u> on α if, for $t_1 \in \alpha$, $t_2 \in \alpha$ and any nontrivial solution vector x, the scalar product $x(t_1)x(t_2)$ is positive. If b is the conjugate point of a, we clearly have $x(a)x(b) = 0$; thus, suborthogonality implies disconjugacy. Suborthogonality is preserved if the coefficient matrix A is replaced by $A_1 = QAQ^{-1}$, where Q is a constant orthogonal matrix. Indeed, if x is a solution of (1.1), then $y = Qx$ is a solution of $y' = A_1 y$, and the assertion follows from the fact that $Qx(t_1)\cdot Qx(t_2) = x(t_1)x(t_2)$. If $x(a)x(b) = 0$, it is easy to see that there exists an orthogonal matrix Q such that $n - 1$ of the components of Qx vanish at a and the remaining component vanishes at b, i.e., b is a conjugate point of a for the coefficient matrix QAQ^{-1} [7]. Hence, equation (1.1) is suborthogonal on α if and only if there exists a Q

such that the equation $x' = QAQ^{-1}$ is disconjugate on α.

2. The conditions for nonoscillation, disconjugacy and suborthogonality to be derived in this paper are all expressed in terms of certain norms of the matrix A or of some of its submatrices. If we employ the Euclidean norm $\|A\|$, it follows from $\|QAQ^{-1}\| = \|A\|$ and the remark made concerning the relation between disconjugacy and suborthogonality that a sufficient condition for disconjugacy which depends only on $\|A\|$ will also guarantee suborthogonality, and vice versa. In the present section we shall show that such a condition is also sufficient to guarantee nonoscillation, although the latter is a more restrictive property than disconjugacy. This will again be achieved by replacing the coefficient matrix A by QAQ^{-1} where Q is orthogonal in the interval under consideration. However, in the present case Q will only be required to be piecewise constant, and the matrix QAQ^{-1} may thus have additional simple discontinuities.

<u>Theorem 2.1. If equation</u> (1.1) <u>is oscillatory on an interval</u> α, <u>then there exists a piecewise constant orthogonal matrix</u> Q <u>such that the equation</u>

$$y' = QAQ^{-1}y \tag{2.1}$$

<u>is not disconjugate (and thus also not suborthogonal) on</u> α. <u>In</u>

particular, Q may be required jto b£ a. diagonal matrix whose diagonal elements $q_1,\ldots,a_n$ have the following property: For each k (l^k^n) there exists 3L t_{fc} e a such that $q_k = 1$ for $t < t_{fc}$, tea and $q_k = -1$ for $t \geq t_k$, tea.

The existence of a matrix Q of the specified form will be needed in Section 4. If it is only desired to establish the assertion made above regarding the role of the norm ||A|| in nonoscillation criteria, it is sufficient to prove the first part of Theorem 2.1. This is easily achieved by means of the following argument. If equation (1.1) is oscillatory on a, there exist a closed interval [a,b] in a and a solution x of (1.1) such that x is oscillatory in [a,b] and two of the components of x vanish, respectively, at a and b. If all the components of x vanish at either of the two points, the assertion is trivial. If there are components x^ which do not vanish at a or b, we have $x,_k(t_k) = 0$ for t_k e (a,b) . We now denote by $Q_1 = Q_1(t)$ the diagonal matrix whose diagonal elements q_k are defined as follows: If ^C31) = 0, $x_k(b) = 0$ or $x_k(a)x_k(b) < 0$, we set $q_k = 1$ in [a,b]; if $x_k(a)x_k(b) > 0$, we take $q_{fc} = 1$ in [a^k) a^d $q_{fc} = -1$ in [t^b] . Q_1 is orthogonal and, clearly, $Q_1(a)x(a)-Q_1(b)x(b) < 0$ (except in the trivial case just mentioned) . By the construction of Q_1, the vector $y(t) = Q_1(t)x(t)$ is continuous on [a,b] (notwithstanding the discontinuity of $Q_1(t)$), and it is a solution of the equation $y' = C^AQ^{-1}y$. But, as just shown, $y(b)y(a) < 0$, and this equation

thus fails to be suborthogonal in $[a,b]$. By continuity (and the fact that $y(a)y(a) > 0$), there exists a point $c \in [a,b]$ such that $y(a)y(c) = 0$. As shown in section 1, this implies the existence of a constant orthogonal matrix Q_2 such that the equation $w' = Q_2[Q_1AQ_1^{-1}]Q_2^{-1}w$ has, at c, a conjugate point with respect to a. This proves our assertion (with $Q=Q_2Q_1$).

Before proving the second part of Theorem 2.1, we illustrate the use of the result just obtained.

<u>Theorem 2.2. Let</u> $\rho(t)$ <u>be positive and piecewise continuous on</u> $[a,b]$ <u>and let</u> $\Lambda = \min[\lambda_1(b), \mu_1(b)]$, <u>where</u> $\lambda_1(b)$ <u>and</u> $\mu_1(b)$ <u>are, respectively, the lowest eigenvalues of the boundary value problems</u>

$$(2.2) \qquad (\rho u')' + \lambda\rho\|A\|^2 u = 0, \qquad u(a) = u'(b) = 0,$$

$$(2.3) \qquad (\rho v')' + \mu\rho\|A\|^2 v = 0, \qquad v'(a) = v(b) = 0,$$

<u>and</u> $\|A\|$ <u>is the Euclidean norm of the matrix</u> A. <u>If</u> $\Lambda > 1$, <u>then equation</u> (1.1) <u>is both nonoscillatory and suborthogonal on</u> $[a,b]$. <u>This bound for</u> Λ <u>is the best possible; indeed, the conclusion does not necessarily follow if</u> $\Lambda = 1$.

