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Research Report 71-11

February 1971

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2/15/71

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ABSTRACT. In analogy to the elementwise definition of von Neumann regular rings an R-module M is called regular if given any element $m \in M$ there exists $f \in \mathrm{Hom}_R(M,R)$ with $(mf)m = m$. Other equivalent definitions are possible; and the basic properties of regular modules are developed. The endomorphism ring E(M) of a regular module ${}_RM$ is examined. It is in general a semiprime ring with a regular center. An immediate consequence of this is the recently observed fact that the endomorphism ring of an ideal of a commutative regular ring is again a commutative regular ring. Certain distinguished subrings of E(M) are also studied. For example, the ideal of E(M) consisting of the endomorphisms with finite-dimensional range is a regular ring, and is simple when the socle of ${}_RM$ is homogeneous. Finally, the self-injectivity of E(M) is shown to depend on the quasi-injectivity of ${}_RM$.

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INTRODUCTION. The notion of (von Neumann) regularity has been extended to modules by D. Fieldhouse [5] and R. Ware [10]. The former author considered arbitrary modules over rings with identity elements while the latter author dealt with projective modules only. Their definitions agree for projectives, and it is in this context that module analogues of ring-theoretic theorems are produced; e.g. [5; Theorem 8.7] and [10; Proposition 2.1 and Theorem 2.12]. In §3 of [10] we additionally find a study of the endomorphism ring of a regular projective module.

In this paper we will identify a class of modules (which we will at the risk of confusion still call regular) somewhat more restricted than the class of modules introduced in [5], somewhat broader than the regular projectives, but for which structure theorems similar to those mentioned above are still available. Regular modules are introduced in §1, and their structure examined in §2. Afterwards, the endomorphism ring of a regular module is studied.

§1. Except as indicated, rings will not be assumed to have identity elements, all modules will be left modules, and module homomorphisms will be written on the right. When a ring has an identity element, all modules are assumed to be unitary.

We begin with some basic notation. Given an R-module M, we set $M^* = \mathrm{Hom}_R(M,R)$, $E(M) = \mathrm{Hom}_R(M,M)$. There are R-bilinear functions $(\, , \,) : M\times M^* \longrightarrow R$ and $[\, , \,] : M^*\times M \longrightarrow E(M)$ defined by $(m,f) = mf$ and $m[f,n] = (m,f)n$ for all $m,n\in M$ and $f\in M^*$. Note that then $[f,m]g = f(m,g)$ for all $m\in M$, $f,g\in M^*$; and that these functions induce R-R-bimodule homomorphisms $(\, , \,) : M \underset{E(M)}{\otimes} M^* \longrightarrow R$ and $[\, , \,] : M^* \underset{R}{\otimes} M \longrightarrow E(M)$. We let $T(M)$ denote the two-sided ideal of $E(M)$ generated by the image of $[\, , \,]$.

An R-module is called <u>projective</u> if it is a direct summand of a direct sum of copies of $R^{\#}$, where $R^{\#}$ denotes the ring obtained from R by adjoining an identity element in the customary manner when R fails to have an identity element, and $R^{\#} = R$ otherwise. The usual properties of projective modules then hold. If $T(M) = E(M)$, with say $\sum_{i=1}^{t} [f_i, m_i] = 1\in E(M)$, $f_i\in M^*$, $m_i\in M$, then M is projective and $M = \sum_{i=1}^{t} Rm_i$. Conversely, if $m_1,\ldots,m_t\in M$ and $N = \sum_{i=1}^{t} Rm_i$ is projective, then $T(N) = E(N)$. (The usual proof of the "dual basis lemma" establishes these facts. See [1; p. 132].

Of crucial importance to our discussion is a straightforward variant of a familiar construction. We include the proof for completeness.

LEMMA 1.1. <u>Let</u> M <u>be an</u> R-<u>module and</u> I <u>a left ideal of</u> $E(M)$. <u>Suppose that</u> $\mu = \mu^2\in I$, $\nu = \nu^2\in I$ <u>with</u> $M\mu \cap M\nu = 0$

and $\nu\mu = 0$. *Then there exists* $\lambda = \lambda^2 \in I$ *with* $M\mu \oplus M\nu = M\lambda$. *Consequently* $M\mu \oplus M\nu$ *is a direct summand of* M.

Proof. Set $\eta = (1-\mu)\nu$. Then $\eta \in I$, $\nu\eta = \nu(1-\mu)\nu = \nu^2 = \nu$, $\eta^2 = (1-\mu)\nu\eta = (1-\mu)\nu = \eta$, $\mu\eta = 0 = \eta\mu$. Now $M\eta = M(1-\mu)\nu \subseteq M\nu$ and $M\nu = M\nu\eta \subseteq M\eta$, so that $M\eta = M\nu$. Since μ and η are orthogonal idempotents $M(\mu+\eta) \subseteq M\mu + M\eta = (M\mu+M\eta)(\mu+\eta) \subseteq M(\mu+\eta)$, so that $M\mu + M\nu = M\mu + M\eta = M(\mu+\eta)$ with $\mu + \eta = (\mu+\eta)^2 \in I$. ||

PROPOSITION 1.2. *Let* M *be an* R-*module such that every cyclic submodule of* M *is a direct summand of* M. *Then given any direct summand* N *of* M *and an element* $m \in M$, *there exists* $m' \in M$ *such that* $N + R^{\#}m = N \oplus R^{\#}m'$ *and* $N + R^{\#}m$ *is again a direct summand of* M.

