TWO PAPERS IN

CATEGORICAL TOPOLOGY

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1. On quotient maps in general topology.
2. Convenient categories for topology.

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ON QUOTIENT MAPS IN GENERAL TOPOLOGY

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It is a well-known and sometimes troublesome fact that the class of Hausdorff spaces is not closed under the formation of quotient spaces. We claim that this is the result of a simple (and quite common) misunderstanding: the quotient maps of one category, topological spaces, are used in a quite different category, namely Hausdorff spaces. If quotient maps and spaces are properly defined, then the category of Hausdorff spaces has all the quotient spaces which one needs.

The basic properties of a quotient map or proclusion $f : X \longrightarrow Y$ are the following. f is surjective, and if Z is a space and $u : Y \longrightarrow Z$ a mapping such that $u f : X \longrightarrow Z$ is continuous, then $u : Y \longrightarrow Z$ is continuous. Our definition of a proclusion translates these properties into categorical language.

We define proclusions and quotient spaces, and dually inclusions and subspaces, in section 1, and we obtain some basic properties of proclusions and inclusions in section 2. Section 3 is concerned with subspaces and quotient spaces for epireflective and monocoreflective subcategories, and with topological applications.

This note is mostly pure categorical algebra. This is appropriate since quotient maps exist not only in topology, but also e.g. in universal algebra. Categorical duality is an added benefit of the abstract treatment, and a very welcome one.

1. Proclusions and quotient objects

1.1. Throughout this note, we consider two categories $\mathcal{B}$ and $\mathcal{T}$, and a functor $P : \mathcal{T} \longrightarrow \mathcal{B}$. In topological applications, $\mathcal{B}$ may be the category of sets or a category of algebras, $\mathcal{T}$ a category of topological spaces or of topological algebras, and P the functor which "forgets topologies". We do not assume that P is faithful.

1.2. We say that a map $f : A \longrightarrow B$ in $\mathcal{T}$ is a P-<u>quotient map</u> or a P-<u>proclusion</u>, <u>over</u> an epimorphism e of $\mathcal{B}$, if the following two conditions are satisfied.

(i) $P\,f = u\,e$ for an epimorphism u of $\mathcal{B}$.

(ii) If $g : A \longrightarrow C$ in $\mathcal{T}$ and $P\,g = v\,e$ for a map v in $\mathcal{T}$, then $g = h\,f$ in $\mathcal{T}$ for a unique map $h : B \longrightarrow C$ of $\mathcal{T}$, with $v = (P\,h)\,u$.

Dually, $f : A \longrightarrow B$ in $\mathcal{T}$ is called a P-<u>inclusion</u>, <u>over</u> a monomorphism m of $\mathcal{B}$, if the following two conditions are satisfied.

(i*) $P\,f = m\,u$ for a monomorphism u of $\mathcal{B}$.

(ii*) If $g : C \longrightarrow B$ in $\mathcal{T}$ and $P\,g = m\,v$ for a map v in $\mathcal{B}$, then $g = f\,h$ in $\mathcal{T}$ for a unique map $h : C \longrightarrow A$ of $\mathcal{T}$, with $v = u\,(P\,h)$.

We say that B is a P-<u>quotient object</u> of A, <u>over</u> e, if a P-proclusion $f : A \longrightarrow B$ over e is given, and we define P-<u>subobjects</u> dually.

1.3. Every P-proclusion is epimorphic in $\mathcal{T}$; the converse is usually false. If f is a P-proclusion over an epimorphism e of $\mathcal{B}$, then f is also a P-proclusion over $P\,f$. If e is not explicitely given, then $e = P\,f$ is understood. Dually, every P-inclusion f is a P-inclusion over $P\,f$, and

$m = P f$ is understood if m is not explicitely given.

P-proclusions are P-opfibred morphisms f in the sense of [1; 7.1] for which $P f$ is epimorphic, and P-inclusions are P-fibred morphisms f for which $P f$ is monomorphic. The name proclusion for quotient map has been proposed in [4] to stress the duality between proclusions and inclusions.

If f in $\mathcal{T}$ is a P-proclusion over an epimorphism e of $\mathcal{B}$, then f is also a P-proclusion over $b\,e$ for every isomorphism b of $\mathcal{B}$ such that $b\,e$ is defined, and f may well be a P-proclusion over other epimorphisms of $\mathcal{B}$. If $f : A \longrightarrow B$ in $\mathcal{T}$ is a P-proclusion over e in $\mathcal{B}$, then a map $f' : A \longrightarrow B'$ of $\mathcal{T}$ is a P-proclusion over e if and only if $f' = t\,f$ for an isomorphism $t : B \longrightarrow B'$ of $\mathcal{T}$.

1.4. If $\mathcal{T}$ is an equational category of algebras, and P the forgetful functor from $\mathcal{T}$ to sets, then every monomorphism of $\mathcal{T}$ is a P-inclusion, and f in $\mathcal{T}$ is a P-proclusion if and only if $P f$ is surjective. The example of rings shows that $\mathcal{T}$ may have epimorphisms which are not P-proclusions.

If $\mathcal{T}$ is the category of topological spaces or the category of Hausdorff spaces, and P the forgetful functor from $\mathcal{T}$ to sets, then a map $f : A \longrightarrow B$ of $\mathcal{T}$ is a P-inclusion if and only if $P f$ is injective and f induces a homeomorphism from A to a subspace of B. A P-proclusion is a quotient map in the usual sense for topological spaces; a P-proclusion for Hausdorff spaces need not be a quotient map in the usual sense.

2. Properties of proclusions

2.1. We consider a diagram in $\mathcal{T}$ as a map $A : D \longrightarrow \mathcal{T}$ from a diagram scheme D to $\mathcal{T}$. A assigns an object A_α of $\mathcal{T}$ to every vertex α of D, and a map $a_\lambda : A_\alpha \longrightarrow A_\beta$ to every arrow $\lambda : \alpha \longrightarrow \beta$ of D. If $B : D \longrightarrow \mathcal{T}$ is a diagram with the same scheme, then a map $f : A \longrightarrow B$ assigns to every vertex α of D a map $f_\alpha : A_\alpha \longrightarrow B_\beta$ of $\mathcal{T}$ such that $f_\beta a_\lambda = b_\lambda f_\alpha$ for every arrow $\lambda : \alpha \longrightarrow \beta$ of D. If A and B have colimits $A_* = \varinjlim A_\alpha$ and $B_* = \varinjlim B_\alpha$, with maps $a_\alpha : A_\alpha \longrightarrow A_*$ and $b_\alpha : B_\alpha \longrightarrow B_*$, then a map $f : A \longrightarrow B$ has a colimit $f_* = \varinjlim f_\alpha : A_* \longrightarrow B_*$, determined uniquely by the condition that $f_* a_\alpha = b_\alpha f_\alpha$ for every vertex α of D.

With these notations, we have the following result.

Proposition. *If every map* f_α *is a* P-*proclusion, and if* $P f_*$ *is epimorphic in* $\mathcal{B}$, *then* $f_* = \varinjlim f_\alpha$ *is a* P-*proclusion*.

Proof. f_* is epimorphic in $\mathcal{T}$ since f_* is a colimit of epimorphisms. If $g : A_* \longrightarrow C$ in $\mathcal{T}$ with $P g = u\,(P f_*)$ for some u in $\mathcal{B}$, then

$$P\,(g\,a_\alpha) \;=\; u\,P\,(f_*\,a_\alpha) \;=\; u\,(P\,b_\alpha)(P\,f_\alpha)$$

in $\mathcal{B}$. Thus $g\,a_\alpha = h_\alpha f_\alpha$ for a unique map $h_\alpha : B_\alpha \longrightarrow C$ of $\mathcal{T}$, for every vertex α of D. If $\lambda : \alpha \longrightarrow \beta$ is an arrow of D, then

$$h_\alpha f_\alpha \;=\; g\,a_\alpha \;=\; g\,a_\beta\,a_\lambda \;=\; h_\beta f_\beta\,a_\lambda \;=\; h_\beta\,b_\lambda\,f_\alpha$$

in $\mathcal{T}$, and $h_\alpha = h_\beta b_\lambda$ follows. Thus there is a map $h_* : B_* \longrightarrow C$ of $\mathcal{T}$ such that $h_\alpha = h_* b_\alpha$ for every vertex α of D. We have

$$h_*\,f_*\,a_\alpha \;=\; h_*\,b_\alpha\,f_\alpha \;=\; h_\alpha\,f_\alpha \;=\; g\,a_\alpha$$

and $g = h_* f_*$ follows ▌

2.2. We consider a pair of functors $U : \mathcal{S} \longrightarrow \mathcal{T}$ and $F : \mathcal{T} \longrightarrow \mathcal{S}$ such that F is left adjoint to U, with unit (or front adjunction) $\eta : \mathrm{Id} \longrightarrow U F$, and counit (or back adjunction) $\varepsilon : F U \longrightarrow \mathrm{Id}\, \mathcal{S}$. We recall that

$$g = \varepsilon_B (F f) \iff f = (U g) \eta_A$$

for $f : A \longrightarrow U B$ in $\mathcal{T}$ and $g : F A \longrightarrow B$ in $\mathcal{S}$, and that this defines a natural bijection

$$\varphi_{AB} : \mathcal{S}(F A, B) \longrightarrow \mathcal{T}(A, B U) ,$$

for all objects A of $\mathcal{T}$ and B of $\mathcal{S}$.