<u>Proof</u>. Suppose (1.1) is oscillatory on $[a,b]$. By Theorem 2.1, there exists a $c \in [a,b]$ such that c is the conjugate point of a with respect to an equation $y' = By$ with $\|B\| = \|A\|$, i.e., each component y_k of y vanishes at either a or c. From $y' = By$ we obtain

$$\|y'\|^2 = \|By\|^2 \leq \|B\|^2\|y\|^2 = \|A\|^2\|y\|^2,$$

and thus, after multiplying by the positive function $\rho(t)$ and integrating over $[a,c]$,

$$\sum_{k=1}^{n} [\int_a^c \rho y_k'^2 dt - \int_a^c \rho\|A\|^2 y_k^2 dt] \leq 0.$$

It follows that there must exist at least one index k for which

$$\int_a^c \rho y_k'^2 dt \leq \int_a^c \|A\|^2 y_k^2 dt.$$

The function y_k vanishes at either a or c. If $y_k(a) = 0$ then, by classical results,

$$\lambda(c) \int_a^c \rho\|A\|^2 y_k^2 dt \leq \int_a^c \rho y_k' dt,$$

where $x(c)$ is the lowest eigenvalue of the problem (2.2) for the interval $[a,c]$. Comparing the two last inequalities, we find that $\lambda_1(c) \leq 1$. Similarly, $y_k(c) = 0$ leads to the conclusion $\mu_1(c) \leq 1$, where $\mu_1(c)$ is the lowest eigenvalue of the problem (2.3) for the interval $[a,c]$. Since $\lambda_1(c)$ and $\mu_1(c)$ are nonincreasing for increasing c, we thus find that if (1.1) has an oscillatory solution on $[a,b]$ we must have $\Lambda = \min[\lambda_1(b), \mu_1(b)] \leq 1$, contrary to our assumption. Hence, (1.1) is nonoscillatory on $[a,b]$.

That the assumptions of Theorem (2.2) also imply suborthogonality follows by observing that if (1.1) is not suborthogonal, then there exists a matrix C, with $\|C\| = \|A\|$, such that $w' = Cw$ is

not disconjugate. The rest of the argument is the same as before.

The special case of Theorem 2.2 corresponding to the choice $p(t) = ||A(t)||^{1}$ provides a new proof for a previous result [7] according to which (1.1) is both nonoscillatory and suborthogonal on [a,b] if

$$\int_a ||A(t)||dt < \bar{\pi}. \tag{2.4}$$

Indeed, the eigensolutions of (2.2) and (2.3) are in this case

$$u = \sin[A \int_a ||A(s)||ds], \quad v = \cos[A \int_a ||A(s)||ds],$$

respectively, where

$$A \int_a^b ||A(t)||dt = |$$

and $A_1(b) = /i(b) = A$. The condition $A > 1$ is therefore in this case equivalent to (2.4). Since equality in (2.4) is not sufficient to guarantee nonoscillation or suborthogonality (cf.[7]), this example also shows that the condition $A > 1$ in Theorem 2.2 is the best possible of its kind.

3. Before proving the second half of Theorem 2.1, we have to devote some attention to what may be called "minimal intervals of oscillation" associated with an equation (1.1) which is known to be oscillatory on an interval a. By this we mean closed intervals [a,b] e a such that the equation is oscillatory on [a,b] but

not on any subinterval of $[a,b]$. The existence of at least one such minimal interval is elementary. The example discussed at the end of Section 1 shows that in the case in which α is a closed interval $[a',b']$, it is possible that $a \neq a'$ and $b \neq b'$.

A minimal interval of oscillation $[a,b]$ is associated with at least one nontrivial solution x of (1.1) which is oscillatory on $[a,b]$. Evidently, x must have at least two components which vanish at a and b, respectively. A more accurate description of x is given in the following statement.

<u>Theorem 3.1.</u> <u>If</u> $[a,b]$ <u>is a minimal interval of oscillation of equation</u> (1.1), <u>then there exists a nontrivial solution</u> x <u>of</u> (1.1) <u>such that each component</u> x_k <u>of</u> x <u>has one of the following three properties</u>:

(a) $x_k(a) = 0$;

(b) $x_k(b) = 0$;

(c) $x_k(t)$ <u>vanishes at some point of</u> $[a,b]$, <u>but</u> $x_k(t) \geq 0$ <u>or</u> $x_k(t) \leq 0$ <u>throughout</u> $[a,b]$.

<u>Proof</u>. If there is more than one minimal solution, we confine our attention to that (or those) for which the number m of components which vanish at either a or b is as large as possible. We assume that $m < n$, since otherwise the assertion of the theorem is trivial. We now choose a number $c \in [a,b]$ which is close enough to b so that the zeros of the components of x with the property

(c) are in (a,c) , and we define a nontrivial solution y of (1.1) in the following manner: $Y_k(^a) \to \circ$ for those k for which $y_k(a) = 0$; $Y_v(^c) \to$" $\circ$ for k such that ^(k) == $\circ$*, with one exception, the remaining components y_k are to vanish at one of the internal zeros of the corresponding components $x,_k$. For the exceptional component y_k we require, say, $y_{k!}(^a) \to !\bullet$ Since (1.1) is nonoscillatory in [a,c], this solution y is uniquely determined and, moreover, y is a continuous function of c (as long as CG (a,b)) [2]. Elementary considerations show that if c-*b through a suitable sequence of values $c_1, c_2, \ldots,$ y will have a uniform limit $\tilde{y}$ which is a nontrivial solution of (1.1), and which is such that all its components $\tilde{y}$. (k^k¹) have zeros in [a,b] which coincide with zeros of the corresponding components of x. We assert that $\tilde{y}$ is a constant multiple of x. Indeed, if this were not the case, we could construct a solution w = x + |3$\tilde{y}$ of (1.1) where the scalar constant p is so chosen that $w,_{k''}(a) = 0$, Xj^n (b) ^ 0. where k" is such that x^tf (a) ^ 0, /\ The solution w would thus have m + 1 components which vanish at either a or b, and this conflicts with our definition of m. Hence, $\tilde{y}$ = yx, where y is a constant.