Proof. Let M, N, m be as above. Since N is a direct summand of M, there exists a surjection $\mu : M \longrightarrow N$ with $\mu^2 = \mu$. Now $R^{\#}m \subseteq R^{\#}m\mu + R^{\#}m(1-\mu) \subseteq N + R^{\#}m(1-\mu)$, and the latter sum is direct since $N \cap M(1-\mu) = 0$. So $N + R^{\#}m \subseteq N + R^{\#}m(1-\mu)$, and since $R^{\#}m(1-\mu) \subseteq R^{\#}m + R^{\#}m\mu \subseteq R^{\#}m + N$ we have $N + R^{\#}m = M\mu \oplus R^{\#}m(1-\mu)$, proving the first statement. Since $R^{\#}m(1-\mu)$ is cyclic, there exists by hypothesis a surjection $\nu : M \longrightarrow R^{\#}m(1-\mu)$ with $\nu^2 = \nu$. So $N + R^{\#}m = M\mu \oplus M\nu$. Now apply the previous lemma. ||

The reader should note that this Proposition remains valid with $R^{\#}$ replaced by R, provided that every submodule of the form Rm is a direct summand of R. An easy induction now establishes the following result.

COROLLARY 1.3. Let M be an R-module which has the property that every cyclic submodule is a direct summand. Then every countably (or finitely) generated submodule of M is a direct sum of cyclic modules and every finitely generated submodule of M is a direct summand of M.

We define an R-module M to be regular if and only if given any $m \in M$ there exists $f \in M^*$ with $m = (mf)m = m[f,m]$. This provides a natural extension of the customary element-wise description of a von Neumann regular ring, it being obvious that a regular ring R is regular as a left R-module. As an immediate consequence of the definition:

(1.4) A submodule of a regular module is regular.

In particular every left ideal of a regular ring is regular. This provides an ample source of regular modules which are not projective. For example, the ring of linear transformations of a countable dimensional vector space contains non-projective left ideals. The maximal regular ideal of a ring is clearly a regular module, as is any regular ideal. Some interesting examples of regular projective modules over non-regular rings appear in [10].

Observe that if m is an element of a regular module, then $m \in Rm$.

(1.5) _A cyclic regular R-module is projective and is isomorphic to a left ideal of R generated by an idempotent._

Proof. Let m be a generator of the cyclic regular module $M = Rm$. Then from the existence of $f \in M^*$ with $m[f,m] = m$ we have that $[f,m]$ is the identity homomorphism on M. Hence $T(M) = E(M)$ and M is projective. Furthermore, f is clearly a monomorphism and $Mf = R(mf)$ with $(mf)^2 = (mf)$. ||

THEOREM 1.6. _Let M be a regular module. Then every finitely generated submodule of M is a direct summand of M. Every countably (or finitely) generated submodule of M is a direct sum of cyclic regular modules._

Proof. In view of Corollary 1.3 we have only to prove that a cyclic submodule Rm of a regular module M is a direct summand. But $m = m[f,m]$ for some $f \in M^*$, which implies that $M = Rm \oplus \ker[f,m]$. ||

COROLLARY 1.7. _A countably (or finitely) generated regular module is projective._

We will complete this section with an examination of regular modules which satisfy a chain condition. The upshot of the matter is summarized below.

THEOREM 1.8. _Suppose that M is a regular R-module which contains no infinite direct sums of submodules. Then M is isomorphic to a finite direct sum of minimal left ideals generated by idempotents._

Proof. Because of Theorem 1.6, every submodule of such a module M must be finitely generated and hence a direct summand of M. It follows that M is a direct sum of simple modules [9; p. 61], and this sum must of necessity involve only a finite number of non-trivial components. Hence $M = Rm_1 \oplus \ldots \oplus Rm_s$, with each Rm_i simple. By (1.5) each Rm_i is isomorphic to a minimal left ideal of R generated by an idempotent. ||

COROLLARY 1.9. *If R is a commutative ring with identity element then a Noetherian or Artinian regular R-module is injective*.

Proof. For such a module satisfies the hypothesis of the previous Theorem, hence is a finite direct sum of simple projective modules. But over a commutative ring with identity, a simple module is flat if and only if it is injective [10; Lemma 2.6]; and a finite direct sum of injective modules is injective. ||

An ideal of a ring is called a *regular ideal* if it is regular as a subring. Each ring R has a unique largest regular ideal M(R) [7; p. 112]. We call a ring (or module) *finite-dimensional* if it contains no infinite direct sums of left ideals (or submodules).

COROLLARY 1.10. *If R is a finite-dimensional ring, then* $R = M(R) \oplus T$ *with* M(R) *semisimple Artinian and where* T *is an ideal of* R *which has no non-zero regular ideals*.

Proof. The proof in [7] goes through verbatim, once it is recognized that a finite-dimensional regular ring is semi-simple Artinian. This follows from Theorem 1.8. ||

We note from the above that when R is a finite-dimensional ring M(R) is in particular a projective R-module. The same is now seen to be true for every regular R-module.

THEOREM 1.11. *If R is either left perfect or is finite-dimensional then every regular R-module is projective.*

Proof. Recall that a left perfect ring is a ring for which direct limits of projective modules are projective [9; p. 170]. Since a module is a direct limit of its finitely generated submodules, the desired conclusion follows from Corollary 1.7.

Suppose now that R contains no infinite direct sums of left ideals, and let M be a regular R-module. Consider $\mathcal{S} = \{\sum_{\alpha\in A} \oplus Rm_\alpha \mid m_\alpha \in M\}$ partially ordered via $\sum_{\alpha\in A} \oplus Rm_\alpha \leq \sum_{\beta\in B} \oplus Rm_\beta$ if and only if $\{m_\alpha\}_{\alpha\in A} \subseteq \{m_\beta\}_{\beta\in B}$. By Zorn's Lemma there exists $N = \sum_{\alpha\in A} \oplus Rm_\alpha$ maximal in $\mathcal{S}$. Clearly then $N \cap Rm \neq 0$ for every $0 \neq m \in M$. We claim that $N = M$.