We put $Q = P U : \mathcal{S} \longrightarrow \mathcal{B}$.

<u>Proposition</u>. <u>If</u> $f : A \longrightarrow B$ <u>in</u> $\mathcal{T}$ <u>is a</u> P-<u>proclusion, and if</u> $Q F f$ <u>is epimorphic in</u> $\mathcal{B}$, <u>then</u> $F f$ <u>is a</u> Q-<u>proclusion</u>.

<u>Proof</u>. Suppose that $Q g = u (Q F f)$ for $g : F A \longrightarrow C$ in $\mathcal{S}$ and some u in $\mathcal{B}$. If $\bar{g} = \varphi_{AC}(g) = (U g) \eta_A$, then

$$P \bar{g} = u (Q F f)(P \eta_A) = u (P \eta_B)(P f) ,$$

by naturality of $P \eta : P \longrightarrow Q F$. Thus $\bar{g} = \bar{h} f$ for a unique $\bar{h} : B \longrightarrow U C$ in $\mathcal{T}$. If $\bar{h} = \varphi_{BC}(h)$, then $\bar{g} = \bar{h} f$ if and only if $g = h (F f)$, by naturality of φ_{AC} in A. Thus $g = h (F f)$ for a unique h in $\mathcal{S}$ ▌

2.3. If U is full and faithful in the situation of 2.2, then $U(\mathcal{S})$ is a full reflective subcategory of $\mathcal{T}$, equivalent to $\mathcal{S}$ by U, with reflections $\eta_A : A \longrightarrow U F A$, and $\varepsilon : F U \longrightarrow \mathrm{Id}\, \mathcal{S}$ is a natural equivalence. Thus U has a left inverse left adjoint, i.e. one for which $F U = \mathrm{Id}\, \mathcal{S}$, and $\varepsilon_A = \mathrm{id}\, A$

for every object A of $\mathcal{S}$. $\eta_{UA} = \mathrm{id}\ U\ A$ follows since $(U\varepsilon)(\eta U) = \mathrm{id}\ U$. We consider P-inclusions $f : A \longrightarrow B\ U$ in this situation.

We recall a definition. An epimorphism e of a category $\mathcal{C}$ is called <u>extremal</u> if $e = v\ u$ in $\mathcal{C}$, for epimorphic u and monomorphic v in $\mathcal{C}$, always implies that v is isomorphic in $\mathcal{C}$. Surjective mappings, and surjective homomorphisms of algebras, always are extremal epimorphisms.

<u>Proposition</u>. <u>If</u> $f : A \longrightarrow U\ B$ <u>is a</u> P-<u>inclusion and</u> $f = (U\ \bar{f})\ \eta_A$ <u>for</u> $\bar{f} : F\ A \longrightarrow B$ <u>in</u> $\mathcal{S}$, <u>and if</u> $P\ \eta_A$ <u>is an extremal epimorphism of</u> $\mathcal{B}$, <u>then</u> η_A <u>is an isomorphism of</u> $\mathcal{T}$, <u>and</u> $\bar{f}$ <u>is a</u> Q-<u>inclusion</u>.

<u>Proof</u>. Since $P\ f = (Q\ \bar{f})(P\ \eta_A)$ is monomorphic, $P\ \eta_A$ is monomorphic as well as an extremal epimorphism. It follows that $P\ \eta_A$ is isomorphic in $\mathcal{B}$. Now $Q\ \bar{f} = (P\ f)(P\ \eta_A)^{-1}$ in $\mathcal{B}$, and thus $U\ \bar{f} = f\ x$ for some x in $\mathcal{T}$. But then $f\ x\ \eta_A = f$, and $x\ \eta_A = \mathrm{id}\ A$ follows since f is monomorphic. As U is full, $\eta_A\ v = U\ y$ for some $y : F\ A \longrightarrow F\ A$ in $\mathcal{S}$, and

$$(U\ y)\ \eta_A = \eta_A = (U\ \mathrm{id}\ F\ A)\ \eta_A$$

follows. But then $y = \mathrm{id}\ F\ A$. Thus $\eta_A\ v = \mathrm{id}\ U\ F\ A$, and η_A is isomorphic in $\mathcal{T}$.

Now $Q\ \bar{f} = (P\ f)(P\ \eta_A)^{-1}$ is monomorphic in $\mathcal{B}$ since $P\ f$ is, and $\bar{f}$ is monomorphic in $\mathcal{S}$ since $U\ \bar{f} = f\ (\eta_A)^{-1}$ is monomorphic in $\mathcal{T}$ and the faithful functor U reflects monomorphisms. If $Q\ g = (Q\ \bar{f})\ v$ for $g : C \longrightarrow B$ in $\mathcal{S}$ and some v in $\mathcal{B}$, then $P\ U\ g = (P\ f)(P\eta_A)^{-1}\ v$ in $\mathcal{B}$, and it follows that $U\ g = f\ h_1$ for some $h_1 : U\ C \longrightarrow A$ in $\mathcal{T}$. If $\eta_A\ h_1 = U\ h$, with $h : C \longrightarrow F\ A$ in $\mathcal{S}$, then $U\ g = U\ (\bar{f}\ h)$, and $g = \bar{f}\ h$ follows ▮

3. Applications

3.1. In this section, $\mathcal{S}$ denotes a full subcategory of $\mathcal{T}$ and U : $\mathcal{S} \longrightarrow \mathcal{T}$ the embedding functor. We say that $\mathcal{S}$ is P-epireflective in $\mathcal{T}$ if $\mathcal{S}$ is reflective, i.e. U has a left adjoint $F : \mathcal{T} \longrightarrow \mathcal{S}$, and $P\eta_A$ is an extremal epimorphism of $\mathcal{B}$ for every reflection $\eta_A : A \longrightarrow U\,F\,A$ of an object A of $\mathcal{T}$ into $\mathcal{S}$. Dually, we call $\mathcal{S}$ P-monocoreflective if U has a right adjoint G, and $P\varepsilon_A$ is an extremal monomorphism for every coreflection ε_A : $U\,G\,A \longrightarrow A$ into $\mathcal{S}$. $\mathcal{S}$ is P-monocoreflective in $\mathcal{T}$ if and only if the dual category $\mathcal{S}^{op}$ of $\mathcal{S}$ is P^{op}-epireflective in $\mathcal{T}^{op}$.

If P is faithful, then every P-epireflective subcategory is epireflective, and every P-monocoreflective subcategory is monocoreflective. The converse is not always true: compact Hausdorff spaces define an epireflective subcategory of Hausdorff spaces, but not a P-epireflective subcategory for the forgetful functor P from Hausdorff spaces to sets.

3.2. Let $\mathcal{S}$ be a P-epireflective full subcategory of $\mathcal{T}$, and let F be the left inverse left adjoint of U (see 2.3), with $\varepsilon_A = \mathrm{id}\,A = \eta_{UA}$ for every object $A = U\,A$ of $\mathcal{S}$. We replace prefixes P and Q, for the functor $Q = P\,U$, by affixes "in $\mathcal{T}$" and "in $\mathcal{S}$".