Suppose now that the component x^t of x changes its sign at one of its zeros, say t^o, in (a,b). Since y->Yx uniformly, if c->b through a sequence {c }, the component y_T of y must take
r x

both positive and negative values near t_0 if r is large enough, and it therefore must vanish at some point of (a,c_r). Since all the other components of this solution y vanish in $[a,c_r]$ by construction, y is found to be oscillatory in $[a,c_r]$. But $c_r \in (a,c)$, and the assumption that the solution x has a component which satisfies neither of the conditions (a), (b), (c) has thus led to a violation of our hypothesis that $[a,c]$ is a minimal interval of oscillation. This completes the proof of Theorem 3.1.

We also need the following result.

<u>Theorem 3.2.</u> <u>Let equation</u> (1.1) <u>have a solution</u> x <u>on an interval</u> $[a,b]$ <u>such that each of its components</u> x_k <u>has one of the following three properties</u>:

(a) $x_k(a) = 0$;

(b) $x_k(b) = 0$;

(c) $x_k(a)x_k(b) < 0$.

<u>Then</u> a <u>has a conjugate point</u> $c \in (a,b]$ <u>with respect to equation</u> (1.1).

<u>Proof</u>. Suppose there exists no conjugate point in $(a,b]$. For elementary reasons there will then exist a uniquely determined solution y of (1.1) for which each component y_k takes prescribed values (not all zero) at either a or γ, where $\gamma \in (a,b]$. We choose these values in the following manner: We set $y_k(a) = 0$ if $x_k(a) = 0$, $y_k(\gamma) = 0$ if $x_k(b) = 0$, and $y_k(a) = x_k(a)$ in those cases in which

$y_k(a)jc_k(b) < 0$. The solution $y(t) = y(t;y)$ will then be a continuous function of y for y e $(a,b]$. If, beginning from b, Y decreases continuously, we must reach a value $y_0 e$ (a,b) such that, for some k characterized by property (c), we have $y_k(a;Y_0)y_k(t>;Y_{rt}) = 0$. Indeed, $y_k(a, Y)y_k(Y, Y)$ varies continuously with Y and the absence of such a value Y_Q would imply that $Y_v(a' Y)y_v(Y, Y) < 0$ for all ye $(a,b]$ and all k with property (c). Thus $y(t,y)$ would be oscillatory for t e $[a,y]$ with Y arbitrarily close to a, and this is absurd since $y_v(a,Y)$ has a fixed nonzero value if k has property (c). Accordingly, for some k there exists a y_0 such that either $y_{Kl}(a;Y_0) = 0$ or $y_v(Y_{rk}7Y_{rk}) = 0$. If x had a total number m of components which vanish at either a or b, the total number of components of $y(t;Y_Q)$ which vanish at either a or Y_0 will be at least $m+1$. We can now repeat this procedure by letting y decrease beyond Y_0. It is clear that in this way we finally arrive at a point c € $(a,b]$ such that all components of $y(t;c)$ vanish at either a or c. This completes the proof of Theorem 3.2.

We are now finally in a position to prove the second part of Theorem 2.1. If the solution x of (1.1) is oscillatory on a, there exists an interval $[a^b]$ e a which is associated with a solution y of the type described in Theorem 3.1. If y has no components with the property (c), the assertion of Theorem 2.1 holds trivially. If there are such components y_k, we define a diagonal

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matrix Q with the following diagonal elements q_k: If $y_k(t_k) = 0$ $(t_k \in (a,b))$, we set $q_k = 1$ for $t \in [a,t_k)$ and $q_k = -1$ for $t \in [t_k,b]$. For components of y which satisfy properties (a) or (b), we set $q_k = 1$ for $t \in [a,b]$. The (continuous) vector $w = Qy$ will then be a solution of the equation $w' = QAQ^{-1}w$, and it is clear that w satisfies all the assumptions of Theorem 3.2. According to the latter theorem, there will thus exist a conjugate point to a in $(a,b]$ with respect to the equation $w' = QAQ^{-1}w$. This concludes the proof of the second part of Theorem 2.1.

4. In the present section we obtain nonoscillation criteria which depend on the matrix norm $\|A\|_p$ induced by the Hölder norm

$$\|x\|_p = \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} \qquad p > 1,$$

of the vector $x = (x_1,\ldots,x_n)$, i.e., $\|A\|_p = \max \|Ax\|_p$ for $\|x\|_p = 1$. The limiting cases $p = 1,\infty$ correspond to the "maximum-column norm"

$$\|A\|_1 = \max_s \sum_{r=1}^{n} |a_{rs}| \tag{4.1}$$

and the "maximum-row norm"

$$\|A\|_\infty = \max_r \sum_{s=1}^{n} |a_{rs}| \tag{4.2}$$

of the matrix $A = (a_{rs})$; $r,s = 1,\ldots,n$.

The only example of a nonoscillation criterion expressed in terms of a Hölder norm $\|A\|_p$ (other than the Euclidean norm $\|A\| = \|A\|_2$) is due to B. Schwarz [9,10] who showed that (1.1) is nonoscillatory on [a,b] if

$$\int_a^b \|A\|_\infty dt < 2 \log 2. \tag{4.3}$$

An example, to be exhibited later, will show that the constant $2 \log 2$ is the best possible; indeed, the conclusion does not necessarily follow if "<" is replaced by "$\leq$".

