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SOME TRANSLATION-INVARIANT SPACES
OF FUNCTIONS ON TOPOLOGICAL
GROUPS

by

E. J. Risler

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ABSTRACT

We extend some of the results of the theory of \mathfrak{F} - and \mathfrak{T} -spaces due to J. J. Schäffer to function spaces on a locally compact Hausdorff topological group G . The class \mathfrak{T} consists of non-zero translation-invariant solid normed spaces that are stronger than \underline{L} , the space of (equivalence classes of) measurable functions that are integrable on compact subsets of G , with the topology of convergence in the mean on every compact subset. We introduce and study some spaces in \mathfrak{T} , in particular $\underline{M}, \underline{S}, \underline{T}$. Our main results concern the quasi local closedness and local closedness of \underline{T} . We also give a generalization of some results about associate spaces and duality between \underline{M} and \underline{T} . The results may be extended to normed spaces of functions with values in a Banach space.

INTRODUCTION

We present some of the results obtained in the course of research done in order to generalize the theory of \mathfrak{F} - and \mathfrak{T} -spaces introduced by Schäffer [7] and further developed by Massera and Schäffer [6], Chapter 2.

In those works the spaces under consideration consist of functions whose domain is an interval of the real line; we consider here spaces of functions defined on a locally compact Hausdorff topological group. Our main objective is to study the spaces $\mathfrak{M}, \mathfrak{S}$ and \mathfrak{T} (introduced in Section 3) belonging to the class \mathfrak{F} , and our main results concern the quasi local closedness and local closedness of \mathfrak{T} . For this we present in Section 1 the necessary basic notions and results and in Section 2 the new definitions and results concerning the class \mathfrak{F} that we may need in Section 3, which contains the main results. In Section 4 we give the rather straightforward generalization of some results about associate spaces and duality between \mathfrak{M} and \mathfrak{T} .

The theory of \mathfrak{F} -spaces is closely related to the work of several authors; in particular see Luxemburg [4], Luxemburg and Zaanen [5] and several subsequent papers of these two authors. (A resumé of their work is to be found in Zaanen [8], Chapter 15.) We refer to [6] and [7] for more detailed references and explanations of the relations between the theory of \mathfrak{F} -spaces and that of other authors.

This work is expected to constitute part of a Ph.D. dissertation. The subject was suggested to me by Professor J. J. Schäffer, my advisor. I have received from him valuable advice and direction.

1. n-Spaces and F-Spaces

In this section we present all basic concepts, definitions, notations, facts and results on n-spaces and F-spaces that we may need in the following. We follow closely [7] and [6], Chapter 2, extending some concepts to the more general framework in which we are going to work. Proofs, examples and other aspects of the theory are to be found there.

We shall use the usual terminology for vector spaces. Let A be a balanced convex set in a real or complex vector space. We define its radial closure by

$$\text{rad } A = \{x \mid \lambda x \in A, 0 \leq \lambda < 1\} = \bigcap_{t > 1} tA.$$

Obviously $\text{rad } A$ is convex and balanced. If $\text{rad } A = A$ we say that A is radially closed; for any nonempty balanced convex set A , $\text{rad } A$ is radially closed. In a topological vector space $\text{rad } A \subset \text{cl } A$; therefore $\text{rad } A$ is bounded if A is bounded.

If X is a normed space, its norm is denoted by $\|\cdot\|_X$ and its unit ball $\{x \mid x \in X, \|x\|_X \leq 1\}$ by $\Sigma(X)$. If $\lambda > 0$, λX denotes the normed space which is algebraically (and topologically) identical with X but has $\Sigma(\lambda X) = \lambda \Sigma(X)$ and therefore $\|x\|_{\lambda X} = \lambda^{-1} \|x\|_X$ for all $x \in X$. The spaces X and λX are said to be homothetic.

If Y and Z are normed spaces, we shall write $Y \leq Z$ or $Z \geq Y$ if Y is algebraically contained in Z and $\Sigma(Y) \subset \Sigma(Z)$. The relation \leq is clearly transitive and $Y \leq Z \leq Y$ implies $Y = Z$.

Let E be a locally convex space (i.e. a locally convex Hausdorff topological vector space) and Y a normed space contained algebraically as a linear manifold in E . Y is said to be stronger than E (or E weaker than Y) if the norm topology of Y is stronger than the topology induced on Y by E . Equivalently, Y is stronger than E iff: 1) the inclusion map $Y \rightarrow E$ is continuous, iff: 2) $\Sigma(Y)$ is E -bounded, iff: 3) for every continuous seminorm π on E , there is a number $\alpha_\pi \geq 0$ such that $\pi(y) \leq \alpha_\pi \|y\|_Y$ for all $y \in Y$. In the particular case that Y and Z are normed spaces, Y is stronger than Z iff $Y \leq \lambda Z$ for some $\lambda > 0$, and iff $\Sigma(Y) \subset \lambda \Sigma(Z)$ for some $\lambda > 0$. The relation "stronger than" is transitive. If Y and Z are two normed spaces and Y is both stronger than and weaker than Z , then Y and Z coincide as topological vector spaces, but not as normed spaces. In this case, they are said to be norm-equivalent, which is justified by the above facts.

From now on E is a fixed locally convex space. We define $\mathfrak{N}(E)$ as the class of all normed spaces which are stronger than E .

We also define the class $\Gamma(E)$ of all nonempty radially closed E -bounded balanced convex sets in E . $\Gamma(E)$ is a conditionally complete lattice under inclusion, the lattice operations being: meet, $A_1 \wedge A_2 = A_1 \cap A_2$ and join $A_1 \vee A_2 = \text{rad conv}(A_1 \cup A_2)$, for all $A_1, A_2 \in \Gamma(E)$. With the order relation \leq $\mathfrak{N}(E)$ is also a conditionally complete lattice and the mapping $Y \mapsto \Sigma(Y)$ is a complete lattice isomorphism from $\mathfrak{N}(E)$ onto $\Gamma(E)$. The

lattice operations in $\mathfrak{h}(E)$ may be described in the following way: if $Y, Z \in \mathfrak{h}(E)$, $Y \wedge Z$ is algebraically $Y \cap Z$ and $\|u\|_{Y \wedge Z} = \max\{\|u\|_Y, \|u\|_Z\}$; $Y \vee Z$ is algebraically $Y + Z$ and $\|u\|_{Y \vee Z} = \inf\{\|y\|_Y + \|z\|_Z \mid u = y + z, y \in Y, z \in Z\}$. If $\{Y_\xi\}_{\xi \in \Xi}$ is a family in $\mathfrak{h}(E)$, $\bigwedge Y_\xi$ is the submanifold (possibly proper) of $\bigcap Y_\xi$ where the norm $\|u\|_{\bigwedge Y_\xi} = \sup\{\|u\|_{Y_\xi} \mid \xi \in \Xi\}$ is finite; in the case $Y_\xi \leq Y_0 \in \mathfrak{h}(E)$, then $\bigvee Y_\xi$ is algebraically ΣY_ξ (set of finite sums $\Sigma y_\xi, y_\xi \in Y_\xi$) with norm $\|u\|_{\bigvee Y_\xi} = \inf\left\{\sum_{k=1}^m \|y_k\|_{Y_{\xi(k)}} \mid u = \sum_{k=1}^n y_k; y_k \in Y_{\xi(k)}; k=1, 2, \dots, n \text{ for some } n=1, 2, \dots \text{ and some } \xi(1), \dots, \xi(n) \in \Xi\right\}$. Norm-equivalence is a congruence in the lattice $\mathfrak{h}(E)$, i.e. finite lattice operations preserve norm-equivalence.