Proposition. Let $f : A \longrightarrow B$ in $\mathcal{T}$. If f is an inclusion in $\mathcal{T}$ over a monomorphism m of $\mathcal{B}$ and B an object of $\mathcal{S}$, then η_A is an isomorphism of $\mathcal{T}$, and $f\,(\eta_A)^{-1} : F\,A \longrightarrow B$ is an inclusion over m in $\mathcal{S}$. If f is a proclusion in $\mathcal{T}$ over an epimorphism e of $\mathcal{B}$ and A an object of $\mathcal{S}$, then $\eta_B\,f : A \longrightarrow F\,B$ is a proclusion in $\mathcal{S}$ over e.

Proof. In the first part, η_A is isomorphic by 2.3. Thus $f\,(\eta_A)^{-1}$ is an inclusion over m in $\mathcal{T}$, and all the more in $\mathcal{S}$.

If $P\,f = u\,e$ in the second part, then $Q\,(\eta_B\,f) = u'\,e$ for $u' = (P\,\eta_B)\,u$ which is epimorphic in $\mathcal{B}$. If $g : A \longrightarrow C$ in $\mathcal{S}$ satisfies $Q\,g = v\,e$ for a map v in $\mathcal{B}$, then $g = h_1\,f$ in $\mathcal{T}$ for a unique map $h_1 : B \longrightarrow C$ in $\mathcal{T}$. η_B is a reflection for $\mathcal{S}$, and thus $h_1 = h\,\eta_B$ for a unique map $h : F\,B \longrightarrow C$ in $\mathcal{S}$. It follows that $g = h\,\eta_B\,f$ for a unique map h in $\mathcal{S}$ ▮

3.3. Let now $\mathcal{B}$ = ENS , the category of sets, and let P be a forgetful functor. We say that $\mathcal{T}$ has all possible proclusions if for every object A of $\mathcal{T}$ and every surjective mapping e with domain $P\,A$, there is in $\mathcal{T}$ a proclusion $f : A \longrightarrow B$ over e. Having all possible inclusions is defined dually.

For TOP , the category of topological spaces, the following is well known. For every space A and every surjective mapping e with domain $P\,A$, there is a unique proclusion $f : A \longrightarrow B$ with $P\,f = e$. For every space B and every injective mapping m with codomain $P\,B$, there is in TOP a unique inclusion $f : A \longrightarrow B$ with $P\,f = m$. Thus TOP has all possible proclusions and all possible inclusions.

Since a map f in TOP is epimorphic if and only if $P\,f$ is surjective, a subcategory $\mathcal{T}$ of TOP is epireflective if and only if $\mathcal{T}$ is P-epireflective. Thus every epireflective subcategory $\mathcal{T}$ of TOP has all possible proclusions and inclusions, by 3.2. Inclusions in $\mathcal{T}$ are inclusions in TOP , but proclusions in $\mathcal{T}$ are usually not proclusions in TOP .

Let now $\mathcal{T}$ be a full subcategory of TOP and $\mathcal{S}$ a full coreflective subcategory of $\mathcal{T}$. If $\mathcal{S}$ has a non-empty space among its objects, then one sees easily that $P\,\varepsilon_A$ is bijective for every coreflection $\varepsilon_A : G\,A \longrightarrow A$ for $\mathcal{S}$. Thus the dual of 3.2 applies, and $\mathcal{S}$ has all possible proclusions and inclusions if $\mathcal{T}$ has all possible proclusions and inclusions. Proclusions in $\mathcal{S}$ are proclusions in $\mathcal{T}$; inclusions in $\mathcal{S}$ need not be inclusions in $\mathcal{T}$.

Epireflective full subcategories of TOP, and their coreflective full subcategories, have been studied extensively; see e.g. [2] and [3].

3.4. All statements made in 3.3 about TOP and its subcategories remain valid for any top category over ENS, in the sense of [5] and [6], provided that a singleton has only one structure. This condition is usually satisfied for categories of sets with topological structures of some kind.

The categories considered in 3.3 and the preceding paragraph have the following properties in common. (a) A map f of $\mathcal{T}$ is monomorphic in $\mathcal{T}$ if and only if $P\,f$ is injective. (b) All possible inclusions and proclusions exist in $\mathcal{T}$. It follows easily from (a) and (b) that the proclusions in $\mathcal{T}$ are precisely the extremal epimorphisms of $\mathcal{T}$ (see 2.3). The corresponding statement for inclusions may well be false. For example, if A is a subspace of a Hausdorff space B, then the inclusion map $j : A \longrightarrow B$ is always an inclusion as defined in this note, but j is an extremal monomorphism of Hausdorff spaces only if A is a closed subspace of B. One cannot decide once for all whether all subspaces or only closed subspaces are the "right" subobjects for a Hausdorff space; this depends very much on the context in which one works.

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CONVENIENT CATEGORIES FOR TOPOLOGY

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In [21], N. E. Steenrod gave a list of criteria which a convenient category of topological spaces should satisfy. One of these criteria is somewhat vague: the category should be large enough to have among its objects "all of the particular spaces arising in practice." The other two criteria can be made precise. They are that the category should be large enough to be closed under standard operations, and that it should be small enough so that these operations satisfy certain reasonable identities. Using results of R. Brown [3], [4], Steenrod showed in [21] that k-spaces satisfy these criteria.

We study in this paper a class of categories of topological spaces which we call compactly generated, and we obtain criteria for such a category to be convenient, in Steenrod's sense. k-spaces form the largest compactly generated category; they satisfy our criteria. Sequential spaces, introduced by G. Birkhoff [2] and studied intensively by S. P. Franklin [9], [10] and others, form a compactly generated category, and a convenient one. Sequential spaces are of course also convenient in a more immediate way: they are the spaces for which convergence of sequences does all the jobs for which convergence of filters (or of nets) is usually needed. Some of our results have been obtained for ~~Sequence~~ Sequential spaces by J. A. Guthrie [13], [14]. Other convenient compactly generated categories are listed in section 2.

In categorical terms, convenience means that the category should be cartesian closed. Closed categories, i.e. categories with an internal hom functor and a tensor product functor, were introduced by S. Eilenberg and G. M. Kelly [8] and by F. E. J. Linton [20]. Long lists of examples have been given in [8] and [20], and by M. Bunge in [5]. Closed categories have been studied intensively and many useful results have been obtained for them. Thus it is helpful to know that a given category is closed. A closed category is called cartesian if its tensor product is the ordinary (or cartesian) product. The closed categories of interest in topology are the cartesian ones, and categories of pointed spaces with a smash product as tensor product. These are the categories with function space topologies which are, in the language of J. Dugundji [7; XII.10], both splitting (or proper) and conjoining (or admissible).

The methods of this paper are mostly categorical; we use very little "hard" topology. We begin with some general properties of coreflective subcategories of epireflective subcategories of the category of topological spaces. These categories were studied first by A. M. Gleason [12], and then by J. F. Kennison [19], S. P. Franklin [11], H. Herrlich and G. E. Strecker [15], [17], [18], and others. See [16] for a survey of this theory and further references. We present only the facts which we shall need. These facts are well known, but not available in print in the form in which we need them. In section 2, we discuss topologies for function spaces. Our main result is that a compactly generated category admits function space topologies which are proper and admissible if certain simple conditions are satisfied. Using results of [24], we show in section 3 that a compactly generated category which satisfies these conditions is convenient.

All results of section 1 and section 3 remain valid in the general setting of top categories [22], [23]. This is a trivial generalization; we leave it to the interested reader. It is not without interest: C. H. Cook and H. R. Fischer [6], and E. Binz and H. H. Keller [1], have shown that the category of limit spaces is convenient. As this is a very large category, it would be useful to obtain further results on convenient top categories.

1. Coreflective subcategories

1.1. We denote by TOP the category of topological spaces, and by $\mathcal{T}$ an epireflective full subcategory of TOP. We assume for convenience that every topological space which is homeomorphic to an object of $\mathcal{T}$ is itself an object of $\mathcal{T}$. Among the possibilities for $\mathcal{T}$ are: TOP itself, T_1-spaces, Hausdorff spaces, T_3-spaces (without T_1) and regular spaces (with T_1), and completely regular spaces (with or without T_1). The terms space and map will usually refer to objects and morphisms of $\mathcal{T}$; exceptions will be noted. We denote by ENS the category of sets, with mappings as morphisms.

1.2. In most of this paper, $\mathcal{A}$ will be a fixed class of spaces which contains at least one non-empty space. If X is any space, then $\mathcal{A}/X$ will denote the class of all maps $u : A \longrightarrow X$ with $A \in \mathcal{A}$.