We shall prove the following result.

<u>Theorem 4.1. If either</u>

$$\int_a^b \|A\|_p dt < c_p \tag{4.4}$$

<u>or</u>

$$\int_a^b \|A^*\|_p dt < c_p, \tag{4.5}$$

<u>where</u>

$$c_p = \int_0^\infty (1+s^p)^{-\frac{1}{p}}(1+s^q)^{-\frac{1}{q}} ds, \quad 1 < p < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{4.6}$$

<u>and</u> c_1, c_∞ <u>are defined as</u> $\lim c_p$ <u>for</u> $p \to 1$ <u>or</u> $p \to \infty$, <u>then equation</u> (1.1) <u>is both nonoscillatory and suborthogonal on</u> [a,b].

For $p = 2$, both (4.4) and (because of $\|A^*\|_2 = \|A\|_2$) (4.5) reduce to the sharp condition (2.4). Since

$$c_{oo} = \int_0^{\infty} [\max(1,s)]^{-1} (1+s)^{-1} ds = 2 \log 2,$$

(4.4) yields the sharp condition (4.3) for $p = oo$. Because of $c_1 = c_{\infty}$ and $\|A^*\|_{\infty} = \|A\|_1$ (4#5) leads to the **nonoscillation condition**

$$\int_a \|A\|_x \, dt < 2 \log 2, \tag{4.7}$$

where $\|A\|_1$ is the maximum-column norm (4.1). (4.7) is likewise a sharp condition, as will be shown later. The constant (4.6) is thus the best possible for $p = 1,2,00$. There are indications that it is the best possible constant for all $pe [l,oo]$, but the construction of the necessary examples seems to be rather laborious.

It will be shown presently that $c_p \le \frac{\pi}{2}$, with equality only for $p = 2$. It is therefore of interest to note that the conclusion of Theorem 4.1 will also hold if the left-hand sides of both (4.4) and (4.5) are bounded by the larger constant $\frac{\pi}{2}$.

<u>Theorem 4.2. If A satisfies the two conditions</u>

$$\int_a^b \|A\|_p \, dt < \frac{\pi}{2} \tag{4.8}$$

<u>and</u>

$$\int_a^b \|A^*\|_p \, dt < \frac{\pi}{2}. \tag{4.8'}$$

<u>then equation (1.1) is both nonoscillatorv and suborthogonal on</u> [a,b].

Theorem 4.2 is an immediate consequence of the fact that

$$(4.9) \qquad (\|A\|_2)^2 \le \|A^*\|_p \|A\|_p.$$

To prove (4.9), it is only necessary to note that $(\|A\|_2)^2 = A$, where A is the highest eigenvalue of the positive-definite matrix A^*A. If x is the corresponding eigenvector, we have

$$\|A^*Ax\|_p = \|Ax\|_p = A\|x\|_p,$$

and thus

$$(\|A\|_2)^2 = A = \frac{\|A^*Ax\|_p}{\|x\|_p} \le \|A^*A\|_p \le \|A^*\|_p \|A\|_p.$$

If both (4.8) and (4.8^T) are satisfied, it follows from (4.9) that condition (4.4) holds for $p = 2$, and Theorem 4.2 is thus found to be a consequence of Theorem 4.1.

Before we begin the proof of Theorem 4.1 we show that $c_p < 2$ for $p \wedge 2$, as asserted above. If we set

$$0 = (1+s^p)^{-\frac{1}{p}} (1+s^q)^{-\frac{1}{q}},$$

$$p^{-1} = a, \quad q^{-1} = 3, \quad \varphi(x) = x \log(1+s^{\frac{1}{x}})_5$$

we have $\log 0 = \varphi(\alpha) + \varphi(p)$. A computation shows that $\varphi''(x) > 0$ if $s \wedge 1$, and it follows therefore that $-r \log 0 = -r[(p(\varphi(x) - \varphi(3)] > \varphi[\frac{1}{2}(\alpha+\beta)] = \varphi(\frac{i}{2}) = \frac{i}{2} \log(1+t^2)$. Hence, by (4.6)

$$c_p = \int_0^\infty (u)^{i-1} ds < \int_0^\infty (1+s^2)^{-1} ds = \frac{\pi}{2},$$

as asserted. The convexity of the function $\varphi(Y)$ also shows that

c_p increases from $2 \log 2$ to $\frac{\pi}{2}$ as p increases from 1 to 2, and then decreases to $2 \log 2$ as p increases from 2 to ∞.

We now assemble some of the facts needed in the proof of Theorem 4.1.

a) If Q is the diagonal matrix described in Theorem 2.1, then

$$\|QAQ^{-1}\|_p = \|A\|_p. \tag{4.10}$$

Indeed, we have $\|Qy\|_p = \|y\|_p$ for any vector y. Since $Q = Q^{-1}$, it follows that

$$\frac{\|QAQ^{-1}x\|_p}{\|x\|_p} = \frac{\|AQx\|_p}{\|Qx\|_p},$$

and this implies both $\|QAQ^{-1}\|_p \leq \|A\|_p$ and $\|A\|_p \leq \|QAQ^{-1}\|_p$.

b) In order to prove Theorem 4.1, it is sufficient to show that, under the stated hypotheses, equation (1.1) is disconjugate on $[a,b]$. This follows from Theorem 2.1 and item (a).

c) If the conclusion of Theorem 4.1 follows from condition (4.4), it also follows from condition (4.5). This is seen by combining the information in item (b) with the fact that (1.1) is disconjugate on $[a,b]$ if and only if the adjoint equation $y' = -A^*y$ is disconjugate on $[a,b]$. This fact, in turn, follows by observing that

$$(yx)' = yAx - xA^*y = yAx - yAx = 0,$$

i.e., yx is constant in an interval in which A is differentiable.