Let any $m \in M$ be given. Rm is isomorphic to a left ideal of R by (1.5), hence contains no infinite direct sums of submodules. By Theorem 1.8, we can write $Rm = Rn_1 \oplus \ldots \oplus Rn_t$ with each Rn_i simple. Now $Rn_i \cap N \neq 0$ for each i, so $Rn_i \cap N = Rn_i$. $Rm \cap N = (Rn_1 \oplus \ldots \oplus Rn_t) \cap N \supseteq \sum_{i=1}^{t} \oplus (Rn_i \cap N) = \sum_{i=1}^{t} \oplus Rn_i = Rm$.

It follows that $Rm \subseteq N$, and so $M = N$ proving that M is projective. ||

§2. We next develop several equivalent characterizations of regularity. Note the following consequence of Theorem 1.6.

(2.1) *Over a ring with identity element a regular module is flat.*

For a regular module is a direct limit of its finitely generated submodules, each of which is projective. ||

THEOREM 2.2. *For an R-module M, the following conditions are equivalent.*

(1) *M is regular.*

(2) *For every $m \in M$, Rm is projective and is a direct summand of M.*

(3) *For every $m_1, \ldots, m_t \in M$, $\sum_{i=1}^{t} Rm_i$ is projective and is a direct summand of M.*

Proof. It is clear from Corollary 1.3 and the remark preceding it that (2)$\Longleftrightarrow$(3). And (1.5) together with Theorem 1.6 gives (1)$\Longrightarrow$(2). So (2)$\Longrightarrow$(1) remains to be proved.

Assume that M satisfies (2) and let m be an arbitrary element of M. By hypothesis Rm is projective and $Rm \oplus N = M$ for some submodule N of M. Since Rm is projective there exist elements $f_i \in (Rm)^*$ and $m_i \in Rm$, $i = 1,2,\ldots,t$, such that $\sum_{i=1}^{t} [f_i, m_i] \in E(Rm)$ is the identity homomorphism on Rm. Now $m_i = r_i m$ for some $r_1, r_2, \ldots, r_t \in R$ and so $\sum_{i=1}^{t} [f_i, m_i] = \sum_{i=1}^{t} [f_i r_i, m] = [\sum_{i=1}^{t} f_i r_i, m]$.

Let g denote $\sum_{i=1}^{t} f_i r_i$ extended across N to M. Then $g \in M^*$ and $m\,[g,m] = m$, proving that M is regular. ||

THEOREM 2.3. *Let* R *be a ring with identity element and* M *an* R-*module such that every cyclic submodule of* M *is contained in a projective direct summand of* M. *Then any one of the following conditions is equivalent to* M *being regular*.

(4) *Every homomorphic image of* M *is flat*.

(5) $IM \cap N = IN$ *for every submodule* N *of* M *and every right ideal* I *of* R.

(6) *For any submodule* N *of* M *and any right* R-*module* L, *the natural homomorphism* $L \otimes_R N \longrightarrow L \otimes_R M$ *is a monomorphism*.

In the proof we will utilize a result which is due to Chase [2] for free modules and which was extended in [10] to projectives.

LEMMA 2.4. *Let* R *be a ring with identity element. Given a projective* R-*module* M *and a submodule* K *of* M, M/K *is flat if and only if for any* $x \in K$ *there exists a homomorphism* $\alpha_x : M \longrightarrow K$ *with* $x\alpha_x = x$.

Proof of the Theorem. First note that a module for which every cyclic submodule is contained in a projective submodule is a direct limit of projectives, and hence flat. Assume that M is a regular module and let K be any submodule of M. We must prove that M/K is flat, and for this it suffices to prove that every finitely generated submodule of M/K is flat. Such a submodule is of the form $N+K/K$ for some finitely generated submodule N of M. Since $N+K/K \cong N/N \cap K$, $N+K/K$ is a homomorphic image of N, a regular projective module by (3) of the preceding theorem and (1.4). Lemma 2.3 now implies that $N+K/K$ is flat, completing the proof.

Conversely, let M be as in (4) and take any $m \in M$. By hypothesis $Rm \subseteq P$ where P is a projective direct summand of M. Composing the natural homomorphisms $M \longrightarrow P \longrightarrow P/Rm$ we see that P/Rm is a homomorphic image of M and hence is flat. Applying the Lemma to the exact sequence $0 \longrightarrow Rm \longrightarrow P \longrightarrow P/Rm \longrightarrow 0$, we obtain a homomorphism $\alpha : P \longrightarrow Rm$ with $m\alpha = m$. Thus Rm is a direct summand of P; and it follows that Rm is projective and is a direct summand of M. By the previous theorem, M is regular.

In [9; p. 133] it is proved that for a flat module M and N a submodule of M, M/N is flat if and only if $IM \cap N = IN$ for every right ideal I. And P. M. Cohn [3] has shown that for a flat R-module M with submodule N, M/N is flat if and only if the natural homomorphism $L \otimes_R N \longrightarrow L \otimes_R M$ is a monomorphism for every right R-module L. This explains the equivalence of (4), (5) and (6). ||

In case M is itself a projective module, many of the hypotheses of the previous two theorems become redundant. The equivalence of conditions (2) through (5) for a projective module over a ring with identity forms Proposition 2.1 of [10]. Using Theorem 2.2 the following properties of a regular module are easily established. The proofs involve little modification from those given in [10] for regular projective modules, and will therefore be omitted here.