For a space X in $\mathcal{T}$, we denote by αX the space in $\mathcal{T}$ consisting of the underlying set $|X|$ of X with the finest topology for which all maps u in $\mathcal{A}/X$ remain continuous. We note that αX is the limit in TOP of the diagram consisting of all maps $\mathrm{id}\,|X| : X_i \longrightarrow X$, for objects X_i of $\mathcal{T}$ with underlying set $|X|$, and with a topology finer than that of X, but coarse enough so that all maps u in $\mathcal{A}/X$ remain continuous for X_i. By a standard categorical argument, this limit is an object of $\mathcal{T}$.

We call a space X an $\mathcal{A}$-space if $\alpha X = X$. All spaces αX , and all spaces A in $\mathcal{A}$, are $\mathcal{A}$-spaces. The following two results show that $\mathcal{A}$-spaces are the objects of a full coreflective subcategory of $\mathcal{T}$. We note without proof that this is the smallest coreflective subcategory of $\mathcal{T}$ for which all spaces in $\mathcal{A}$, and all spaces homeomorphic to an object of the subcategory, are objects.

1.3. Proposition. *If X is an $\mathcal{A}$-space and Y a space, then a mapping $f : X \longrightarrow Y$ is continuous if and only if $f\,u : A \longrightarrow Y$ is continuous for every map $u : A \longrightarrow X$ in $\mathcal{A}/X$.*

Proof. $f\,u$ is continuous for all u in $\mathcal{A}/X$ if and only if the coarsest topology of $|X|$ for which f is continuous is coarser than the finest topology for which all u in $\mathcal{A}/X$ remain continuous, i.e. if and only if $f : \alpha X \longrightarrow Y$ is continuous ▮

1.4. Theorem. *For every space X , the map $\mathrm{id}\,|X| : \alpha X \longrightarrow X$ is a coreflection for the category of $\mathcal{A}$-spaces.*

Proof. We must show that $f : Y \longrightarrow \alpha X$ remains continuous if $f : Y \longrightarrow X$ is continuous and Y an $\mathcal{A}$-space. If $u : A \longrightarrow Y$ is in $\mathcal{A}/Y$, then $f\,u$ is in $\mathcal{A}/X$, and thus $f\,u : A \longrightarrow \alpha X$ is continuous. But then $f : Y \longrightarrow \alpha X$ is continuous by 1.3 ▮

1.5. The category of $\mathcal{A}$-spaces is generated by $\mathcal{A}$, as already noted in 1.2. There may of course be many classes of spaces which generate the same coreflective full subcategory of $\mathcal{T}$. We say that the category of $\mathcal{A}$-spaces is compactly generated if it can be generated by a class of compact Hausdorff spaces.

We obtain the largest compactly generated coreflective subcategory of $\mathcal{T}$ by letting $\mathcal{A}$ be the class of all compact Hausdorff spaces in $\mathcal{T}$. We call this category the category of k-<u>spaces</u> in $\mathcal{T}$. If $\mathcal{T}$ is the category of Hausdorff spaces, then this is the category of k-spaces in the usual sense.

The smallest compactly generated coreflective subcategory of $\mathcal{T}$ is the category of $\mathcal{T}$-discrete spaces, generated by singletons. αX for this category is $|X|$ with the finest topology of an object of $\mathcal{T}$.

If $\mathcal{A}$ consists only of $N_\omega = N \cup \{\omega\}$, the Alexandroff one-point compactification of the discrete space N of natural numbers, then $\mathcal{A}$-spaces are sequential spaces in $\mathcal{T}$. A continuous map $u : N_\omega \longrightarrow X$ is basically a convergent sequence (u_n) in X with limit u_ω. A convergent sequence in X which has more than one limit defines more than one map $u : N_\omega \longrightarrow X$. In this example, the topology of αX is the finest topology of an object of $\mathcal{T}$ with underlying set $|X|$ for which all convergences $u_n \longrightarrow u_\omega$ of sequences in X remain valid. A mapping $f : X \longrightarrow Y$ from a sequential space X in $\mathcal{T}$ to a space Y is continuous if and only if f preserves the convergence of sequences. If $\mathcal{T}$ is TOP, or the category of T_i-spaces, for $i = 0, 1, 2$, then sequential spaces in $\mathcal{T}$ are sequential spaces in the usual sense. We do not know whether this remains true for other categories $\mathcal{T}$.

1.6. We extend the operator α to maps by putting $\alpha f = f : \alpha X \longrightarrow \alpha Y$ for a map $f : X \longrightarrow Y$. It follows immediately from 1.4 that this is well defined. Thus we have a functor α from $\mathcal{T}$ to $\mathcal{A}$-spaces. By its construction, this functor is a coreflector for $\mathcal{A}$-spaces, i.e. α is right adjoint to the embedding functor from $\mathcal{A}$-spaces to $\mathcal{T}$. This has important consequences.

It is well known that every diagram A in TOP, with vertices A_i, has a limit A^* and a colimit A_*, and that the forgetful functor from TOP to ENS preserves and creates limits and colimits; we refer to [22; 6.2] for an exact statement of this. If A is a diagram in $\mathcal{T}$, then A^* is a limit of A in $\mathcal{T}$, and if $\eta_{A_*} : A_* \longrightarrow A_{**}$ is a reflection for $\mathcal{T}$, then A_{**} is a colimit of A in $\mathcal{T}$. If A is a diagram of $\mathcal{A}$-spaces, then the colimit A_{**} of A in $\mathcal{T}$ is an $\mathcal{A}$-space, and a colimit of A in the category of $\mathcal{A}$-spaces, and $A^{**} = \alpha A^*$ defines a limit A^{**} of A in the category of $\mathcal{A}$-spaces.

Subspaces and inclusion maps behave like limits, and quotient spaces and proclusion maps like colimits; we refer to [24] for this.

For convenience, we denote by $X \otimes Y$ the product of αX and αY in the category of $\mathcal{A}$-spaces. Thus

$$X \otimes Y = \alpha X \otimes \alpha Y = \alpha (X \times Y) = \alpha(\alpha X \times \alpha Y) ,$$

for any spaces X and Y. This expresses the well-known fact that the coreflector α preserves products.

2. Function spaces

2.1. For two spaces X and Y in $\mathcal{T}$, we denote by $C(X,Y)$ the set of all maps from X to Y in $\mathcal{T}$. For $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$ in $\mathcal{T}$, we define a mapping

$$C(f,g) \ : \ C(X',Y) \longrightarrow C(X,Y')$$

by putting $(C(f,g))(h) = g\,h\,f$ for $h : X' \longrightarrow Y$ in $\mathcal{T}$. This defines a hom functor $C : \mathcal{T}^{op} \times \mathcal{T} \longrightarrow ENS$, where $\mathcal{T}^{op}$ is the dual category of $\mathcal{T}$.

2.2. We wish to "lift" the hom functor by providing every set $C(X,Y)$ with a topology in such a way that all mappings $C(f,g)$ become continuous maps. This can be done in many ways; we describe a general procedure which includes most of the function space topologies which have been used.

For $u : A \longrightarrow X$ in $\mathcal{A}/X$ and V open in Y, we put

$$W(u,V) = \{f \in C(X,Y) \ : \ u(A) \subset f^{-1}(V)\} .$$

We denote by $C_{\mathcal{A}}(X,Y)$ the topological space obtained by providing $C(X,Y)$ with the topology for which these sets $W(u,V)$ form a subbasis of open sets.

If $\mathcal{A}$ is the class of compact Hausdorff spaces and X a Hausdorff space, then $C_{\mathcal{A}}(X,Y)$ is $C(X,Y)$ with the usual compact-open topology. If $\mathcal{A}$ consists of singletons, then $C_{\mathcal{A}}(X,Y)$ is $C(X,Y)$ with the topology of pointwise convergence. For $\mathcal{A} = \{N_\omega\}$ (see 1.5), the spaces $C_{\mathcal{A}}(X,Y)$ have been considered in [13] and [14]. The following result is known for these three cases.

2.3. *Proposition. Every mapping* $C(f,g) : C_{\mathcal{A}}(X',Y) \longrightarrow C_{\mathcal{A}}(X,Y')$ *is continuous. If* Y *is a* T_i*-space* $(i = 0, 1, 2)$, *then* $C_{\mathcal{A}}(X,Y)$ *is a* T_i*-space. If* Y *is a* T_3*-space and* $\mathcal{A}$ *consists of compact spaces, then* $C_{\mathcal{A}}(X,Y)$ *is a* T_3*-space.*

Proof. For u in $\mathcal{A}/X'$ and V open in Y', we obviously have

$$(C(f,g))^{-1}(W(u,V)) = W(f\,u,\ g^{-1}(V)) .$$

It follows immediately that $C(f,g)$ is continuous.