Since x and y are continuous on $[a,b]$, we have $yx = \text{const}$, throughout $[a,b]$.

If $c \in [a,b]$ is a conjugate point of a for equation (1.1), the latter has a nontrivial solution $x = (x_1, \ldots, x_n)$ such that $x_k(s_k) = 0$, $k = 1,\ldots,n$, where either $s_k = a$ or $s_k = c$. Since $x \neq 0$, there exists a k' such that $x_{k'}(c) \neq 0$. If it were true that c is not a conjugate point of a for the adjoint equation, the latter would have a unique solution $y = (y_1, \ldots, y_n)$ such that $y_k(a+c-s_k) = 0$ for $k \neq k'$ and $y_{k'}(c) = 1$. We would then have $x(a)y(a) = 0$ and $x(c)y(c) = x_{k'}(c) \neq 0$. Since, as just shown, xy is constant on $[a,b]$, this is absurd. Hence, c must be a conjugate point of a for the adjoint equation.

d) If x is a differentiable vector, then

$$\left|\frac{d}{dt}(\|x\|_p)\right| \leq \|x'\|_p. \tag{4.11}$$

This is a consequence of the triangle inequality. Indeed, for any vector norm $\|\ \|$ we have $\|x(t_2)\| = \|x(t_1)+x(t_2)-x(t_1)\| \leq \|x(t_1)\| + \|x(t_2)-x(t_1)\|$. Interchanging the roles of t_1 and t_2, we obtain

$$\left|\|x(t_2)\|-\|x(t_1)\|\right| \leq \left\|\frac{x(t_2)-x(t_1)}{t_2 - t_1}\right\|, \quad t_2 > t_1,$$

and (4.11) follows.

We now prove Theorem 4.1. According to item (b) on the preceding list it is sufficient to show that, under the stated hypotheses, the interval $[a,b]$ cannot contain a conjugate point $c = c(a)$ for

equation (1.1). Suppose, then, such a conjugate point exists, i.e., suppose (1.1) has a nontrivial solution x such that $m(1 \leq m < n)$ of its components vanish at the point a and $n - m$ components vanish at c. Separating these components, we write $x = u + v$, where $u(a) = v(c) = 0$ and the number of not identically vanishing components of u and v is m and $m - n$, respectively. Setting $\|u\|_p = R$, $\|v\|_p = S$, $\sigma = \frac{R}{S}$, we have $R(a) = S(c) = 0$, and

$$(4.12) \qquad |\sigma'| = |S^{-2}(SR' - RS')| \leq S^{-2}(R^q + S^q)^{\frac{1}{q}}(|R'|^p + |S'|^p)^{\frac{1}{p}}$$

at those points at which σ' exists. An application of (4.11) to u and v shows that

$$|R'|^p + |S'|^p \leq (\|u'\|_p)^p + (\|v'\|_p)^p = (\|x'\|_p)^p$$

and thus, by (1.1),

$$[|R'|^p + |S'|^p]^{\frac{1}{p}} \leq \|Ax\|_p \leq \|A\|_p \|x\|_p = \|A\|_p (R^p + S^p)^{\frac{1}{p}}.$$

Accordingly, (4.12) leads to the inequality

$$(4.13) \qquad |\sigma'| \leq (1+\sigma^q)^{\frac{1}{q}}(1+\sigma^p)^{\frac{1}{p}}\|A\|_p.$$

The function $\sigma(t)$ varies continuously from 0 to ∞ as t increases from a to c, and it thus follows from (4.13) that

$$c_p \leq \int_a^c (1+\sigma^q)^{-\frac{1}{q}}(1+\sigma^p)^{-\frac{1}{p}}|\sigma'|\,dt \leq \int_a^c \|A\|_p\,dt \leq \int_a^b \|A\|_p\,dt$$

where c_p is the constant (4.6). But this contradicts assumption (4.4), and the proof of Theorem 4.1 is complete.

We add here a remark concerning the cases $p = 1$ and $p = \infty$. If u and v are two vectors such that, for every k, either $u_k \equiv 0$ or $v_k \equiv 0$, then we have $\|u\|_1 + \|v\|_1 = \|u+v\|_1$ and $\|u\|_\infty + \|v\|_\infty = \|u+v\|_\infty$. It is of interest to note that the value $2 \log 2$ for both c_1 and c_∞ can be obtained by the sole use of this property and that, as a result, we have the following more general criterion.

<u>Theorem 4.3.</u> <u>Let</u> $|||y|||$ <u>be a vector norm with the property that</u> $|||u+v||| = |||u||| + |||v|||$ <u>for two vectors</u> u, v <u>such that, for every</u> k, <u>either</u> $u_k \equiv 0$ <u>or</u> $v_k \equiv 0$, <u>and let</u> $|||A|||$ <u>be the matrix norm induced by</u> $|||y|||$. <u>If</u>

$$\int_a^b |||A|||\, dt < 2 \log 2, \tag{4.14}$$

<u>then equation</u> (1.1) <u>is disconjugate on</u> $[a,b]$.

Examples of vector norms with this property, in addition to $\|y\|_1$ and $\|y\|_\infty$, are the norms $|||y||| = \alpha_1|y_1| + \ldots + \alpha_n|y_n|$ and $|||y||| = \max_k \alpha_k|y_k|$, where the α_k are given positive constants.