The E -closure of a bounded balanced convex set in E is convex, balanced, radially closed and bounded. Therefore $A \in \Gamma(E)$ implies $\text{cl}_E A \in \Gamma(E)$. The map $A \mapsto \text{cl}_E A$ is a closure operation on the lattice $\Gamma(E)$ and $\text{cl}_E(\lambda A) = \lambda \text{cl}_E A$ for $\lambda > 0$. For every $X \in \mathfrak{h}(E)$ we then define $\text{lc}_E X$, the local closure of X in E , as the space in $\mathfrak{h}(E)$ that satisfies $\Sigma(\text{lc}_E X) = \text{cl}_E \Sigma(X)$. We may omit the reference to E if no confusion can arise. If $\Sigma(X)$ is closed in E (i.e. $\text{lc}_E X = X$), X is said to be locally closed (in E). Clearly $X \leq \text{lc} X$ and $\text{lc} X$ is locally closed for all $X \in \mathfrak{h}(E)$. Spaces homothetic to a locally closed space are locally closed, but local closedness in general is not invariant under norm-equivalence. A space in $\mathfrak{h}(E)$ is called quasi locally closed (in E) if it is norm equivalent to a locally closed space.

An important subclass of $\mathfrak{h}(E)$ is that consisting of all Banach (i.e. complete) spaces in $\mathfrak{h}(E)$; we denote it by $\mathfrak{B}(E)$.

$\mathfrak{B}(E)$ is a sublattice of $\mathfrak{h}(E)$ and contains the meet of any subclass. If E is a complete locally convex space, then $Y = \text{lc } Y$ implies $Y \in \mathfrak{B}(E)$. Conversely if $Y \in \mathfrak{B}(E)$ and $\text{lc } Y$ has the same elements as Y , then Y is quasi locally closed.

If E is complete then for every $Y \in \mathfrak{h}(E)$, the set $\{Z \mid Z \in \mathfrak{B}(E), Z \geq Y\}$ is not empty: it contains $\text{lc } Y$. Then this class has a least element (namely its meet) that we denote by bY , that is

$$bY = \bigwedge \{Z \mid Z \in \mathfrak{B}(E), Z \geq Y\}.$$

We can characterize bY as follows:

THEOREM 1.1. If $Y \in \mathfrak{h}(E)$, then bY consists of all $u \in E$ which are E -limits of Y -Cauchy sequences; the norm is

$$\|u\|_{bY} = \inf \left\{ \lim_{n \rightarrow \infty} \|y_n\|_Y \mid (y_n) \text{ is a } Y\text{-Cauchy sequence, } \lim_{n \rightarrow \infty} y_n = u \right\}.$$

PROOF: Cf. [6], 21.G.

bY is a kind of completion of Y but the trivial injection $Y \rightarrow bY$ need not be isometrical. Denote by \hat{Y} the abstract completion of Y .

THEOREM 1.2. If E is complete and $Y \in \mathfrak{h}(E)$, the following statements are equivalent:

i) if (y_n) is a Y -Cauchy sequence such that $\lim_{n \rightarrow \infty} y_n = y \in Y$,
then $\|y\|_Y \leq \lim_{n \rightarrow \infty} \|y_n\|_Y$;

ii) if (y_n) is a Y -Cauchy sequence such that $\lim_{n \rightarrow \infty} y_n = 0$,
then $\lim_{n \rightarrow \infty} \|y_n\|_Y = 0$;

iii) the canonical embedding of Y into \hat{Y} may be extended
to an isometrical isomorphism of bY into \hat{Y} ;

iv) the trivial injection from Y into bY is isometrical.

PROOF: [7] Theorem 2.4, and the above Theorem 1.1.

If $Y \in \mathcal{H}(E)$ and satisfies conditions i)-iv) of Theorem 1.2, it shall be called E -completable. E -completeness is invariant under norm-equivalence.

We are going to deal with certain spaces of functions on topological groups, which we always assume to be Hausdorff. G shall denote a locally compact σ -compact (Hausdorff) topological group and μ shall denote an (essentially unique) left-invariant (regular) Haar measure on G defined on the σ -algebra of Borel sets. We remark that every locally compact group has an open-closed σ -compact subgroup ([3] Theorem 5.7) and, hence, the group is a disjoint union of σ -compact open-closed cosets; these facts allow us to extend the results of the theory to all locally compact groups in a rather direct way; we omit the details of the argument.

We shall study spaces of (classes of equivalence of) strongly measurable functions from G into X , a real or complex Banach space with norm $\|\cdot\|$. We identify functions equal a.e. and sometimes confuse functions with classes of equivalence. The characteristic function of a set $E \subset G$ is denoted by χ_E ; a

subset of G is said to be bounded if it is contained in a compact subset. We shall consider the space of all measurable real-valued functions with its usual vector lattice structure; equalities and inequalities between functions should always be understood as holding almost everywhere. To any strongly measurable function f from G into X , it corresponds the measurable real-valued function $\|f\|$ defined by $\|f\|(t) = \|f(t)\|$ for all t in G .

Let $\underline{L}(X)$ be the space of all (equivalence classes of) strongly measurable from G into X which are (Bochner) integrable on compact subsets of G , endowed with the topology of convergence in mean on every compact subset of G . $\underline{L}(X)$ is a locally convex Fréchet space, i.e. complete and metrizable. In case $X = \mathbb{R}$ we shall write $\underline{L}(\mathbb{R}) = \underline{L}$. It is to be remembered that $f \in \underline{L}(X)$ if and only if f is strongly measurable and $\|f\| \in \underline{L}$.

We introduce now the class \mathfrak{F} , in full ${}_G\mathfrak{F}$, consisting of all normed spaces \underline{F} (with norm $\|\cdot\|_{\underline{F}}$) that satisfy:

(N) \underline{F} is stronger than \underline{L} (i.e. $\underline{F} \in \mathfrak{h}(\underline{L}) = \mathfrak{h}$) i.e. for every compact subset C of G , there is a number $\alpha_C \geq 0$ such that $\int_C |f(t)| d\mu(t) \leq \alpha_C \|f\|_{\underline{F}}$ for all $f \in \underline{F}$

and

(F) if $\varphi \in \underline{F}$ and ψ is a real-valued measurable function G such that $|\psi| \leq |\varphi|$, then $\psi \in \underline{F}$ and $\|\psi\|_{\underline{F}} \leq \|\varphi\|_{\underline{F}}$.