If $f \neq f'$ in $C(X,Y)$, then $f(x) = f'(x)$ for at least one $x \in X$. If Y is Hausdorff, then the points $f(x)$ and $f(x')$ have disjoint open neighborhoods V and V'. If A in $\mathcal{A}$ is non-empty and $u : A \longrightarrow X$ is the constant map to x, then $f \in W(u,V)$ and $f' \in W(u,V')$, and these sets are open in $C_{\mathcal{A}}(X,Y)$, and disjoint.

Every neighborhood of f is a neighborhood of g if and only if this is true for every neighborhood $W(u,V)$ of f , for every u and for every V , and this is the case if and only if $f(x) \in V$ implies $g(x) \in V$, for every $x \in X$ and every open V in Y . If Y is a T_1-space, it follows that $f(x) = g(x)$ for all x , i.e. $f = g$, and $C_{\mathcal{A}}(X,Y)$ is a T_1-space. If Y is a T_0-space, it follows that $f = g$ if f and g have the same neighborhoods in $C_{\mathcal{A}}(X,Y)$, i.e. $C_{\mathcal{A}}(X,Y)$ is a T_0-space ▌

If A in $\mathcal{A}$ is compact and $f \in W(u,V)$, then $f(u(A))$ is a compact subset of the open set V of Y . If Y satisfies T_3 , this implies that

$$f(u(A)) \subset V_1 \subset \overline{V}_1 \subset V$$

for an open set V_1 . If g is in the closure of $W(u,V_1)$, and if x denotes a point of $u(A)$ and the **induced constant map from** A , then $W(x,V')$ is a neighborhood of g , and intersects $W(u,V_1)$, for every open neighborhood V' of $g(x)$. This implies that V' intersects V_1 . Thus $g(x) \in \overline{V}_1$, and g maps $u(A)$ into $\overline{V}_1$. This shows that

$$f \in W(u,V_1) \subset \overline{W(u,V_1)} \subset W(u,V) \quad ,$$

and verifies T_3 for subbasis neighborhoods, if $\mathcal{A}$ consists of compact spaces and Y is a T_3-space. T_3 for all neighborhoods in $C_{\mathcal{A}}(X,Y)$ follows immediately ▌

2.4. For any spaces X , Y , we have an <u>evaluation mapping</u>

$$e_{XY} : C(X,Y) \times |X| \longrightarrow |Y| \quad ,$$

defined by $e_{XY}(f,x) = f(x)$. We have also a mapping

$$h_{XY} : |X| \longrightarrow C(Y, X \times Y) \quad ,$$

obtained by putting $h_{XY}(x)(y) = (x,y)$, for all $x \in X$ and $y \in Y$. The evaluation mapping e_{XY} is of course well known, but the closely related mapping h_{XY} has received so little attention that it does not even have a name.

We are interested in situations in which the mappings e_{XY} and h_{XY} become continuous maps. A preliminary step for this is to require that $C_{\mathcal{A}}(X,Y)$ is a space in $\mathcal{T}$ for all spaces X and Y in $\mathcal{T}$. 2.3 states the known results in this direction; further results would obviously be useful. A second step, strongly suggested by known results, is to restrict ourselves to $\mathcal{A}$-spaces. In general, $C_{\mathcal{A}}(X,Y)$ is not an $\mathcal{A}$-space for $\mathcal{A}$-spaces X and Y . This is well known for k-spaces and the compact-open topology. The remedy is simple: replace the space $C_{\mathcal{A}}(X,Y)$ in $\mathcal{T}$ by the $\mathcal{A}$-space $\alpha C_{\mathcal{A}}(X,Y)$. The maps $C(f,g)$ remain continuous if we do this. If Y is an $\mathcal{A}$-space, then $h_{XY}(x) : Y \longrightarrow X \otimes Y$ remains continuous for every $x \in X$, and we have a mapping

$$h_{XY} \; : \; |X| \longrightarrow C(Y, X \otimes Y) \; .$$

This leads us to the statement of our main result.

2.5. <u>Theorem</u>. <u>Assume that $\mathcal{A}$ consists of compact Hausdorff spaces, and that $C_{\mathcal{A}}(X,Y)$ is a space in $\mathcal{T}$ for all spaces X and Y in $\mathcal{T}$. Consider in addition the following two conditions</u>.

CL. <u>Every point of a space A in $\mathcal{A}$ has a basis of closed neighborhoods in A which are continuous images of spaces in $\mathcal{A}$</u> .

CP. <u>For any two spaces A and B in $\mathcal{A}$, the product $A \times B$ is an $\mathcal{A}$-space</u>.

<u>If</u> CL <u>is satisfied, then</u>

$$e_{XY} \; : \; C_{\mathcal{A}}(X,Y) \otimes X \longrightarrow Y$$

<u>is continuous for all spaces</u> X <u>and</u> Y <u>in</u> $\mathcal{T}$. <u>If</u> CP <u>is satisfied, then</u>

$$h_{XY} \; : \; X \longrightarrow C_{\mathcal{A}}(Y, X \otimes Y)$$

<u>is continuous for all</u> $\mathcal{A}$<u>-spaces</u> X <u>and</u> Y .

<u>Proof</u>. We must show for the first part that $e_{XY}\,u$ is continuous for every continuous map $u : A \longrightarrow C_{\mathcal{A}}(X,Y) \times X$ with A in $\mathcal{A}$. Let $u(a) = (f,x)$ for some $a \in A$, and assume that $f(x) \in V$, where V is open in Y . Let

$$C_{\mathcal{A}}(X,Y) \xleftarrow{\;v\;} A \xrightarrow{\;w\;} X$$

be the component maps of u . $w^{-1}(f^{-1}(V))$ is an open neighborhood of a , and there is by CL a map $j : B \longrightarrow A$ with B in $\mathcal{A}$ such that $j(B)$ is a closed neighborhood of a contained in $w^{-1}(f^{-1}(V))$. Then f is in $W(w\,j, V)$ which is open in $C_{\mathcal{A}}(X,Y)$. Since v is continuous, the set

$$N \;=\; j(B) \cap v^{-1}(W(w\,j,\, V))$$

is a neighborhood of a in A . If $a' \in N$ and $u(a) = (f',x')$, then clearly $f'(x') \in V$. Thus $e_{XY}\,u$ is continuous.

We must show for the second part that $h_{XY}\,u$ is continuous for every continuous map $u : A \longrightarrow X$ with A in $\mathcal{A}$. Thus assume that $h_{XY}(u(a))$ is in $W(v,V)$ for an open set V of $X \otimes Y$ and a map $v : B \longrightarrow Y$ with B in $\mathcal{B}$. This means that $(u(a), v(b)) \in V$ for every $b \in B$. By CP, $A \times B$ is an $\mathcal{A}$-space, and thus $u \times v : A \times B \longrightarrow X \otimes Y$ is continuous. Since this maps $\{a\} \times B$ into V and B is compact, there is a neighborhood N of a in A such that $u(N) \times v(B) \subset V$. This means that $h_{XY}(u(a')) \in W(v,V)$ for every $a' \in N$. Thus $h_{XY}\,u$ is continuous ∎

2.6. We give a list of classes $\mathcal{A}$ which satisfy CL and CP. All spaces in $\mathcal{A}$ must of course be compact Hausdorff spaces in $\mathcal{T}$.

2.6.1. A singleton.

2.6.2. $\{N_\omega\}$, if N_ω is in $\mathcal{T}$.

2.6.3. $\{[0,1]\}$, if $[0,1]$ is in $\mathcal{T}$.

2.6.4. All compact metrizable spaces in $\mathcal{T}$.

2.6.5. All compact Hausdorff spaces in $\mathcal{T}$.

2.6.6. All compact Hausdorff spaces X in $\mathcal{T}$ with $|X|$ countable.

2.6.7. All totally disconnected compact Hausdorff spaces in $\mathcal{T}$.

The list can be made much longer, without great effort. CL is obvious for all seven examples, and CP is obvious for five examples. $N_\omega \times N_\omega$ is metrizable and hence sequential, and it is easily verified that a mapping f from the unit square to a space X is continuous if f is continuous on every path.