To prove Theorem 4.3 we set $R = |||u|||$, $S = |||v|||$, $\sigma = \frac{R}{S}$, where u and v have the same meaning as in the proof of Theorem 4.1. We then have

$$\begin{aligned} |\sigma'| &\leq S^{-2}(S|R'| + |S'|) \\ &\leq S^{-2}(S|||u'||| + R|||v'|||), \end{aligned}$$

where the last inequality follows from (4.11). Hence

$$|o« I \ £ \ S"^2 \max[S,R](||| u' ||| + |||v' |||).$$

By hypothesis $||| u^1 ||| + ||| v^1 ||| = ||| y' |||$ and, by (1.1), $\text{III } y« \text{ III } 1 \text{ III } A||| \text{ III } y||| = \text{III } A||| (R+S)$. Thus

$$|\sigma'| \leq (1+\sigma)\max[1,\sigma]||| A||| .$$

As before, the assumption that (1.1) is oscillatory implies that $a(a) = 0$, $a(c) = oo$, where $c€ (a,bj$. Since

$$\int_0^{P^{oo}} ((I4o)\max[l,a])^{-1} da = 2 \log 2,$$

we obtain

$$2 \log 2 \wedge \int_a ||| A||! dt £ \int_a HI A||| dt.$$

This contradiction to (4.4) proves Theorem 4.3.

We now exhibit an example which shows that the constant $2 \log 2$ appearing in conditions (4.3) and (4.7) (the special cases $p = ao$ and $p = 1$ of condition (4.4)) is the best possible. Using the abbreviation $Y = \frac{1}{2}r(a+b)$, we choose a function $r(t)$ which is increasing and differentiable in $[a,y)j$ and for which $r(a) = 0$, $\lim_{t\to Y} r(t) = \log 2$. In $[y,b]_3$ we define r by

$$r(Y+t) = -r(Y-t), \ Y*e(Y,b], \text{ and } r(y) = -\log 2.$$

We then define an nxn-matrix A as follows: All elements of A other than a_{11} a_{12}, a_{21}, a_{22} are identically zero. In $[a,y)j$ we set $a_{i:L} = a_{21} = 0$, $a_{12} = r^1$, $a_{22} = -r^1$; in $[y,1o]$, we take $a_{12} = a_{22} = 0$, $a_{22} = r^1$, $a_{22} = -r^!$. It is readily confirmed that the equation $x^1 = Ax$ is solved by the following continuous vector

function $x = (x_1,\ldots,x_n)$: $x_2 = x_3 = \ldots = x_n = 0$; $x_1 = 1-e^{-r}$ for $t \in [a,\gamma)$, $x_1 = e^r$ for $t \in [\gamma,b]$; $x_2 = e^{-r}$ for $t \in [a,\gamma)$, $x_2 = 1 - e^r$ for $t \in [\gamma,b]$. Since $x_1(a) = x_2(b) = 0$ and all other components of x vanish identically, the equation is oscillatory in $[a,b]$. The maximum-row norm is $\|A\|_\infty = r'$, and therefore

$$\int_a^b \|A\|_\infty \, dt = \int_a^b r' \, dt = 2 \log 2.$$

This shows that the constant appearing in (4.3) cannot be improved.

The same example also shows that inequality (4.7) is the best possible. The point b is a conjugate point of a for the equation $x' = Ax$. As shown above, it therefore is also a conjugate point of a with respect to the adjoint equation $y' = -A^*y$. Since $\|A\|_\infty = \|A^*\|_1$, the equation $y' = -A^*y$ has the required properties.

5. A conjugate point $b = b(a)$ of an equation (1.1) is associated with a solution vector $x = (x_1,\ldots,x_n)$ such that k $(1 \leq k \leq n-1)$ of its components vanish at a and the remaining $n - k$ components vanish at b. Without loss of generality we may assume that $x_1(a) = x_2(a) = \ldots = x_k(a) = x_{k+1}(b) = x_{k+2}(b) = \ldots = x_n(b) = 0$; this can always be achieved by re-numbering the components of x. Generalizing a similar concept which has proved to be useful in the study of the oscillatory behavior of linear n-th order equations

[3,5], we shall say that $b = b(a)$ is a $(k,n-k)$-conjugate point of (1.1). The absence, on an interval a, of a point which possesses a $(k,n-k)$-conjugate point on a will be referred to as $(k,n-k)$-disconjugacy of (1.1) on a. Evidently, (1.1) is disconjugate on a if and only if it is $(k,n-k)$-disconjugate on a for all $k = 1,\ldots,n-l$.

The main result of this section is the following sufficient condition for $(k,n-k)$-disconjugacy.

Theorem J5.JL. Let A^ A^ Ay A^ be defined b^ partitioning the matrix A according to the scheme

$$A = \begin{pmatrix} A_1 A_2 \\ A_3 A_4 \end{pmatrix}, \tag{5.1}$$

where $A_{\perp}$ is ja kxk-matrix, and set

$$HA_r ll_p = \langle P_r, \qquad r = 1,2,3,4; \quad 1 \text{ £ } p \text{ £ } co. \tag{5.2}$$

$$Tj(t) = \exp\{\int_a^t (\varphi_1+\varphi_4)\,dt\}. \tag{5.2*}$$

Denote by w the solution of the second-order differential equation

$$(\langle P \sim \tfrac{1}{2} Tjw^{!})' + \langle P_3 r)w = 0 \tag{5*3}$$

with the initial conditions $w(a) = 0$, $w^{!}(a) = 1$. JJP $w^{f} > 0$ in the interval $[a,b]$, then equation (1.1) JLS $(k_5 n-k)$-disconjugate on $[a,b]$.