We shall always denote by α_C the least number with the property expressed in (N).

For spaces of the class \mathfrak{F} it is possible to prove some useful results concerning the local closures. With proofs analogous to those in [7], we obtain:

THEOREM 1.3. Let $F \in \mathfrak{F}$ be given. If $\varphi \in \text{lc } F$, there exists an F -bounded increasing sequence (φ_n) of positive elements in F , $\varphi_n \leq |\varphi|$ such that $\lim_{L} \varphi_n = |\varphi|$ and $|\varphi|_{\text{lc } F} = \lim_{n \rightarrow \infty} |\varphi_n|_F$.

Conversely if (φ_n) is an F -bounded increasing sequence of positive terms in F , it converges in L to a function $\varphi \in \text{lc } F$ and $|\varphi|_{\text{lc } F} \leq \lim_{n \rightarrow \infty} |\varphi_n|_F$ (cf. [7] Theorem 3.5).

COROLLARY 1.4. If $F \in \mathfrak{F}$, then $\text{lc } F \in \mathfrak{F}$ (cf. [7] Corollary 3.2).

COROLLARY 1.5. A space $F \in \mathfrak{F}$ is locally closed if and only if for every F -bounded increasing sequence (φ_n) of positive terms in F its L -limit φ lies in F and $|\varphi|_F = \lim_{n \rightarrow \infty} |\varphi_n|_F$ (cf. [7] Corollary 3.3).

THEOREM 1.6. A space $F \in \mathfrak{F}$ is quasi locally closed if and only if for every F -bounded increasing sequence (φ_n) of positive terms in F its L -limit φ lies in F (cf. [7] Theorem 3.6).

COROLLARY 1.7. A space $F \in \mathfrak{F}$ is quasi locally closed if and only if F and $\text{lc } F$ consist of the same elements. (Cf. [7] Corollary 3.4.)

The following theorem is also proved in a similar way to [7] Theorem 3.2.

THEOREM 1.8. For any non-empty index set Ξ , if $F_{\xi} \in \mathfrak{F}$ for all $\xi \in \Xi$ then $\bigwedge F_{\xi} \in \mathfrak{F}$ and if $\bigvee F_{\xi}$ exists (for instance if is finite or $F_{\xi} \leq F_0$ for some $F_0 \in \mathfrak{h}(\underline{L})$ and all $\xi \in \Xi$) then also $\bigvee F_{\xi} \in \mathfrak{F}$.

We shall also need:

THEOREM 1.9. If $F \in \mathfrak{F}$, then $bF \in \mathfrak{F}$.

PROOF: As in [7] Theorem 3.4, simplified by Theorem 1.1.

2. \mathfrak{J} -Spaces

The group G operates to the left on $\underline{L}(X)$ in the following way: for any $s \in G$, $f \in \underline{L}(X)$, $sf = L_s f$ is defined by

$$(sf)(x) = f(s^{-1}x) \quad \text{for all } x \in G.$$

To see that sf is strongly measurable it is enough to realize that since the map $x \mapsto s^{-1}x$ is a homeomorphism on G , for any $s \in G$, it maps Borel sets on Borel sets, and that since μ is left invariant $\mu(s^{-1}B) = \mu(B)$, for any measurable set B . To see that sf is integrable on compact sets, for any compact C we have:

$$\begin{aligned} (2.1) \quad \int_C \|sf(x)\| d\mu(x) &= \int_C \|f(s^{-1}x)\| d\mu(x) = \int_{s^{-1}C} \|f(x)\| d\mu(sx) = \\ &= \int_{s^{-1}C} \|f(x)\| d\mu(x) < \infty \end{aligned}$$

since $f \in \underline{L}(X)$ and $s^{-1}C$ is compact.

It is easy to see that the following relations hold:

$$(2.2) \quad L_{s^{-1}} L_s = L_e = \text{identity} \quad \text{for every } s \in G$$

$$(2.3) \quad \|L_s f\| = L_s \|f\| \quad \text{for all } s \in G$$

$$(2.4) \quad L_{st} = L_s L_t \quad \text{for all } s, t \in G.$$

Hence for every $s \in G$, L_s is a continuous linear bijective mapping from $\underline{L}(X)$ onto $\underline{L}(X)$.

We consider now the class \mathfrak{J} , or in full ${}_G\mathfrak{J}$, consisting of all spaces $\mathfrak{F} \in \mathfrak{J}$ satisfying the following additional conditions:

$$(Z) \quad \underline{F} \neq \{0\}$$

$$(T) \quad \text{If } \varphi \in \underline{F}, s \in G, \text{ then } s\varphi \in \underline{F} \text{ and } \|s\varphi\|_{\underline{F}} \leq \|\varphi\|_{\underline{F}}.$$

We remark that in (T) inequality may be replaced by equality, since by (2.2) $\|\varphi\|_{\underline{F}} = \|s^{-1}s\varphi\|_{\underline{F}} \leq \|s\varphi\|_{\underline{F}} \leq \|\varphi\|_{\underline{F}}$.

Classes $\underline{\mathfrak{F}}$ and $\underline{\mathfrak{J}}$ have their counterpart for strongly measurable functions defined on G with values in X , a given Banach space. For every $\underline{F} \in \underline{\mathfrak{F}}$ we denote by $\underline{F}(X)$ the set of $f \in \underline{L}(X)$ such that $\|f\| \in \underline{F}$ with the norm $\|f\|_{\underline{F}(X)} = \|\|f\|\|_{\underline{F}}$. By (F) this definition means that $\underline{F}(\mathbb{R}) = \underline{F}$. The class of all spaces $\underline{F}(X)$, $\underline{F} \in \underline{\mathfrak{F}}$, for a fixed X , is written as $\underline{\mathfrak{F}}(X)$. In the same way we define the class $\underline{\mathfrak{J}}(X)$, a subclass of $\underline{\mathfrak{F}}(X)$.

Every property of spaces of functions with values in X can be deduced from the corresponding property of spaces of real-valued functions, as it is allowed by the content of results coincident with [7] Thm. 3.1, Thm. 3.3, Corollary 3.1, Thm. 3.8, and Thm. 4.1. The proofs in our case would be literally the same. Hence from now on we shall restrict ourselves to the study of real-valued functions, with the knowledge that the results of this section also hold for spaces in the class $\underline{\mathfrak{J}}(X)$.