(2.5) _If M is a regular module, then the Jacobson radical of M equals zero._

(2.6) _If M is a faithful regular R-module, then R is Jacobson-semisimple. Consequently, for any regular R-module M, the annihilator of M in R is an intersection of maximal left ideals._

(2.7) _The singular submodule of a regular module equals zero._

THEOREM 2.8. $\sum_{\alpha \in A} \oplus M_\alpha$ _is regular if and only if each_ M_α _is a regular module._

Proof. Assume that each M_α is a regular module. It clearly suffices to show that $M_1 \oplus M_2$ is regular whenever M_1 and M_2 are. In fact, given $m_1 \in M_1$ and $m_2 \in M_2$ it suffices by Theorem 2.2 to prove $R(m_1+m_2)$ is then projective and a direct summand of $M_o = Rm_1 \oplus Rm_2$.

For $i = 1,2$, let $\pi_i : M_o \longrightarrow Rm_i$ denote the canonical projection homomorphism, and set $\pi = \pi_1 |_{R(m_1+m_2)}$. There is an exact sequence

$$0 \longrightarrow \ell(m_1)m_2 \longrightarrow R(m_1+m_2) \xrightarrow{\pi} Rm_1 \longrightarrow 0$$

where $\ell(m_1) = \{r \in R | rm_1 = 0\}$ and $\ell(m_1)m_2 = \ell(m_1)(m_1+m_2)$ is the kernel of π. Since Rm_1 is projective there exists a homomorphism $\mu : Rm_1 \longrightarrow R(m_1+m_2)$ with $\mu\pi = 1 \in E(Rm_1)$ and $R(m_1+m_2) = Rm_1\mu \oplus \ell(m_1)m_2$.

Next, there is an exact sequence

$$0 \longrightarrow \ell(m_1) \longrightarrow R \longrightarrow Rm_1 \longrightarrow 0,$$

the homomorphism $R \longrightarrow Rm_1$ being defined via $r \longmapsto rm_1$. And this sequence is clearly split by any $f \in (Rm_1)^*$ with $m_1 = m_1[f,m_1]$, so that $R = \ell(m_1) \oplus Rm_1f$. Since $m_1f = (m_1f)^2$ it follows that $\ell(m_1) = R(1-(m_1f))$, and hence that $\ell(m_1)m_2$ is a cyclic submodule of Rm_2. This already proves that $R(m_1+m_2) = Rm_1\mu \oplus \ell(m_1)m_2 \cong Rm_1 \oplus \ell(m_1)m_2$ is projective.

Finally, $\ell(m_1)m_2$ is a direct summand of Rm_2, so there exists a surjection $\nu : Rm_2 \longrightarrow \ell(m_1)m_2$ with $\nu^2 = \nu$. Then $R(m_1+m_2) = Rm_1\mu \oplus Rm_2\nu = M_o(\pi_1\mu) \oplus M_o(\pi_2\nu)$. Since $(\pi_1\mu)^2 = \pi_1\mu$, $(\pi_2\nu)^2 = \pi_2\nu$ and $(\pi_2\nu)(\pi_1\mu) = 0$, $R(m_1+m_2)$ is by Lemma 1.1 a direct summand of M_o. ||

We remark that as a consequence of this theorem every submodule of a free module over a regular ring is a regular module.

§3. The endomorphism ring of a regular module need not be regular. Indeed, Cukerman [4] and Ware [10] have noted that the endomorphism ring of an infinitely generated free module over a regular ring R is regular if and only if R is Artinian. On the other hand, Wiegand [11] has observed that the endomorphism ring of an ideal of a commutative regular ring is again a commutative regular ring. This is not true for non-commutative rings, as is seen in the next example.

EXAMPLE 3.1. <u>A left ideal of a regular ring whose endomorphism ring is not regular</u>.

Let R be a ring of column-finite countable matrices over a field. Let $\{e_{ij} \mid 1 \leq i,j < \infty\}$ denote the usual matrix units of R. Set $J = \prod_{j \text{ odd}} Re_{1j} \oplus \sum_{j \text{ even}} \oplus Re_{1j}$, a left ideal of R. Define $\alpha : J \longrightarrow J$ by setting $(\prod_{j \text{ odd}} Re_{1j})\alpha = 0$ and $e_{1j}\alpha = e_{1(j-1)} \in J$ for j even. Then α extends linearly to an R-endomorphism of J and $J\alpha = \sum_{j \text{ odd}} \oplus Re_{1j}$. Were E(J) regular, $J\alpha$ would be a direct summand of J and hence of $\prod_{j \text{ odd}} Re_{1j}$. But this is easily seen not to be the case: for since $R = \prod_{j \text{ odd}} Re_{1j} \oplus \prod_{j \text{ even}} Re_{1j}$, $J\alpha = \sum_{j \text{ odd}} \oplus Re_{1j}$ would then be a principal left ideal, in clear violation of the fact that it is an infinite direct sum.

What then can one say affirmatively about the endomorphism ring of a regular module or ideal. One minor observation:

(3.2) The endomorphism ring of a regular module is a semiprime ring.

For if $0 \neq \alpha \in E(M)$ with $_{R}M$ regular, choose $m \in M$ so that $m\alpha \neq 0$. Since M is regular there exists $f \in M^*$ with $m\alpha = m\alpha[f,m\alpha] = m(\alpha[f,m]\alpha)$. It follows that $\alpha[f,m]\alpha \neq 0$, and hence that $E(M)$ is semiprime. ||

Before we can turn to our next result we need an observation which is essentially contained in [6].