2.7. A given coreflective full subcategory of $\mathcal{T}$ is generated by more than one class $\mathcal{A}$. The topology of $C_{\mathcal{A}}(X,Y)$, and even that of $\alpha\, C_{\mathcal{A}}(X,Y)$, depends in general on the choice of $\mathcal{A}$. The author's student B. V. S. Thomas has provided the following example for sequential spaces.

Let $X = Y$ be the half-open interval $[0,1)$, with the usual topology. Put $f_n(x) = x^n$ for $x \in X$ and $n \in N$, with $f_0(0) = 1$. For $\mathcal{A} = \{N_\omega\}$, $\lim f_n = f_\omega$ means that $\lim f_n(u_n) = f_\omega(u_\omega)$ for every convergent sequence $u : N_\omega \to X$. This is the case for $f_\omega = 0$. If X is in $\mathcal{A}$, then $\lim f_n = 0$ requires uniform convergence on X which is not the case.

If we restrict ourselves to classes $\mathcal{A}$ which satisfy CL and CP in 2.5, then the situation changes, at least for $\mathcal{A}$-spaces. It is well known that $C(Y,Z)$, for $\mathcal{A}$-spaces Y and Z, has at most one $\mathcal{A}$-space topology such that 3.1 is true for all $\mathcal{A}$-spaces X.

3. Convenient categories

We assume in this section that $C_{\mathcal{A}}(X,Y)$ is a space in $\mathcal{T}$ for all spaces X , Y in $\mathcal{T}$, and that the conclusions of Theorem 2.5 hold. We say that $\mathcal{A}$-spaces form a convenient coreflective subcategory of $\mathcal{T}$ in this situation.

3.1. Theorem. Assume that $\mathcal{A}$-spaces form a convenient coreflective subcategory of $\mathcal{T}$, and let X , Y , Z be $\mathcal{A}$-spaces. For maps

$$f : X \otimes Y \longrightarrow Z \quad \text{and} \quad g : X \longrightarrow C_{\mathcal{A}}(Y,Z) ,$$

the following conditions are logically equivalent.

(i) $f = e_{YZ} \cdot (g \otimes \mathrm{id}\, Y)$.

(ii) $g = C(\mathrm{id}\, Y, f) \cdot h_{XY}$.

(iii) $f(x,y) = g(x)(y)$ for all $x \in X$ and $y \in Y$.

Putting $\mu_{XYZ}(f) = g$ if these conditions are met defines a homeomorphism

$$\mu_{XYZ} : \alpha\, C_{\mathcal{A}}(X \otimes Y,\ Z) \longrightarrow \alpha\, C_{\mathcal{A}}(X, \alpha C_{\mathcal{A}}(Y,Z))$$

which is natural in X , Y , Z .

Proof. If g is a map and f given by (i), then f is a map since e_{YZ} is continuous, and f and g satisfy (iii). If f is a map and g given by (ii), then g is a map since h_{XY} is continuous, and (iii) holds. f and g determine each other in (iii), and thus we have a bijection

$$\mu_{XYZ} : C(X \otimes Y,\ Z) \longrightarrow C(X,\ C_{\mathcal{A}}(Y,Z)) .$$

μ_{XYZ} clearly is natural in X , Y , Z . Using the fact that μ_{XYZ} is a bijection for any three $\mathcal{A}$-spaces, we prove that μ_{XYZ} is a homeomorphism. We omit all subscripts, and we denote by

$$a : (X \otimes Y) \otimes Z \longrightarrow X \otimes (Y \otimes Z)$$

the natural homeomorphism, given by $a((x,y),z) = (x,(y,z))$.

To show that μ_{XYZ} is continuous, we begin with the evaluation map

$$e : C_{\mathcal{A}}(X \otimes Y, Z) \otimes (X \otimes Y) \longrightarrow Z .$$

We have $$(\mu(e \cdot a))(f,x)(y) = e(f,(x,y)) = f(x,y) ,$$

for $f : X \otimes Y \longrightarrow Z$, $x \in X$, $y \in Y$, and $\mu(\mu(e \cdot a)) = \mu_{XYZ}$ follows. This is continuous. For the evaluation map

$$e : C_{\mathcal{A}}(X, \propto C_{\mathcal{A}}(Y,Z)) \otimes X \longrightarrow \propto C_{\mathcal{A}}(Y,Z) ,$$

we have $$(\mu^{-1}(e))((g,x),y) = e(g,x)(y) = g(x)(y) ,$$

for $x \in X$, $y \in Y$, and $g : X \longrightarrow \propto C_{\mathcal{A}}(Y,Z)$. $\mu(\mu^{-1}(e) \cdot a^{-1}) = (\mu_{XYZ})^{-1}$ follows, and thus $(\mu_{XYZ})^{-1}$ is continuous ▌

3.2. <u>Corollary</u>. <u>Let</u> $\bigotimes_{i \in I} Y_i$ <u>be a product of</u> $\mathcal{A}$<u>-spaces, with projections</u> $p_i : \bigotimes_{i \in I} Y_i \longrightarrow Y_i$. If $\pi(f) = (p_i\, f)_{i \in I}$, <u>for an</u> $\mathcal{A}$<u>-space</u> X <u>and all maps</u> $f : X \longrightarrow \bigotimes_{i \in I} Y_i$, <u>then</u>

$$\pi : \propto C_{\mathcal{A}}(X, \bigotimes_{i \in I} Y_i) \longrightarrow \bigotimes_{i \in I} \propto C_{\mathcal{A}}(X,Y_i)$$

<u>is a homeomorphism of</u> $\mathcal{A}$<u>-spaces</u>.

<u>Proof</u>. This follows immediately from the fact that the functor $\propto C_{\mathcal{A}}(X, -)$ on $\mathcal{A}$-spaces is right adjoint to the functor $- \otimes X$, by a standard categorical argument which can be found e.g. in [1; p. 9] ▌

Consider now a diagram $A : D \longrightarrow \mathcal{T}$ in $\mathcal{T}$ with scheme D , with vertices A_i and arrows $a_\lambda : A_i \longrightarrow A_j$ corresponding to the vertices i and arrows λ :

$i \longrightarrow j$ of D, and let $A_{**} = \varinjlim A_i$ be a colimit of A in $\mathcal{T}$, with maps $a_i : A_i \longrightarrow A_{**}$. If X is a space, then the spaces $A_i \otimes X$ are the vertices, and the maps $a_\lambda \otimes \mathrm{id}\, X$ the arrows, of a diagram with scheme D which we denote by $A \otimes X : D \longrightarrow \mathcal{T}$. We denote by $Z = \varinjlim (A_i \otimes X)$ a colimit of $A \otimes X$, with maps $z_i : A_i \otimes X \longrightarrow Z$. The relations $F z_i = a_i \otimes \mathrm{id}\, X$ determine a map $F : Z \longrightarrow A_{**} \otimes X$ uniquely.

3.3. Corollary. If all vertices A_i of A are $\mathcal{A}$-spaces, then

$$F \ : \ \varinjlim (A_i \otimes X) \longrightarrow (\varinjlim A_i) \otimes X$$

is a homeomorphism of $\mathcal{A}$-spaces.

Proof. Since $\mathcal{A}$-spaces form a coreflective subcategory of $\mathcal{T}$ which is closed under homeomorphisms, the colimits are $\mathcal{A}$-spaces. 3.3 follows immediately from the fact that the functor $- \otimes X$ on $\mathcal{A}$-spaces is left adjoint to the functor $\alpha C_{\mathcal{A}}(X, -)$; we sketch the standard argument. We define

$$G \ : \ \varinjlim A_i \longrightarrow \alpha\, C_{\mathcal{A}}(X, \varinjlim (A_i \otimes X))$$

by putting $G a_i = \mu(b_i)$, for the colimit maps $b_i : A_i \otimes X \longrightarrow B$. One verifies easily that $G = \mu(F^{-1})$ defines the desired inverse of F ▌

Let us now consider proclusions in $\mathcal{T}$, as defined in [24], for the forgetful functor $P : \mathcal{T} \longrightarrow \mathrm{ENS}$.

3.4. Corollary. If $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$ are P-proclusions in $\mathcal{T}$, and if X and Y are $\mathcal{A}$-spaces, then X' and Y' are $\mathcal{A}$-spaces, and $f \otimes g : X \otimes Y \longrightarrow X' \otimes Y'$ is a P-proclusion.