From now on we shall work with a triple (G, K, μ) where G is, as before, a locally compact σ -compact Hausdorff topological group; K is a regularly closed compact symmetrical neighborhood of the origin $e \in G$; and μ , as before, is a left-invariant Haar measure on G , but normalized by $\mu(K) = 1$.

We now define an important space \tilde{M} , in full \tilde{M}_K , which is the set of all functions $f \in L_1$ that satisfy $\sup_{s \in G} \int_{sK} |f(t)| d\mu(t) < \infty$, with this supremum as norm.

LEMMA 2.1. $\tilde{M} \in \mathfrak{J}$ and it is locally closed.

PROOF. We prove first that \tilde{M} satisfies (N). Let C be any compact set of G , $C \subset FK$ for some finite set $F \subset G$. Then for all $\varphi \in \tilde{M}$

$$\int_C |\varphi(t)| d\mu(t) \leq \sum_{s \in F} \int_{sK} |\varphi(t)| d\mu(t) \leq \text{card } F \cdot \|\varphi\|_{\tilde{M}}$$

\tilde{M} satisfies (F) and (Z) trivially. To see that it satisfies (T), let $r \in G$ and $\varphi \in \tilde{M}$ be arbitrary, then

$$\sup_{s \in G} \int_{sK} |(r\varphi)(t)| d\mu(t) = \sup_{s \in G} \int_{rsK} |\varphi(t)| d\mu(t) = \|\varphi\|_{\tilde{M}}.$$

Thus $\tilde{M} \in \mathfrak{J}$.

The second part is proved, using the criterion given by Corollary 1.5 in the same way as in [7] Lemma 4.1, with some obvious changes.

We shall need the following.

THEOREM 2.2. (Density) Let G be a locally compact group, ν a right-invariant Haar measure on G . Let $E \subset G$ be a bounded Borel set. If, for every $x \in G$, and every bounded neighborhood U of $e \in G$, $f_U(x) = \frac{\nu(E \cap Ux)}{\nu(Ux)}$, then f_U converges in the mean (and hence in measure) to χ_E as $U \rightarrow e$.

PROOF. Sketched in [2], p. 268 Ex. 5.

We use this theorem now to prove a lemma that we think is well known but for which we have not found a concrete reference.

LEMMA 2.3. Let G and μ be as in our general context, A and B two Borel sets, $\mu(A) > 0$. Then $\mu(B \cap As) = 0$ for all $s \in G$ (or $\mu(B \cap sA) = 0$ for all $s \in G$) implies $\mu(B) = 0$.

PROOF. We may assume that A is bounded without loss of generality, and, G being σ -compact, we may also assume that B is bounded.

Let ν be a right-invariant Haar measure and $f_U = \frac{\nu(A \cap Ux)}{\nu(Ux)}$ for every bounded neighborhood U . $A = \{x | x \in A, f_U(x) < \frac{3}{4}\} \cup \{x | x \in A, f_U(x) \geq \frac{3}{4}\}$; since $f_U \leq 1$ and since, by Theorem 2.2, f_U converges in measure to χ_A as $U \rightarrow e$,

$$\nu(\{x | x \in A, f_U(x) < \frac{3}{4}\}) = \nu(\{x | x \in A, |f_U(x) - \chi_A(x)| > \frac{1}{4}\}) \rightarrow 0$$

as $U \rightarrow e$. Hence $\nu(\{x | x \in A, f_U(x) \geq \frac{3}{4}\}) \rightarrow \nu(A) > 0$ as $U \rightarrow e$.

Suppose now $\nu(B) > 0$. Then similarly for $g_U(x) = \frac{\nu(B \cap Ux)}{\nu(Ux)}$,

$$\nu(\{x \in B | g_U(x) \geq \frac{3}{4}\}) \rightarrow \nu(B) > 0 \text{ as } U \rightarrow e.$$

Choose V such that $\nu(\{f_V(x) \geq \frac{3}{4}\}) > 0$ and $\nu(\{g_V(x) \geq \frac{3}{4}\}) > 0$, and then points s_1 and s_2 respectively in each of these sets i.e. points such that

$$\nu(A \cap Vs_1) \geq \frac{3}{4}\nu(Vs_1) = \frac{3}{4}\nu(V)$$

$$\nu(B \cap Vs_2) \geq \frac{3}{4}\nu(Vs_2) = \frac{3}{4}\nu(V).$$

Set $s = s_1^{-1}s_2$. Then

$$\begin{aligned} \nu(As \cap B) &\geq \nu(As \cap B \cap \nu s_2) = \nu(As \cap \nu s_2) + \nu(B \cap \nu s_2) - \nu((As \cup B) \cap \nu s_2) \geq \\ &\geq \nu(A \cap \nu s_2 s_1^{-1}) + \nu(B \cap \nu s_2) - \nu(\nu s_2) \geq \nu(A \cap \nu s_1) + \\ &+ \nu(B \cap \nu s_2) - \nu(V) \geq \frac{1}{2}\nu(V) > 0. \end{aligned}$$

Hence if $\mu(B) > 0$, then $\nu(B) > 0$, and we have shown above that there is an $s \in G$ such that $\nu(B \cap As) > 0$, thus $\mu(B \cap As) > 0$. The first implication is proved. For the second implication, we use that for E Borel, $\mu(E) > 0$ iff $\mu(E^{-1}) > 0$. Then $\mu(B \cap sA) = 0$, for every $s \in G$, implies $\mu(B^{-1} \cap A^{-1}s^{-1}) = 0$, for all $s \in G$, and by the first part $\mu(B^{-1}) = 0$, hence $\mu(B) = 0$.

LEMMA 2.4. If $F \in \mathfrak{F}$ there exists a non-null measurable set E such that $\chi_E \in \mathfrak{F}$. Moreover for every measurable set A , $\mu(A) > 0$, there is a non-null measurable $E' \subset A$ such that $\chi_{E'} \in \mathfrak{F}$.

PROOF. By (Z), $F \neq \{0\}$; then there is $\phi \in F$, $\phi \neq 0$. Hence there is a $\sigma > 0$ such that $E = \{t \mid |\phi(t)| \geq \sigma\}$ has positive measure. Then $0 \leq \chi_E \leq \sigma^{-1}|\phi|$ and (F) implies $\chi_E \in \mathfrak{F}$. Since $\mu(A) > 0$, by Lemma 2.3, there is $x \in G$ such that $\mu(A \cap xE) > 0$, and since $\chi_{xE} = x\chi_E \in \mathfrak{F}$ by (T) and $\chi_{A \cap xE} \leq \chi_{xE}$, then $\chi_{A \cap xE} \in \mathfrak{F}$. Take $E' = A \cap xE$.

THEOREM 2.5. Let Ξ be a non-void index set.