LEMMA 3.3. *Let $\alpha \in$ center of $E(M)$. Then there exists an element $\beta \in$ center of $E(M)$ with $\alpha\beta\alpha = \alpha$ if and only if $M = M\alpha \oplus \ker \alpha$.*

Proof. Suppose that such a β exists. Set $\pi = \beta\alpha = \alpha\beta$. Then $\pi^2 = \pi$; $M\alpha = M\alpha\pi \subseteq M\pi$ and $M\pi = M\beta\alpha \subseteq M\alpha$ so that $M\alpha = M\pi$; and $\ker \alpha = \ker \pi$ since $\pi = \alpha\beta$ and $\alpha = \pi\alpha$. Clearly $M = M\pi \oplus M(1-\pi) = M\pi \oplus \ker \pi$, and so $M = M\alpha \oplus \ker \alpha$.

Conversely, suppose that $\alpha \in$ center of $E(M)$ and $M = M\alpha \oplus \ker \alpha$. Given $m \in M$ write $m = n\alpha + k$ with $n \in M$ and $k \in \ker \alpha$; and in turn write $n = n_1\alpha + k_1$ with $n_1 \in M$ and $k_1 \in \ker \alpha$. Then $m\alpha = ((n_1\alpha + k_1)\alpha + k)\alpha = (n_1\alpha)\alpha^2$. Set $x_m = n_1\alpha \in M\alpha$. Observe that x_m is the unique element of $M\alpha$ such that $m\alpha = x_m\alpha^2$; for if $y \in M\alpha$ with $m\alpha = y\alpha^2$, then $(y-x_m)\alpha^2 = 0$ so that $(y-x_m)\alpha \in M\alpha \cap \ker \alpha = 0$, and then also $y - x_m \in M\alpha \cap \ker \alpha = 0$ so that $y = x_m$. It is easy to check that $x_{rm} = rx_m$, $x_{m+n} = x_m + x_n$, $x_{m\gamma} = x_m\gamma$ for any $m,n \in M$, $r \in R$, $\gamma \in E(M)$. Consequently there is a homomorphism $\beta \in E(M)$ defined by $m\beta = x_m$. For any $m \in M$, $m\alpha\beta\alpha = (m\alpha\beta)\alpha = x_{m\alpha}\alpha = x_m\alpha^2 = m\alpha$, so that $\alpha\beta\alpha = \alpha$. It remains only to show that β is in the center of $E(M)$. But this is easy, for given any $\gamma \in E(M)$ and any $m \in M$, $m\gamma\beta = x_{m\gamma} = x_m\gamma = m\beta\gamma$. ||

THEOREM 3.4. _Suppose that_ $_RM$ _is a regular module. Then the center of_ $E(M)$ _is a regular ring._

Proof. Let $\alpha \in$ center of $E(M)$. Given any $m \in M$ choose $f \in M^*$ with $m\alpha = m\alpha[f,m\alpha] = m[f,m]\alpha^2$. Then $m = m[f,m]\alpha + (m-m[f,m]\alpha)$ with $m[f,m]\alpha \in M\alpha$ and $m-m[f,m]\alpha \in \ker \alpha$. So $M = M\alpha + \ker \alpha$. If $m\alpha \in M\alpha \cap \ker \alpha$, then from the above equation it follows that $m\alpha = m[f,m]\alpha^2 = (m,f)(m\alpha^2) = 0$. Hence $M\alpha \cap \ker \alpha = 0$ and $M = M\alpha \oplus \ker \alpha$. Now apply the previous lemma. ||

COROLLARY 3.5. _Suppose that_ J _is a regular left ideal of_ R. _If_ J _is commutative then_ $E(J)$ _is a commutative regular ring_.

Proof. First recall that a regular (left) ideal is a (left) ideal which is itself a regular ring. (For example, any ideal of a regular ring is a regular ideal.) A regular left ideal J of R is in particular a regular left R-module. To complete the proof it suffices to show that $E(J)$ is commutative when J is. Let $\alpha,\beta\in E(J)$ and let x be any element of J. Choose $y\in J$ with $xyx = x$. Then $x\alpha\beta = (xyx)\alpha\beta = (xy(x\alpha))\beta = (y(x\alpha)x)\beta = y(x\alpha)(x\beta) = y(x\beta)(x\alpha) = (y(x\beta)x)\alpha = (xy(x\beta))\alpha = (xyx)\beta\alpha = x\beta\alpha$ proving that $\alpha\beta = \beta\alpha$. ||

N. Funayama has recently observed that Lemma 3.3 can easily be used to prove that the ring of R-R-bimodule endomorphisms of a regular ideal of R is a commutative regular ring.

We have not been able to determine precisely which regular modules have commutative (and hence necessarily regular) endomorphism rings. The next result gives some information about this situation.

THEOREM 3.6. _Suppose that_ ${}_RM$ _is a regular module with_ $E(M)$ _commutative. If either_ R _is left self-injective or_ ${}_RM$ _is projective, then_ M _is isomorphic to a left ideal of_ R.

Proof. If ${}_RM$ is a module with $E(M)$ commutative, then $N\alpha \subseteq N$ for every endomorphic image N of M and every $\alpha\in E(M)$. (Let $N = M\beta$ with $\beta\in E(M)$; then $N\alpha = M\beta\alpha = M\alpha\beta \subseteq M\beta = N$.) Consequently, if ${}_RM$ is additionally a regular module, then $N\alpha \subseteq N$ for every submodule N of M and every $\alpha\in E(M)$. (For given any $n\in N$, Rn is a direct summand of M, so a fortiori is an endomorphic image of M. Hence $n\alpha\in Rn\alpha \subseteq Rn \subseteq N$.)