Suppose there exists a point $a' \in [a,c)$ which has a $(k,n-k)$-conjugate point $b' \in (a,c]$. If the coefficient matrix A is partitioned in accordance with (5.1) and if u, v denote the vectors $u = (x_1, \ldots, x_k)$ and $v = (x_{k+1}, \ldots, x_n)^*$ respectively, (1.1) may be replaced by the system

$$(5.4) \qquad \begin{aligned} u' &= A_1 u + A_2 v \\ v' &= A_3 u + A_3 v, \end{aligned}$$

where $u(a') = v(b') = 0$. If we set $R = \|u\|_p$, $S = \|v\|_p$ and observe (4.11) and (5.2), we find that (5.4) leads to the system of inequalities $|R'| \le \varphi_1 R + \varphi_2 S$, $|S'| \le \varphi_3 R + \varphi_4 S$, where $R(a') = S(b') = 0$. Hence, the function $a = \frac{R}{S}$ is subject to the inequality

$$\sigma' \le S^{-2}[S|R'| + R|S'|] \le S^{-2}[S(\varphi_1 R + \varphi_2 S) + R(\varphi_3 R + \varphi_4 S)],$$

i.e.,

$$(5.5) \qquad a' < \varphi_3 a^2 + (\varphi_1 - \varphi_4)a + \varphi_2.$$

We note that $a(a') = 0$, and $a(t) \to +\infty$ for $t \to b'$.

If w is a solution of (5.3), then the function $r = \varphi_2 w(w')^{-1}$ is a solution of the Riccati equation

$$(5.6) \qquad \tau' = \varphi_3 \tau^2 + (\varphi_1 + \varphi_4)\tau + \varphi_2.$$

By hypothesis, (5.3) has a solution w such that $w(a) = 0$ and $w' > 0$ in $[a,b]$. Because of the Sturm separation theorem (or,

rather, a trivial modification of it) the solution w_1 of (5.3) defined by $w_1(a') = 0$, $w_1'(a') = 1$ will have a positive derivative in $[a',b]$. The function $\tau = \varphi_2 w_1 (w_1')^{-1}$ will thus be a solution of (5.6) which vanishes at a' and remains finite on $[a',b']$. Subtracting (5.6) from (5.5), we obtain

$$\frac{d}{dt}\left[(\sigma-\tau)\exp\{-\int_{a'}^{t}[(\sigma+\tau)\varphi_3 + \varphi_1 + \varphi_4]ds\}\right] \leq 0.$$

Since $\sigma(a') = \tau(a') = 0$, it follows that $\sigma \leq \tau$ throughout $[a',b']$. But this is absurd, since τ remains finite in this interval and $\sigma \to +\infty$ as $t \to b'$. The assumption that the interval $[a,b]$ contains a point and its $(k,n-k)$-conjugate has thus led to a contradiction. This completes the proof of Theorem 5.1.

We illustrate the use of Theorem 5.1 by two examples. First, we consider the scalar n-th order equation

$$y^{(n)} + r_{n-2}y^{(n-2)} + \ldots + r_1 y' + r_0 y = 0 \tag{5.7}$$

which, in the usual manner, we replace by a vector equation of the form (1.1), where x is the vector $(y,y',\ldots,y^{(n-1)})$, and A is the $n \times n$ matrix whose only non-zero components are $a_{m,m+1}=1 (m=1,\ldots,n-1)$, $a_{nm} = r_{m-1} (m=1,\ldots,n-1)$. If A is partitioned in the manner indicated by (5.1) and we set $k = n - 1$ (i.e., A_1 is an $(n-1)\times(n-1)$-matrix), it is easily seen that, for all $p \in [1,\infty]$, $\|A_1\|_p = \|A_2\|_p = 1$, $\|A_4\|_p = 0$ and $\|A_3\|_p = \|r\|_p$, where r is the vector $(r_0,r_1,\ldots,r_{n-1},0)$. The function (5.2')

is $\eta = e^t$, and an application of Theorem (5.1) yields the following result.

<u>Let</u> w <u>be the solution of the second-order equation</u>

$$w'' + w' + \|r\|_p w = 0 \tag{5.8}$$

<u>determined by</u> $w(a) = 0$, $w'(a) = 1$, <u>and let</u> y <u>be the solution of</u> (5.7) <u>satisfying the initial conditions</u> $y(a) = y'(a) = \ldots = y^{(n-2)}(a) = 0$, $y^{(n-1)}(a) = 1$. <u>If</u> $w' > 0$ <u>in</u> $[a,b]$, <u>then</u> $y^{(n-1)}(t) > 0$ (<u>and therefore, as shown by a repeated application of Rolle's theorem</u>, $y(t) > 0$) <u>in the same interval</u>.

Our second example deals with the second-order vector-matrix equation

$$(B^{-1}u')' + Cu = 0, \tag{5.9}$$

where B and C are continuous $n\times n$-matrices, and B is nonsingular, on $[a,b]$. We wish to obtain a condition which prevents the existence of a solution vector u of (5.9) such that $u(a') = u'(b') = 0$, where $a \leq a' < b' \leq b$. Writing (5.9) as a first-order system for a $(2n)$-dimensional vector (u,v), we have $u' = Bv$, $v' = -Cu$, where v is subject to the condition $v(b') = 0$. Partitioning the $2n \times 2n$ coefficient matrix of the system in accordance with (5.1) (with $k=n$), we obtain $A_1 = A_4 = 0$, $A_2 = B$, $A_3 = -C$. The existence of a solution of (5.9) with the indicated properties corresponds to the existence of an (n,n)-conjugate point for the first-order system. Accordingly, Theorem 5.1 leads to the following criterion (cf. [1,8]).