(1) If $F_\xi \in \mathfrak{F}$, for every $\xi \in \Xi$, then either $\bigwedge F_\xi = \{0\}$ or $\bigwedge F_\xi \in \mathfrak{F}$. In particular, if Ξ is finite $\bigwedge F_\xi \neq \{0\}$.

(2) If $F_\xi \in \mathfrak{F}$ for every $\xi \in \Xi$, and if $\bigvee F_\xi$ exists - in particular if Ξ finite-, then $\bigvee F_\xi \in \mathfrak{F}$.

PROOF. (1) By Theorem 1.8 $\bigwedge_{\mathbb{F}_\xi \in \mathfrak{F}}$, i.e. satisfies (N) and (F). It is clear that $\bigwedge_{\mathbb{F}_\xi}$ satisfies (T). Thus if $\bigwedge_{\mathbb{F}_\xi} \neq \{0\}$, we have $\bigwedge_{\mathbb{F}_\xi} \in \mathfrak{F}$.

For the case of finite Ξ , we may just consider two spaces \mathbb{F}_1 and $\mathbb{F}_2 \in \mathfrak{F}$. By Lemma 2.4, there are sets E_1, E_2 of positive measure, such that $\chi_{E_1} \in \mathbb{F}_1$ and $\chi_{E_2} \in \mathbb{F}_2$. By Lemma 2.3, there is $x \in G$ such that $\mu(E_2 \cap xE_1) > 0$. Then $0 \neq \chi_{E_2 \cap xE_1} \leq \chi_{E_2} \in \mathbb{F}_2$ and also $0 \neq \chi_{E_2 \cap xE_1} \leq \chi_{xE_1} \in \mathbb{F}_1$. Therefore $\mathbb{F}_1 \wedge \mathbb{F}_2 \neq \{0\}$.

(2) By Theorem 1.8, if $\bigvee_{\mathbb{F}_\xi}$ exists, it belongs to \mathfrak{F} . It clearly satisfies (Z).

Let $\varphi \in \bigvee_{\mathbb{F}_\xi}$ and $s \in G$ be given. If $\varphi = \sum \varphi_\xi$, $\varphi_\xi \in \mathbb{F}_\xi$, finite sum, then $s\varphi = \sum s\varphi_\xi \in \bigvee_{\mathbb{F}_\xi}$ and $\|s\varphi\|_{\bigvee_{\mathbb{F}_\xi}} \leq \sum_{\xi} \|s\varphi_\xi\|_{\mathbb{F}_\xi} = \sum \|\varphi_\xi\|_{\mathbb{F}_\xi}$ and taking infimum over all possible finite sums such that $\varphi = \sum \varphi_\xi$, $\|s\varphi\|_{\bigvee_{\mathbb{F}_\xi}} \leq \|\varphi\|_{\bigvee_{\mathbb{F}_\xi}}$. Hence $\bigvee_{\mathbb{F}_\xi}$ satisfies (T).

THEOREM 2.6. If $\mathbb{F} \in \mathfrak{F}$, then $b\mathbb{F} \in \mathfrak{F}$.

PROOF. We use Theorem 1.1 to prove that $b\mathbb{F}$ satisfies (T), which suffices since by Theorem 1.9, $b\mathbb{F} \in \mathfrak{F}$, and $b\mathbb{F} \geq \mathbb{F} \neq \{0\}$.

Let $\varphi \in b\mathbb{F}$ and $s \in G$ be given. For every \mathbb{F} -Cauchy sequence (φ_n) with \mathbb{L} -limit φ we have $s\varphi_n \in \mathbb{F}$, for all n , and $\|s\varphi_n - s\varphi_m\|_{\mathbb{F}} \leq \|s(\varphi_n - \varphi_m)\|_{\mathbb{F}} \leq \|\varphi_n - \varphi_m\|_{\mathbb{F}}$ and $\lim_{n \rightarrow \infty} s\varphi_n = s\varphi$. Thus $(s\varphi_n)$ is an \mathbb{F} -Cauchy sequence with \mathbb{L} -limit $s\varphi$ so that $s\varphi \in b\mathbb{F}$ and

$$\|s\varphi\|_{b\mathbb{F}} \leq \lim_{n \rightarrow \infty} \|s\varphi_n\|_{\mathbb{F}} \leq \lim_{n \rightarrow \infty} \|\varphi_n\|_{\mathbb{F}}.$$

Taking infimum of the last member for all \mathbb{F} -Cauchy sequences with \mathbb{L} -limit ϕ we obtain $\|\phi\|_{b\mathbb{F}} \leq \|\phi\|_{b\mathbb{F}}$. Therefore $b\mathbb{F} \in \mathfrak{J}$.

THEOREM 2.7. If $\mathbb{F} \in \mathfrak{J}$, then $lc \mathbb{F} \in \mathfrak{J}$.

PROOF. By Corollary 1.4, $lc \mathbb{F} \in \mathfrak{J}$, and since $lc \mathbb{F} \geq \mathbb{F} \neq \{0\}$, it remains to prove that $lc \mathbb{F}$ satisfies (T). This is done using Theorem 1.3 in a way similar to the proof of Theorem 2.6.

LEMMA 2.8. If $\mathbb{F} \in \mathfrak{J}$ then \mathbb{F} is stronger than \mathbb{M} .

PROOF. Let $\phi \in \mathbb{F}$ and $s \in G$ be given. Then $s\phi \in \mathbb{F}$ and

$$\begin{aligned} \int_{sK} |\phi(t)| d\mu(t) &= \int_K |\phi(s^{-1}t)| d\mu(s^{-1}t) = \int_K |s\phi(t)| d\mu(t) \leq \\ &\leq \alpha_K \|s\phi\|_{\mathbb{F}} \leq \alpha_K \|\phi\|_{\mathbb{F}}, \quad \text{since } \mathbb{F} \text{ satisfies (N) and (T)}. \end{aligned}$$

Thus $\phi \in \mathbb{M}$ and $\|\phi\|_{\mathbb{M}} \leq \alpha_K \|\phi\|_{\mathbb{F}}$.

REMARK. The preceding lemma shows that \mathbb{M} is the weakest element of the class \mathfrak{J} .

3. Very strong spaces in \mathfrak{F}

Our next purpose is to determine a class of "very strong" spaces in \mathfrak{F} , i.e. the strongest spaces in \mathfrak{F} containing one given nonzero function. By Lemma 2.8 any such given function must belong to \tilde{M} . By (F) the desired space must then contain the absolute value of the function, by (T) all translates of the absolute value and finite linear combinations, as well as measurable functions dominated, in absolute value, by any such linear combination. We shall show that only these functions are necessary.