Proof. The first assertion follows immediately from the dual of [24; 3.2];

see also [24; 3.3]. Since

$$f \otimes g = (\mathrm{id}\, X' \otimes g)(f \otimes \mathrm{id}\, Y) \quad ,$$

and the composition of proclusions is a proclusion, we must prove only that $f \otimes \mathrm{id}\, Y$ is a proclusion. The map $f \otimes \mathrm{id}\, Y$ clearly is surjective if f is, and the functor $- \otimes Y$ on $\mathcal{A}$-spaces is left adjoint to $\propto C_{\mathcal{A}}(Y, -)$. Thus $f \otimes \mathrm{id}\, Y$ is a proclusion by [24; 2.2] ▮

We have verified four of the five "test propositions" of [21]. The remaining test proposition is valid in $\mathcal{T}$ as well as for $\mathcal{A}$-spaces.

3.5. Proposition. *If $A : D \longrightarrow \mathcal{T}$ and $B : D \longrightarrow \mathcal{T}$ are diagrams in $\mathcal{T}$ with the same scheme D, and with vertices A_i and B_i, and if a map $f : A \longrightarrow B$ consists of proclusions $f_i : A_i \longrightarrow B_i$, then a colimit*

$$\varinjlim f_i \; : \; \varinjlim A_i \longrightarrow \varinjlim B_i$$

of f in $\mathcal{T}$ is a proclusion.

Proof. By [24; 2.1], we must only show that $\varinjlim f_i$ is surjective. The surjective maps f_i are epimorphic in TOP, and thus a colimit $f_* : A_* \longrightarrow B_*$ of f in TOP is epimorphic, i.e. surjective. If $\eta_{A_*} : A_* \longrightarrow A_{**}$ and $\eta_{B_*} : B_* \longrightarrow B_{**}$ are reflections for $\mathcal{T}$, then $\eta_{B_*} f_* = f_{**} \eta_{A_*}$ for a unique map $f_{**} : A_{**} \longrightarrow B_{**}$ of $\mathcal{T}$, and f_{**} is surjective since the other three maps in the relation defining f_{**} are surjective. A_{**} and B_{**} are colimits of A and B in $\mathcal{T}$, and f_{**} is a colimit of f ▮

In 3.1 through 3.4, we have used the continuity of the evaluation map e_{XY} (see 2.5) only for $\mathcal{A}$-spaces X and Y. We conclude the paper with a useful proposition which uses the full strength of 2.5.

3.6. <u>Proposition.</u> <u>If</u> X <u>is an</u> $\mathcal{A}$ <u>-space and</u> Y <u>a space in</u> $\mathcal{T}$, <u>then</u> $C(X,Y) = C(X,\alpha Y)$ <u>and</u> $\alpha C_{\mathcal{A}}(X,Y) = \alpha C_{\mathcal{A}}(X,\alpha Y)$.

<u>Proof.</u> $C(X,Y) = C(X,\alpha Y)$ follows immediately from 1.4. $C_{\mathcal{A}}(X,\alpha Y)$ has the finer topology than $C_{\mathcal{A}}(X,Y)$. On the other hand, if $u : A \longrightarrow C_{\mathcal{A}}(X,Y)$ is continuous with A in $\mathcal{A}$, then

$$\bar{u} = e_{XY} \cdot (u \otimes \mathrm{id}\ X) \ : \ A \otimes X \longrightarrow Y$$

defines a map $\bar{u}$. The map $\bar{u} : A \otimes X \longrightarrow \alpha Y$ remains continuous, and thus

$$u = C(X,\bar{u}) \cdot h_{AX} \ : \ A \longrightarrow C_{\mathcal{A}}(X,\alpha Y)$$

remains continuous. This shows that $\alpha C_{\mathcal{A}}(X,\alpha Y) = \alpha C_{\mathcal{A}}(X,Y)$ ▌

3.7. The present paper raises some topological questions which it does not answer. Here are three of these questions.

3.7.1. For which epireflective subcategories $\mathcal{T}$ of TOP and for which classes $\mathcal{A}$ of compact Hausdorff spaces in $\mathcal{T}$ is it true that $C_{\mathcal{A}}(X,Y)$ is in $\mathcal{T}$ for all spaces X , Y in $\mathcal{T}$? In other words, can 2.3 be extended?

3.7.2. If $\mathcal{T}'$ is an epireflective subcategory of TOP contained in $\mathcal{T}$, and $\mathcal{A}$ a class of compact Hausdorff spaces in $\mathcal{T}'$, then a space X in $\mathcal{T}'$ has $\mathcal{A}$-space modifications αX in $\mathcal{T}$ and $\alpha' X$ in $\mathcal{T}'$. How are these related? The example $\mathcal{T}$ = Hausdorff spaces, $\mathcal{T}'$ = regular spaces, $\mathcal{A}$ = all compact Hausdorff spaces, shows that αX and $\alpha' X$ may be distinct.

3.7.3. If $\mathcal{A}$ -spaces satisfy the conditions of this section, is the class of compact $\mathcal{A}$-spaces closed under the formation of finite products and of closed subspaces in $\mathcal{T}$? This is the case for k-spaces and for sequential spaces. A related question: does 2.5 have a converse?

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CONCERNING CONVENIENT CATEGORIES

Note by Oswald Wyler

The notations of this note will be those of the author's report: Convenient Categories for Topology, and all references will be to this report. We obtain a converse of Theorem 2.5, and we generalize the well-known result that the product of a k-space with a locally compact space is a k-space.

We recall that we operate in an epireflective subcategory $\mathcal{T}$ of TOP , and that $\mathcal{A}$-spaces are the objects of the coreflective subcategory of $\mathcal{T}$ generated by a class $\mathcal{A}$ of compact Hausdorff spaces in $\mathcal{T}$. These subcategories are assumed to be full and replete (in the non-Freydian sense). α denotes the coreflector for $\mathcal{A}$-spaces, and $X \otimes Y$ is the $\mathcal{A}$-space product of αX and αY. This must be distinguished carefully from the product $X \times Y$ in $\mathcal{T}$.

We redefine convenient categories by saying that $\mathcal{A}$-spaces form a <u>convenient category</u> if they have a convenient basis, and we say that $\mathcal{A}$ is a <u>convenient basis</u> for $\mathcal{A}$-spaces if the following two conditions are met.

CE. If $A \in \mathcal{A}$, then $e_{AZ} : \alpha C_{\mathcal{A}}(A,Z) \times A \longrightarrow Z$ is continuous for every space Z in $\mathcal{T}$.

CP. For any two spaces A and B in $\mathcal{A}$, the product $A \times B$ in $\mathcal{A}$ is an $\mathcal{A}$-space.

The results of this note show that this definition of a convenient category is stronger than that of the report cited above, and enhances convenience.

Before proving the main results of this note, we derive an important consequence of condition CL of Theorem 2.5, and we show that CE suffices for the conclusion of Theorem 2.5.

Proposition 1. *If $\mathcal{A}$ satisfies CL, then $C_{\mathcal{A}}(A,Z)$ is $C(A,Z)$ with the compact-open topology, for every $A \in \mathcal{A}$ and every space Z in $\mathcal{T}$.*

Corollary. *CL implies CE.*

Proof. The compact-open topology of $C(A,Z)$ obviously is finer than the topology of $C_{\mathcal{A}}(A,Z)$. For the converse statement, consider a set $W(K,V)$ with $K \subset A$ compact and $V \subset Z$ open. If $f \in W(K,V)$, then $f^{-1}(V)$ is a neighborhood of K. Since K is compact, it follows from CL that K is covered by a finite number of sets $u_i(B_i)$ contained in $f^{-1}(V)$, for maps $u_i : B_i \longrightarrow A$ in $\mathcal{A}/A$. The set $\bigcap W(u_i,V)$ then is a neighborhood of f in $C_{\mathcal{A}}(A,Z)$, and contained in $W(K,V)$. Thus $W(K,V)$ is open in $C_{\mathcal{A}}(A,Z)$, and the topology of $C_{\mathcal{A}}(A,Z)$ is finer than the compact-open topology.