Let w *be the solution of*

$$[(\|B\|_p)^{-1}w'] + \|C\|_p w = 0,$$

defined by $w(a) = 0$, $w'(a) = 1$. *If* $w' > 0$ *in* $[a,b]$, *then the equation* (5.9) *cannot have a nontrivial solution vector* u *for which* $u(a') = u'(b') = 0$, $a \leq a' < b' < b$.

Theorem 5.1 will yield more accurate criteria if it is possible to obtain fundamental solutions of the equations

$$C' = A_1 C, \qquad D' = A_4 D, \tag{5.10}$$

where A_1 and A_4 are the square matrices appearing in (5.1). Since $(C^{-1})' = -C^{-1}C'C^{-1} = -C^{-1}A_1$, the first equation (5.4) can then be written in the form $(C^{-1}u)' = [(C^{-1})' + C^{-1}A_1]u + C^{-1}A_2 v = C^{-1}A_2 v$. Similarly, the second equation (5.4) transforms into $(D^{-1}v)' = D^{-1}A_3 u$. Accordingly, if we set

$$C^{-1}u = U, \qquad D^{-1}v = V,$$

the system (5.4) may be replaced by

$$U' = C^{-1}A_2 DV$$

$$V' = D^{-1}A_4 CU,$$

and Theorem 5.1 leads to the following result.

Theorem 5.2. Let A_k $(k=1,2,3,4)$ *have the same meaning as in Theorem* 5.1, *and let* C *and* D *denote fundamental solution matrices of the equations* (5.10). *Let* w *denote the solution of*

$$(5.11) \qquad (\varphi_2^{-1} w')' + \varphi_3 w = 0, \quad w(a) = 0, \quad w'(a) = 1,$$

<u>where</u>

$$(5.12) \qquad \varphi_2 = \|C^{-1} A_2 D\|_p, \quad \varphi_3 = \|D^{-1} A_4 C\|_p, \quad p \in [1,\infty].$$

<u>If</u> $w' > 0$ <u>for</u> $t \in [a,b]$, <u>then</u> <u>equation</u> (1.1) <u>is</u> $(k,n-k)$-<u>disconjugate</u> <u>on</u> $[a,b]$.

As an application of this result, we consider the equation (1.1) corresponding to the n-th order equation

$$(5.13) \qquad y^{(n)} - r(t)y = 0$$

in the manner described in the discussion of equation (5.7). The only non-zero elements $b_{\nu\mu}$ of the $k\times k$-matrix A_1 are $b_{\nu,\nu+1} = 1$ $(\nu=1,\ldots,k-1)$. Since A_1 is constant and $A_1^k = 0$, the solution C of the first equation (5.10) with the initial condition $C(a) = I$ is

$$C = \exp[A_1 t] = I + \sum_{\nu=1}^{k-1} \frac{A_1^\nu t^\nu}{\nu!},$$

and we have

$$C^{-1} = e^{-A_1 t} = I + \sum_{\nu=1}^{k-1} (-1)^\nu \frac{A_1^\nu t^\nu}{\nu!}.$$

Similar expressions (with k replaced by $n-k$) are obtained for D and D^{-1}. The matrices A_2 and A_3 have each only one nonvanishing element -- 1 and $p(t)$, respectively -- which appears at the bottom of the first column. Combining these facts, we find that the $(k\times n-k)$-matrix $C^{-1}A_2D$ has the non-zero elements

$$\delta_{\nu\mu} = (-1)^{k-\nu} \frac{t^{k+\mu-\nu-2}}{(k-\nu-1)!\,(\mu-1)!}, \quad \nu = 1,\ldots,k; \quad \mu = 1,\ldots,n-k.$$

Hence,

$$\|C^{-}\ {}_{2}\ {}_{\cdot\cdot\cdot\cdot} = \max \sum_{m=0}^{k-1} \frac{2m}{(ml)^2} \sum_{\ell=0}^{n-k-1} \left|\gamma_\ell \frac{t^\ell}{\ell!}\right|^2$$

for $\gamma_0^2 + \ldots + \gamma_{n-1}^2 = 1$. Accordingly, if $\langle p_2$ is the quantity defined in (5.12) (for p=2) we have, for $t \wedge 0$,

$$\varphi_2 \leq \sum_{m=0}^{k-1} \frac{4^{2m}}{(ml)^2} \sum_{\wedge O}^{n-k-1} \frac{.21}{(U)^2} \leq {}^{n}(1+t^2)^{-} \leq (1+t)^{o\ -A_4},$$

i.e., $\langle p_2 \leq (1+t)^{n\ 2}$. A similar computation shows that $\varphi_3 \leq |r(t)|(1+t)^{n-2}$.

The assertion of Theorem 5.2 remains valid if the quantities $\langle p_2$ and cp_3 are replaced, respectively, by upper bounds for these quantities (this follows either from the proof of Theorem 5.1, or else by applying the Sturm comparison theorem to equation (5.11)). By combining our estimates for $(p_2, \langle p_3$ with Theorem 5.2 we therefore obtain the following criterion.

Let w be the solution of the differential equation

$$[(1+t)^{2\sim n} W]^{!} + (1+t)^{n\,''\sim 2} |p(t)| w = 0$$

determined by the initial conditions $w(0) = 0$, $w^{!}(0) = 1$. JEf $w^{!}(t) > 0$ in $[0,b]$, then the n-th order equation (5.13) cannot have a. solution y for which $y(a) = y^1(a) = \ldots = y^{\nu}\cdot(a) = y^{(k+1)}(b)$ ss9 ^ ^ y(n-U (b) = 0, 0 ^. a < b, 0 ^ k ^ n-2.

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