Let then $\varphi \in \tilde{M}$, $\varphi \neq 0$ be given. We consider the vector space S_{φ} consisting of all measurable real-valued functions ψ on G such that

$$(3.1) \quad |\psi| \leq \sum_1^n a_i |\tau_i \varphi| = \sum_1^n a_i \tau_i |\varphi|$$

for some non-negative integer n , a finite sequence (a_i) of real numbers, $a_i \geq 0$, $i = 1, \dots, n$, and a finite sequence (τ_i) , of point G , endowed with the norm $\|\psi\|_{S_{\varphi}} = \inf \sum_1^n a_i$, where the infimum is taken over all choices of n , (a_i) , (τ_i) satisfying (3.1). The expression is obviously a seminorm, and, it is easily seen to be a norm; indeed, every element in S_{φ} belongs to \tilde{M} and

$$(3.2) \quad \|\psi\|_{\tilde{M}} \leq \|\varphi\|_{\tilde{M}} \sum_1^n a_i$$

which implies $\|\psi\|_{\mathcal{S}_\varphi} \geq \frac{\|\psi\|_M}{\|\varphi\|_M}$, as desired. In particular $\varphi \in \mathcal{S}_\varphi$

($n=1$, $a_1=1$, $\tau_1=e$) with $\|\varphi\|_{\mathcal{S}_\varphi} = 1$.

Following the proofs of [7], with obvious changes in notation we have the following results which show that \mathcal{S}_φ is the desired space.

LEMMA 3.1. $\mathcal{S}_\varphi \in \mathfrak{J}$ for every $\varphi \in M$, $\varphi \neq 0$ (cf. [7] Lemma 4.3).

THEOREM 3.2. If $F \in \mathfrak{J}$ and $\varphi \in F$, $\varphi \neq 0$, then $\|\varphi\|_F^{-1} \mathcal{S}_\varphi \leq F$; hence \mathcal{S}_φ is stronger than F (cf. [7] Theorem 4.8).

COROLLARY 3.3. If $F \in \mathfrak{J}$, then $F = \bigvee \{ \|\varphi\|_F^{-1} \mathcal{S}_\varphi \mid \varphi \in F \}$ (cf. [7], Corollary 4.5).

Clearly, if $0 \neq \varphi, \varphi' \in M$, then $|\varphi| \leq |\varphi'|$ implies $\mathcal{S}_\varphi \leq \mathcal{S}_{\varphi'}$. Because of this fact we are able to consider, among the spaces \mathcal{S}_φ , a subclass of spaces which is still "strongest", in the sense that it contains spaces stronger than any given space in \mathfrak{J} .

For every bounded measurable set $E \subset G$, $\mu(E) > 0$, we write $\mathcal{S}_E = \mathcal{S}_{\chi_E}$. In particular we write $\mathcal{S} = \mathcal{S}_K$.

THEOREM 3.4. If $F \in \mathfrak{J}$, there is a bounded measurable nonnull set $E \subset G$, such that \mathcal{S}_E is stronger than F .

PROOF. By Lemma 2.4 there is a measurable nonnull set $E \subset G$, which without loss of generality we may assume bounded, such that $\chi_E \in F$. By Theorem 3.2, \mathcal{S}_E is stronger than F .

We want now to identify the strongest among all complete

spaces in \mathfrak{J} . Clearly the spaces bS_{φ} are the ones we need (see Theorem 3.6 below), but we shall give a direct description of these spaces, similar to the definition of S_{φ} .

Let $\varphi \in \underline{M}$, $\varphi \neq 0$, be given. Consider the vector space T_{φ} of all measurable real-valued functions ψ on G such that

$$(3.3) \quad |\psi| \leq \sum_1^{\infty} a_i \tau_i |\varphi|$$

for some sequence $(a_i)_{i \in \mathbb{N}}$, $a_i \geq 0$, such that $\sum_1^{\infty} a_i < \infty$, and a sequence $(\tau_i)_{i \in \mathbb{N}}$ in G , endowed with norm $\|\varphi\|_{T_{\varphi}} = \inf \sum_1^{\infty} a_i$ where the infimum is taken over all possible choices of sequences (a_i) , (τ_i) satisfying (3.3).

Using the fact that \underline{M} is complete, we see that $\psi \in \underline{M}$,

$$\|\psi\|_{T_{\varphi}} \geq \frac{\|\psi\|_{\underline{M}}}{\|\varphi\|_{\underline{M}}}, \text{ and hence } \|\cdot\|_{T_{\varphi}} \text{ is a norm. As above it follows}$$

that $\varphi \in T_{\varphi}$ and $\|\varphi\|_{T_{\varphi}} = 1$. We also write $T_E = T_{\chi_E}$, for every bounded measurable non-null set $E \subset G$, and $T = T_K$.

THEOREM 3.5. For every $\varphi \in \underline{M}$, $\varphi \neq 0$, $T_{\varphi} \in \mathfrak{J}$ and $T_{\varphi} = bS_{\varphi}$, hence T_{φ} is a Banach space (cf. [7] Theorem 4.10).

Results analogous to Theorems 3.2 and 3.4 also hold.

THEOREM 3.6. If $\underline{F} \in \mathfrak{J}$ is a Banach space and $\varphi \in \underline{F}$, $\varphi \neq 0$, then $\|\varphi\|_{\underline{F}}^{-1} T_{\varphi} \leq \underline{F}$. Hence T_{φ} is stronger than \underline{F} . Moreover $\underline{F} = \bigvee \{ \|\varphi\|_{\underline{F}}^{-1} T_{\varphi} \mid \varphi \in \underline{F} \}$ (cf. [7] Theorem 4.11).

THEOREM 3.7. If $F \in \mathcal{J}$ is a Banach space there exists a bounded measurable nonnull set $E \subset G$ such that \mathcal{T}_E is stronger than F . (Cf. [7] Theorem 4.12).

Another property we shall need in the sequel is

THEOREM 3.8. For any bounded measurable non-null set E , \mathcal{S}_E is L -completable and \mathcal{T}_E is its L -completion. (Cf. [7], Theorem 4.13).

For the relative strength of the various \mathcal{S}_E for different E , we have

LEMMA 3.9. Let E, E' be bounded measurable non-null sets in G . Then \mathcal{S}_E [resp. \mathcal{T}_E] is stronger than $\mathcal{S}_{E'}$ [resp. $\mathcal{T}_{E'}$] if and only if there is a finite set F such that $E \subset FE'$ except for perhaps a null set (cf. [7] Lemma 4.4).

COROLLARY 3.10. For any bounded measurable non-null set E \mathcal{S}_E [resp. \mathcal{T}_E] is stronger than \mathcal{S} [resp. \mathcal{T}] (cf. [7] Corollary 4.6).

Therefore \mathcal{S} is the weakest space of the type \mathcal{S}_E (and \mathcal{T} is the weakest space of the type \mathcal{T}_E) up to norm-equivalence. Our main objective is to show that \mathcal{T} is quasi locally closed. For this we now need to establish a covering property that is satisfied by all locally compact groups. In [1], Theorem 2.1.2, Emerson and Greenleaf have proved that all locally compact groups satisfy the following covering property:

(C) For at least one relatively compact set C with non-empty interior, there is an indexed family $(x_\alpha)_{\alpha \in J}$ in G such that $\{Cx_\alpha\}$ is a covering for G whose covering index at each point g (the number of $\alpha \in J$ with $g \in Cx_\alpha$) is uniformly bounded throughout G .