Now let $N = \sum_{i\in I} \oplus Rm_i$ be any submodule of M which is a direct sum of cyclic submodules. For each $i\in I$ write $Rm_i = M\alpha_i$ where $\alpha_i \in E(M)$, and set $N_i = \sum_{j\in I, j\neq i} Rm_j$. By

By Lemma 3.3, $M = M\alpha_i \oplus \ker \alpha_i = Rm_i \oplus \ker \alpha_i$ for each $i\in I$. By the first paragraph, $N_i\alpha_i \subseteq N_i \cap Rm_i = 0$, so that $N_i \subseteq \ker \alpha_i$ for each $i\in I$. Also $\mathrm{Hom}_R(Rm_i, N_i) = 0$ since any $\gamma\in \mathrm{Hom}_R(Rm_i, N_i)$ can be extended across $\ker \alpha_i$ via $(\ker \alpha_i)\gamma = 0$ to an element $\gamma\in E(M)$; and then $(Rm_i)\gamma \subseteq Rm_i \cap N_i = 0$ so that $\gamma = 0$. In particular, $\mathrm{Hom}_R(Rm_i, Rm_j) = 0$ whenever $i,j\in I$ with $i \neq j$.

For each $i\in I$, there exists by (1.5) an R-isomorphism $f_i : Rm_i \longrightarrow Re_i$ where $m_i f_i = e_i = e_i^2 \in R$ and $(m_i f_i)m_i = m_i$. We can extend f_i to an R-homomorphism $g_i \in M^*$ by defining $g_i = 0$ on $\ker \alpha_i$. $g = \sum_{i\in I} (g_i|_N)$ is then a well-defined R-homomorphism from N to R, and $Ng = \sum_{i\in I} Rm_i f_i = \sum_{i\in I} Re_i$. We claim that g is a monomorphism, and to see this it suffices to prove that $\sum_{i\in I} Re_i$ is a direct sum.

Set $L = \sum_{i\in I} Re_i$, $L_i = \sum_{j\in I, j\neq i} Re_j$, and suppose that $a\in Re_i \cap L_i$. Since $Re_i \cong Rm_i$ is regular, Ra is a direct summand of Re_i. Hence if $a \neq 0$, then $0 \neq \mathrm{Hom}_R(Re_i, Ra) \subseteq \mathrm{Hom}_R(Re_i, L_i)$. Since $L_i = \sum_{j\in I, j\neq i} \oplus Re_j$, $\mathrm{Hom}_R(Re_i, L_i) \neq 0$ implies that there exists $j\in I$, $j\neq i$ with $\mathrm{Hom}_R(Re_i, Re_j) \neq 0$. But $\mathrm{Hom}_R(Re_i, Re_j) \cong \mathrm{Hom}_R(Rm_i, Rm_j) = 0$. Hence $a = 0$ and $\sum_{i\in I} Re_i$ is a direct sum, establishing the fact that g is a monomorphism of N into R.

If ${}_RM$ is projective then we can choose $N = M$ giving the desired conclusion. For a projective module is a direct sum of countably generated projectives (by [8; Theorem 1] applied to a regular projective module), each of which is by Corollary 1.3 a direct sum of cyclic submodules. In general, we may use a Zorn's Lemma argument as in the proof of Theorem 1.11 to choose N with the property that $N \cap Rm \neq 0$ for every nonzero element $m \in M$. If now ${}_RR$ is injective we can extend g to an R-homomorphism $g' : M \longrightarrow R$. Since $\ker g' \cap N = \ker g = 0$, it follows that $\ker g' = 0$ and M is therefore isomorphic to a left ideal of R. ||

One final comment on a related result is in order. It follows directly from Corollary 4.3 of [12] together with Theorem 1.8 of this paper that over a prime ring R a regular module with a commutative endomorphism ring is isomorphic to a minimal left ideal of R.

(3.7) Let R be a commutative ring and P a projective R-module such that $E(P)$ is a regular ring. Then P is a regular module.

For rings with identity elements this is Theorem 3.9 of [10]. We note that with but minor modification the proof given there is valid for rings without identity elements. It can be extended as follows.

THEOREM 3.8. *Let* R *be a commutative ring, and let* M *be an* R-*module with the property that every cyclic submodule of* M *is contained in a projective direct summand of* M. *If* $E(M)$ *is a regular ring then* M *is a regular module*.

Proof. Let $m \in M$ be given. By hypothesis $R^{\#}m \subseteq P$ where P is a projective direct summand of M. Letting π denote the canonical projection homomorphism of M onto P, $E(P) \cong \pi E(M)\pi$, so that $E(P)$ is a regular ring. By (3.7), P is regular. Hence there exists $g \in P^{*}$ with $(mg)m = m$. Extending g across a complementary summand of P in M to a homomorphism $f \in M^{*}$, we have $(mf)m = m$, proving that M is regular. ||

§4. We have already seen that the center of $E(M)$ is a regular ring when M is a regular R-module. In this section it is our intention to investigate other distinguished subsets of $E(M)$. Three come immediately to mind; namely, $T(M)$, $F(M) = \{\alpha \in E(M) \mid M\alpha \text{ is finitely generated}\}$, and $G(M) = \{\alpha \in E(M) \mid M\alpha \text{ is finite-dimensional}\}$. $T(M)$ is an ideal of $E(M)$, $F(M)$ is a multiplicative subsemigroup of $E(M)$, and $G(M)$ will shortly be seen to be an ideal of $E(M)$.

THEOREM 4.1. *If* M *is a regular* R-*module then given* $\alpha \in F(M)$ *there exists* $\beta \in F(M)$ *with* $\alpha = \alpha\beta\alpha$.