It is well known that $e_{AZ} : C(A,Z) \times A \longrightarrow Z$ is continuous if A is a compact Hausdorff space and $C(A,Z)$ has the compact-open topology. Used for all spaces A in $\mathcal{A}$, this is somewhat stronger than CE ▌

Prop. 1 is useful not only because of its corollary, but mainly because the spaces $C_{\mathcal{A}}(A,Z)$ with A in $\mathcal{A}$ determine all spaces $C_{\mathcal{A}}(Y,Z)$. To see this, let $C_p(Y,Z)$ be the set $C(Y,Z)$ with the topology of pointwise convergence. The sets $W(\{x\}, V)$, for $x \in X$ and V open in Z, form a subbase of open sets for this topology, and $C_{\mathcal{A}}(Y,Z)$ has a finer topology than $C_p(Y,Z)$. Thus the following diagram (all arrows) is a commutative diagram in TOP, for every map $u : A \longrightarrow Y$ in $\mathcal{A}/Y$.

$$(1)\qquad \begin{array}{ccc} C_{\mathcal{A}}(Y,Z) & \xrightarrow{C(u,Z)} & C_{\mathcal{A}}(A,Z) \\ \downarrow{\scriptstyle \mathrm{id}\,C(Y,Z)} & & \downarrow{\scriptstyle \mathrm{id}\,C(A,Z)} \\ C_p(Y,Z) & \xrightarrow{C(u,Z)} & C_p(A,Z) \end{array}$$

The class of all diagrams (1) determines $C_{\mathcal{A}}(Y,Z)$ as follows.

Proposition 2. *Let D be the diagram in TOP consisting of all solid arrows of diagrams* (1). $C_{\mathcal{A}}(Y,Z)$, *with all broken arrows of diagrams* (1), *is a limit of D in TOP* .

Proof. $C(Y,Z)$, with the broken arrows, is trivially a limit of D at the set level. Thus we need only show that $C_{\mathcal{A}}(Y,Z)$ has the coarsest topology for which all broken arrows of diagrams (1) remain continuous. This is the topology for which the sets $(C(u,Z))^{-1}(V)$, for arbitrary $u : A \longrightarrow Y$ in $\mathcal{A}/Y$ and V in a subbase of $C_{\mathcal{A}}(A,Z)$, form a subbase of open sets. If $V = W(v,U)$ for a map $v : B \longrightarrow A$ in $\mathcal{A}/A$ and U open in Z , then clearly

$$(C(u,Z))^{-1}(V) = W(u\,v,\ U) \ .$$

These sets form a subbase of open sets for $C_{\mathcal{A}}(Y,Z)$ ▌

We note that D is a large diagram if $\mathcal{A}/Y$ is a proper class. This does not matter for many applications, including the one given below, and in any case, D can easily be replaced by a small subdiagram with the same limit.

Proposition 3. *If Z is a space in* $\mathcal{T}$ *such that* $C_{\mathcal{A}}(A,Z)$ *is in* $\mathcal{T}$ *for every space* $A \in \mathcal{A}$, *then* $C_{\mathcal{A}}(Y,Z)$ *is in* $\mathcal{T}$ *for every topological space* Y .

Proof. $C_p(Y,Z)$ is homeomorphic to a subspace of the product space $Z^{|Y|}$ and thus is in $\mathcal{T}$ regardless of the topology of Y . Now $C_{\mathcal{A}}(Y,Z)$ is the limit in TOP of a diagram with vertices in the epireflective subcategory $\mathcal{T}$. Since $\mathcal{T}$ is full and replete, it follows that $C_{\mathcal{A}}(Y,Z)$ is a space in $\mathcal{T}$ ∎

After this digression, we return to the main theme of the present note.

<u>Proposition 4</u>. <u>If</u> $\mathcal{A}$ <u>satisfies</u> CE, <u>then</u> $e_{YZ} : C_{\mathcal{A}}(Y,Z) \otimes Y \longrightarrow Z$ <u>is continuous for all spaces</u> Y <u>and</u> Z <u>in</u> $\mathcal{A}$.

<u>Proof</u>. We must show that $e_{YZ}\, u$ is continuous for every map $u : A \longrightarrow C_{\mathcal{A}}(Y,Z) \times Y$ with A in $\mathcal{A}$. If $u = \{p,q\}$ for $p : A \longrightarrow C_{\mathcal{A}}(Y,Z)$ and $q : A \longrightarrow Y$, and if $v = \{p, \mathrm{id}\, A\} : A \longrightarrow \alpha\, C_{\mathcal{A}}(Y,Z) \times A$, then v is continuous, and the following diagram clearly commutes at the set level.

$$(2)\qquad \begin{array}{ccccc} A & \xrightarrow{\ v\ } & \alpha C_{\mathcal{A}}(Y,Z) \times A & \xrightarrow{\ C(q,Z)\ \ \mathrm{id}\, A\ } & \alpha C_{\mathcal{A}}(A,Z) \times A \\ & \searrow^{u} & \downarrow{\scriptstyle \mathrm{id} \times q} & & \downarrow{\scriptstyle e_{AZ}} \\ & & \alpha C_{\mathcal{A}}(Y,Z) \times Y & \xrightarrow{\ e_{YZ}\ } & Z \end{array}$$

The upper leg of (2) is continuous by CE, and thus $e_{YZ}\, u$ is continuous ∎

We note that the product in $\mathcal{T}$ appears in CE, and the product of $\mathcal{A}$-spaces in Prop. 4. Although our next result eliminates this distinction, it is vital in its proof.

<u>Theorem 1</u>. <u>If</u> $\mathcal{A}$-<u>spaces form a convenient category, and if an</u> $\mathcal{A}$-<u>space</u> Y <u>is a locally compact Hausdorff space, then the product</u> $X \times Y$ <u>in</u> $\mathcal{T}$ <u>is an</u> $\mathcal{A}$-<u>space for every</u> $\mathcal{A}$-<u>space</u> X.

<u>Proof</u>. The topology of $X \otimes Y$ is finer than that of $X \times Y$; we must prove the converse. We do this first for $Y = A$ in a convenient basis $\mathcal{A}$ for $\mathcal{A}$-spaces. The identity mapping from $X \times A$ to $X \otimes A$ is the composite map

$$X \times A \xrightarrow{h_{XA} \times id} \propto C_{\mathcal{A}}(A, X \otimes A) \times A \xrightarrow{e_{A, X\otimes A}} X \otimes A$$

which is continuous by CE and 2.5.

In the general case, let W be a neighborhood of (x,y) in $X \otimes Y$. Since $h_{XY}(x) : Y \longrightarrow X \otimes Y$ is continuous, $(h_{XY}(x))^{-1}(W)$ is a neighborhood of y in Y. Thus $\{x\} \times N \subset W$ for a compact neighborhood N of y in Y. Let

$$V = \{x' \in X : \{x'\} \times N \subset W\} .$$

Clearly $x \in V$ and $V \times N \subset W$, and we are done if V is a neighborhood of x in X. Since X is an $\mathcal{A}$-space, this means that, for every pair (u,a) with $u : A \longrightarrow X$ in $\mathcal{A}/X$ and $a \in A$ such that $u(a) = x$, there is a neighborhood U of a in A such that $u(U) \subset V$. Now $u \times id\ Y : A \times Y \longrightarrow X \otimes Y$ is continuous since $A \times Y$ is an $\mathcal{A}$-space, and thus $W_u = (u \times id\ Y)^{-1}(W)$ defines a neighborhood W_u of (a,y) in $A \times Y$. Clearly $\{a\} \times N \subset W_u$. Since N is compact, $U \times N \subset W$ for some neighborhood U of a in A. But $u(U) \subset V$ for such a neighborhood U ▮

__Theorem 2__. If $\mathcal{A}$-spaces form a convenient category, then the class of all compact Hausdorff $\mathcal{A}$-spaces is a convenient basis for $\mathcal{A}$-spaces.

Proof. Denote this class by $\mathcal{B}$, and assume that $\mathcal{A}$ is a convenient basis. By Theorem 1 and Prop. 4, $\mathcal{B}$ satisfies CP, and $e_{BZ} : \propto C_{\mathcal{A}}(B,Z) \times B \longrightarrow Z$ is continuous for every space Z in $\mathcal{T}$. Since $C_{\mathcal{B}}$-topologies are finer than $C_{\mathcal{A}}$-topologies, e_{BZ} remains continuous if $C_{\mathcal{A}}(B,Z)$ is replaced by $C_{\mathcal{B}}(B,Z)$. Thus $\mathcal{B}$ also satisfies CE ▮

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