Moreover, in the same paper [1], Lemma 2.1.1, it is shown that in property (C) we may replace, "For at least one..." with, "For every...". It is clear that, by using the inversion symmetry $x \mapsto x^{-1}$, we may verify that the same property holds for coverings by left translates of every relatively compact C with non-empty interior. Following Emerson's and Greenleaf's arguments we prove

LEMMA 3.11. Let G be a locally compact group and let C be a symmetric compact neighborhood of $e \in G$. Let $(x_\alpha)_{\alpha \in J}$ be an indexed family such that $\{x_\alpha C\}$ is a covering of G with covering index uniformly bounded throughout G . Then $\{x_\alpha C^2\}$ is also a covering of G with uniformly bounded covering index.

PROOF. Since $e \in C$, $C^2 \supset C$. Since C^2 is compact we may choose $\{g_1, g_2, \dots, g_m\} \subset G$ such that $C^2 \subset \bigcup_{i=1}^m Cg_i$.

We assert that if $\{x_\alpha C\}$ is a covering whose index is $\leq N$, then $\{x_\alpha C^2\}$ is a covering whose index is $\leq Nm$. Evidently $\bigcup_{\alpha} x_\alpha C^2 \supset \bigcup_{\alpha} x_\alpha C = G$. Also for each $i = 1, 2, \dots, m$, and for all $x \in G$, $xg_i^{-1} \in x_\alpha C$ for at most N choices of α ; this implies that $x \in \bigcup_{i=1}^m x_\alpha Cg_i$ for at most Nm choices of α .

Since $x_\alpha C^2 \subset \bigcup_{i=1}^m x_\alpha C g_i$ for all α , this implies $x \in x_\alpha C^2$ for at most Nm choices of α , as claimed.

Combining the previous results, we conclude that every locally compact group satisfies the following covering property:

(C[#]) For every compact symmetric neighborhood C of the origin $e \in G$, there is an indexed family $(x_\alpha)_{\alpha \in J}$ in G such that $\{x_\alpha C\}$ and $\{x_\alpha C^2\}$ are coverings of G whose covering index is uniformly bounded throughout G .

It is clear that if the group is also σ -compact the indexed family can be chosen countable.

We are now able to prove

THEOREM 3.12. \mathcal{T} is quasi locally closed.

PROOF. We shall exhibit a locally closed space which is norm-equivalent to \mathcal{T} . By property (C[#]) and the above observation, there is a countable family $(x_n)_{n \in \mathbb{N}}$ such that $\{x_n K\}_{n \in \mathbb{N}}$ and $\{x_n K^2\}_{n \in \mathbb{N}}$ are coverings with uniformly bounded index.

Let $\mathcal{F} = \{\varphi \mid \varphi \text{ measurable such that } \sum_{n=1}^{\infty} \text{ess. sup}_{t \in x_n K} |\varphi(t)| < \infty\}$

normed with $\|\varphi\|_{\mathcal{F}} = \sum_{n=1}^{\infty} \text{ess. sup}_{t \in x_n K} |\varphi(t)|$. It is obvious that this

is a norm and that $\mathcal{F} \in \mathfrak{F}$.

We prove first that \mathcal{F} is locally closed, using the criterion of Corollary 1.5. Let (φ_n) be a \mathcal{F} -bounded increasing sequence

of positive elements of \tilde{F} such that $\lim_{n \rightarrow \infty} \|\varphi_n\|_{\tilde{F}} = \lambda$. Let φ be the \tilde{L} -limit as well as the pointwise limit a.e. of this sequence. If $a_{mn} = \text{ess. sup}_{t \in x_n K} \varphi_m(t)$, the sequence (a_{mn}) , for each n fixed, is nondecreasing and bounded by λ . Denote by a_n its limit.

Since φ is the pointwise limit a.e. of (φ_m) , an increasing sequence, we have $a_n = \text{ess. sup}_{t \in x_n K} \varphi(t)$. For any N , $\sum_{n=1}^N a_n =$
 $= \lim_{m \rightarrow \infty} \sum_{n=1}^N a_{mn} \leq \lambda$. Hence $\sum_{n=1}^{\infty} a_n \leq \lambda$. Thus, $\varphi \in \tilde{F}$ and $\|\varphi\|_{\tilde{F}} \leq \lambda$.

Therefore, \tilde{F} is locally closed.

We prove now that \tilde{T} and \tilde{F} are norm-equivalent. Let $\varphi \in \tilde{F}$. Let $a_n = \text{ess. sup}_{t \in x_n K} |\varphi(t)|$. Then $|\varphi| \leq \sum_{n=1}^{\infty} a_n \chi_{x_n K}$; hence $\varphi \in \tilde{T}$ and $\|\varphi\|_{\tilde{T}} \leq \|\varphi\|_{\tilde{F}}$.

For the converse, let $\varphi \in \tilde{T}$ and assume $|\varphi| \leq \sum_{i=1}^{\infty} a_i \chi_{s_i K}$, for some sequences (a_i) and (s_i) . We remark that $\{x_n K^2\}$ is a covering whose covering index is $\leq N$ throughout G if and only if for all $t \in G$, $tK \cap x_n K \neq \emptyset$ for at most N choices of n .

Assume $t \in x_n K$. Then

$$|\varphi(t)| \leq \sum_i \{a_i | t \in s_i K\} \leq \sum_i \{a_i | x_n K \cap s_i K \neq \emptyset\}.$$

Then

$$\text{ess. sup}_{t \in x_n K} |\varphi(t)| \leq \sum_i \{a_i | x_n K \cap s_i K \neq \emptyset\}.$$

Hence

$$\begin{aligned}
|\varphi|_{\tilde{F}} &\leq \sum_{n=1}^{\infty} \sum_i \{a_i |x_n^K \cap s_i^K \neq \emptyset\} = \\
&= \sum_{i=1}^{\infty} a_i \text{card}\{n | x_n^K \cap s_i^K \neq \emptyset\} \leq N \sum_1^{\infty} a_i.
\end{aligned}$$

Consequently $|\varphi|_{\tilde{F}} \leq N|\varphi|_{\tilde{T}}$, which ends the proof.

We shall give a sufficient condition for \tilde{T} to be locally closed. For this purpose we now define a property (P) that may or may not be satisfied by K .