Proof. Let $\alpha \in F(M)$, M a regular module. One can write $M\alpha = Rm_1\alpha \oplus \ldots \oplus Rm_t\alpha$, $M = M\alpha \oplus N$ for some elements $m_1, \ldots, m_t \in M$ and N a submodule of M. Since each $Rm_i\alpha$ is regular, there exist homomorphisms $g_i \in (Rm_i\alpha)^*$ for $i = 1, \ldots, t$ such that $(m_i\alpha g_i)m_i\alpha = m_i\alpha$. Set $M_i = Rm_1\alpha \oplus \ldots \oplus Rm_{i-1}\alpha \oplus Rm_{i+1}\alpha \oplus \ldots \oplus Rm_t\alpha \oplus N$; $Rm_i\alpha \oplus M_i = M$ for each $i = 1, \ldots, t$. Let $f_i \in M^*$ denote g_i extended across M_i to M via $(M_i)f_i = 0$; and set $\beta = \sum_{i=1}^{t} [f_i, m_i] \in T(M)$.

Then $\alpha\beta\alpha = \alpha$; for, given any $m \in M$, write $m\alpha = \sum_{i=1}^{t} r_i m_i \alpha$

with $r_1, \ldots, r_t \in R$, and then $m\alpha\beta\alpha = (\sum_{i=1}^{t} r_i m_i \alpha)(\sum_{j=1}^{t} [f_j, m_j\alpha]) =$

$\sum_{i,j=1}^{t} r_i(m_i\alpha, f_j)m_j\alpha = \sum_{i=1}^{t} r_i(m_i\alpha g_i)m_i\alpha = \sum_{i=1}^{t} r_i m_i \alpha = m\alpha$. Finally

observe that $\beta \in F(M)$ since $M\beta = (Rm_1\alpha \oplus \ldots \oplus Rm_t\alpha \oplus N)\beta =$
$\sum_{i=1}^{t} R(m_i\alpha g_i)m_i$ is finitely generated. ||

COROLLARY 4.2. [10; Theorem 3.6] _If_ M _is a finitely generated regular module, then_ $E(M)$ _is a regular ring_.

THEOREM 4.3. $G(M)$ _is a regular ideal of_ $E(M)$ _and_ $G(M) \subseteq F(M) \subseteq T(M)$.

Proof. For any $\alpha \in G(M)$, $M\alpha$ is by Theorem 1.8 a finite direct sum of simple modules. Hence $G(M) \subseteq F(M)$. For any $\alpha, \beta \in G(M)$ and $\gamma \in E(M)$, $M(\alpha+\beta) \subseteq M\alpha + M\beta$, $M\gamma\alpha \subseteq M\alpha$, and $M\alpha\gamma = (M\alpha)\gamma$. Since finite sums, submodules and homomorphic images of a finite direct sum of simple modules are again of the same type, it follows that $\alpha + \beta$, $\gamma\alpha$, $\alpha\gamma \in G(M)$ proving that $G(M)$ is an ideal of $E(M)$.

To see that $G(M)$ is regular, simply use Theorem 4.1 to choose for a given $\alpha \in G(M)$ an element $\beta \in F(M)$ with $\alpha = \alpha\beta\alpha$. Then $\alpha = \alpha(\beta\alpha\beta)\alpha$ with $\beta\alpha\beta \in G(M)$ since $G(M)$ is an ideal of $E(M)$. It remains to prove that $F(M) \subseteq T(M)$. Let $\alpha \in F(M)$ and choose $\beta \in T(M)$ as in the proof of Theorem 4.1 with $\alpha = \alpha\beta\alpha$. Since $T(M)$ is an ideal of $E(M)$, $\alpha = \alpha\beta\alpha \in T(M)$. ||

COROLLARY 4.4. _If_ R _is finite-dimensional and_ M _is a regular_ R-_module then_ $G(M) = F(M) = T(M)$.

Proof. First note that every finitely generated submodule N of M is finite-dimensional. For by Theorem 1.6, N is isomorphic to a finite direct sum of left ideals of R, each of which is finite-dimensional. In particular, $F(M) \subseteq G(M)$.

Next let $\alpha = \sum_{i=1}^{t} [f_i, m_i] \in T(M)$, $f_i \in M^*$, $m_i \in M$. Then $M\alpha \subseteq \sum_{i=1}^{t} (Mf_i)m_i \subseteq \sum_{i=1}^{t} Rm_i$. Since $\sum_{i=1}^{t} Rm_i$ is finite-dimensional, so is $M\alpha$. Hence $T(M) \subseteq G(M)$. Now apply the preceding theorem. ||

In analogy to the situation for the ring of linear transformations of a vector space, we will show that $G(M)$ is a simple ring when the simple submodules of M are all of the same isomorphism type (such a module is said to have a <u>homogeneous socle</u>). The proof for vector spaces is easily adapted, the key step being the following lemma, whose proof is sketched below for the sake of completeness.

LEMMA 4.5. *Assume that M is an R-module with the property that every finite direct sum of simple submodules of M is a direct summand of M. Suppose that $\mu = \mu^2 \in G = G(M)$ and that N is a submodule of M with both $M\mu$ and N finite direct sums of simple modules of the same isomorphism type. Then there exists $\nu = \nu^2 \in G\mu G$ with $M\nu = N$.*

Proof. Write $M\mu = M_1 \oplus \ldots \oplus M_t$ with each M_i simple. First assume N is simple. Choose an isomorphism $\alpha : N \longrightarrow M_1$, and extend α across a complementary summand of N in M, so that $\alpha \in G(M)$. Define $\beta : M\mu \longrightarrow N$ via $\beta|_{M_1} = \overline{\alpha}^{1}$, $\beta|_{M_i} = 0$ $(i = 2,\ldots,t)$, and extend β across a complementary summand of $M\mu$ in M, so that $\beta \in G(M)$. Set $\nu = \alpha\mu\beta \in G\mu G$. Then $M\nu = N$ and $\nu|_N = 1$ so that $\nu^2 = \nu$.

Suppose now $N = N_1 \oplus \ldots \oplus N_t$ with each N_i simple. By induction, there exists $\nu' = (\nu')^2 \in G\mu G$ with $M\nu' = N_1 \oplus \ldots \oplus N_{t-1} = N'$. Write $N_t = Rn_t$ and say $n_t\nu' = \sum_{i=1}^{t-1} n_i$, $n_i \in N_i$. Then clearly $n = n_t - \sum_{i=1}^{t-1} n_i \in \ker \nu'$ and $N = N' \oplus Rn$, so that Rn is simple. By the first paragraph, there exists $\nu'' = (\nu'')^2 \in G\mu G$ with $M\nu'' = Rn$. Set $\nu = \nu' + \nu'' - \nu'\nu'' \in G\mu G$. For any $n' \in N'$, $n'\nu = n' + n'\nu'' - n'\nu'' = n'$, while $n\nu = n\nu'' = n$, so that

$\nu|_N = 1$. Hence $M\nu \supseteq N$. On the other hand $M\nu \subseteq M\nu' + M\nu'' \subseteq N$, so that $M\nu = N$ and $\nu^2 = \nu$. ||

THEOREM 4.6. *If M is a regular R-module with homogeneous socle then $G(M)$ is a simple ring.*

Proof. Let $0 \neq \alpha \in G = G(M)$; we must show that $G\alpha G = G$. Since $G(M)$ is regular we can choose $\beta \in G(M)$ with $\alpha = \alpha\beta\alpha$. Then $(\beta\alpha)^2 = \beta\alpha \in G(M)$. Given any $\gamma \in G(M)$ we apply the lemma to choose $\nu = \nu^2 \in G(\beta\alpha)G$ with $M\nu = M\gamma$. Then $\gamma = \gamma\nu \in \gamma G(\beta\alpha)G \subseteq G\alpha G$, proving that $G\alpha G = G$. ||

For M an arbitrary regular module, the situation is not much more complicated. $G(M)$ is then the direct sum of ideals $G_i(M)$, one for each homogeneous component L_i of the socle of M. Each $G_i(M)$ is simple, and in fact $G_i(M) = G(M) \cap \mathrm{Hom}_R(M,L_i)$. The details are elementary.

COROLLARY 4.7. *If R is a simple Artinian ring and M is any R-module then $G(M)$ is a simple regular ring*.

We conclude this section with a description of the regular modules with semisimple endomorphism rings.

THEOREM 4.8. *Let M be a regular module. Then $E(M)$ is semisimple with minimum condition if and only if ${}_RM$ is finite-dimensional*.

Proof. If M contains no infinite direct sums, then by Theorem 1.8 M is a direct sum of finitely many simple modules and so has a semisimple endomorphism ring. Conversely, suppose that $E(M)$ is semisimple with M a regular module, and that $\sum_{i\in I} \oplus M_i \subseteq M$ for some collection of nonzero submodules M_i of M. It is easy to see that $\sum_{i\in I} [M^*, M_i]$ then forms a direct sum of nonzero left ideals of $E(M)$. So I must be a finite index set. ||

§5. In this brief concluding section we determine exactly when the endomorphism ring of a regular module is left self-injective. The next result is of some independent interest.

THEOREM 5.1. *Assume that M is an R-module with the property that for every cyclic submodule L of M there exists a surjection of M onto L. If $E(M)$ is left self-injective, then ${}_RM$ is quasi-injective.*

Proof. Let N be an R-submodule of M and let $\alpha \in \mathrm{Hom}_R(N,M)$. We must prove that α can be extended to an element of $E = E(M)$. Set $I = \mathrm{Hom}_R(M,N)$, a left ideal of E. Define $\theta : I \longrightarrow E$ via $(\gamma)\theta = \gamma \circ \alpha$, $\gamma \in I$. Clearly $\theta \in \mathrm{Hom}_E(I,E)$. Since ${}_EE$ is injective and contains an identity element, there exists a homomorphism $\alpha' \in E$ with $(\gamma)\theta = \gamma \circ \alpha'$ for all $\gamma \in I$. The proof is concluded by demonstrating that $\alpha'|_N = \alpha$.

For any $n \in N$ there exists by hypothesis a surjection $\beta \in \mathrm{Hom}_R(M, R^{\#}n)$. Choose $m \in M$ with $m\beta = n$. Note that $\beta \in I$. Then $n\alpha' = (m\beta)\alpha' = m(\beta \cdot \alpha') = m(\beta \circ \alpha) = (m\beta)\alpha = n\alpha$, so that $\alpha'|_N = \alpha$. ||

THEOREM 5.2. _For_ M _a_ _regular_ R-_module_, E(M) _is_ _left_ _self_-_injective_ _if_ _and_ _only_ _if_ $_RM$ _is_ _quasi_-_injective_; _and_ _when_ _this_ _is_ _the_ _case_ E(M) _is_ _regular_.

Proof. The "only if" part is a consequence of the previous result, it being evident that a regular module satisfies the hypothesis of Theorem 5.1. While it is possible to give an independent proof of the remainder of this theorem, it is actually a special case of a more general result. It is known that the endomorphism ring E of an injective or quasi-injective module $_RM$ is regular and left self-injective when J(E), the Jacobson radical of E equals zero. Furthermore, $J(E) = \{\alpha \in E \mid \ker \alpha \cap N \neq 0$ for every R-submodule $N \neq 0$ of $M\}$. (See for example [9; pp. 102-104].) When M is a regular R-module it is a simple matter using (2.7) to verify that $J(E) = 0$. ||

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CARNEGIE-MELLON UNIVERSITY
PITTSBURGH, PENNSYLVANIA 15213