(P) For each $\rho > 1$ and each $\varphi \in \tilde{S} = \tilde{S}_K$, there is a positive integer $N = N(\rho, \varphi)$ such that for every function ψ , $0 \leq \psi \leq |\varphi|$, there is a finite set $\Xi \subset G$ of cardinality at most N and a positive real-valued function $(a_{\xi})_{\xi \in \Xi}$ satisfying

$$i) \quad 0 \leq \psi \leq \sum_{\xi \in \Xi} a_{\xi} \chi_{\xi K}$$

$$ii) \quad \sum_{\xi \in \Xi} a_{\xi} \leq \rho |\varphi|_{\tilde{S}}.$$

We do not know whether or not this property (P) is satisfied by all G and all K .

THEOREM 3.13. Let $\{\varphi_n\} \subset G$ be such that $0 \leq \varphi_n \uparrow \varphi \in \tilde{S}$. If K satisfies (P), then $\lim_{n \rightarrow \infty} |\varphi_n|_{\tilde{S}} = |\varphi|_{\tilde{S}}$.

PROOF. By (P), we can choose, for each $n \in \mathbb{N}$ and each $\rho > 1$, coefficients $a_{in} \geq 0$ and points $\xi_{in} \in G$, $i = 1, \dots, N$; $N = N(\rho, \varphi)$ such that

$$(3.4) \quad 0 \leq \varphi_n \leq \sum_{i=1}^N a_{in} \xi_{in} \chi_K = \sum_{i=1}^N a_{in} \chi_{\xi_{in} K}$$

$$(3.5) \quad \sum_{i=1}^N a_{in} \leq \rho \|\varphi_n\|_S.$$

Without loss of generality we may consider $\text{supp } \varphi$ to be compact, and it is also possible to consider that for every i and every n ξ_{in} is contained in a fixed compact set C (e.g. $C = (\text{supp } \varphi)K$). In $[0, \rho \|\varphi\|_S]^N \times C^N \subset \mathbb{R}^N \times G^N$, a compact space with the topology induced by the product topology, the sequence $((a_{1n}, \dots, a_{Nn}, \xi_{1n}, \dots, \xi_{Nn}))_{n \in \mathbb{N}}$ has a cluster point, say $(a_1, \dots, a_N, \xi_1, \dots, \xi_N)$.

We claim that the following inequality holds

$$(3.6) \quad \liminf_n \left(\sum_{i=1}^N a_{in} \xi_{in} \chi_K \right) \leq \sum_{i=1}^N a_i \xi_i \chi_K.$$

Since $(a_1, \dots, a_N, \xi_1, \dots, \xi_N)$ is a cluster point of the above sequence, there is a subnet converging to it, say $((a_{1n(\alpha)}, \dots, a_{Nn(\alpha)}, \xi_{1n(\alpha)}, \dots, \xi_{Nn(\alpha)}))_{\alpha \in A}$. In particular, for every $i = 1, \dots, N$, $a_{in(\alpha)} \xrightarrow{\alpha} a_i$ in \mathbb{R} and $\xi_{in(\alpha)} \xrightarrow{\alpha} \xi_i$ in $C \subset G$.

We prove now that for any index i , $i = 1, \dots, N$; if $t \notin \text{Fr}(\xi_i K)$ then $\lim_{\alpha} \xi_{in(\alpha)} \chi_K(t) = \xi_i \chi_K(t)$.

If $t \in (\xi_i K)^{\circ} = \xi_i K^{\circ}$, then there exists a neighborhood U of e such that $Ut \subset \xi_i K$. Since $\xi_{in(\alpha)} \xrightarrow{\alpha} \xi_i$, there is α_0 such that, for all $\alpha \succ \alpha_0$, $\xi_i (\xi_{in(\alpha)})^{-1} \in U$. Then for all

$\alpha > \alpha_0$, $\xi_i(\xi_{in(\alpha)})^{-1}t \in \xi_i K$, and consequently $t \in \xi_{in(\alpha)} K$.

Therefore $\lim_{\alpha} \xi_{in(\alpha)} \chi_K(t) = 1 = \xi_i \chi_K(t)$.

Similarly, if $t \notin \xi_i K$, there is a neighborhood U of e such that $Ut \subset G \setminus \xi_i K$. Since $\xi_{in(\alpha)} \rightarrow \xi_i$, there is α_0 such that $\xi_i(\xi_{in(\alpha)})^{-1} \in U$ for every $\alpha > \alpha_0$. Hence $\xi(\xi_{in(\alpha)})^{-1}t \in G \setminus \xi_i K = \xi_i(G \setminus K)$ for all $\alpha > \alpha_0$, and therefore $t \in \xi_{in(\alpha)}(G \setminus K) = G \setminus \xi_{in(\alpha)} K$. Thus $\lim_{\alpha} \xi_{in(\alpha)} \chi_K(t) = 0 = \xi_i \chi_K(t)$.

From here we deduce that for every $t \notin \bigcup_{i=1}^N \text{Fr}(\xi_i K)$,

$$\sum_{i=1}^N a_{in(\alpha)} \xi_{in(\alpha)} \chi_K(t) \xrightarrow{\alpha} \sum_{i=1}^N a_i \xi_i \chi_K(t).$$

Since $\mu(\bigcup_{i=1}^N \text{Fr}(\xi_i K)) = 0$ on account of the regularity of the Haar measure, this implies in particular that for almost

all t , $\sum_{i=1}^N a_i \xi_i \chi_K(t)$ is a cluster point of the sequence

$$\left(\sum_{i=1}^N a_{in} \xi_{in} \chi_K(t) \right)_{n \in \mathbb{N}}.$$

Hence for almost all t

$$\liminf_n \left(\sum_{i=1}^N a_{in} \xi_{in} \chi_K(t) \right) \leq \sum_{i=1}^N a_i \xi_i \chi_K(t)$$

and (3.6) is proved.

From (3.4) and (3.6) we deduce

$$(3.7) \quad 0 \leq \varphi = \lim_{n \rightarrow \infty} \varphi_n \leq \liminf_n \left(\sum_{i=1}^N a_{in} \xi_{in} \chi_K \right) \leq \sum_{i=1}^N a_i \xi_i \chi_K.$$

Since $\sum_{i=1}^N a_i$ is a cluster point of the sequence $\left(\sum_{i=1}^N a_{in} \right)_{n \in \mathbb{N}}$, (3.5) implies

$$(3.8) \quad \sum_{i=1}^N a_i \leq \limsup_n \left(\sum_{i=1}^N a_{in} \right) \leq \rho \lim_{n \rightarrow \infty} \|\varphi_n\|_{\mathfrak{S}}.$$

From (3.7) and (3.8), it follows that

$$\|\varphi\|_{\mathfrak{S}} \leq \sum_{i=1}^N a_i \leq \rho \lim_{n \rightarrow \infty} \|\varphi_n\|_{\mathfrak{S}}.$$

Since $\rho > 1$ is arbitrary, $\|\varphi\|_{\mathfrak{S}} \leq \lim_{n \rightarrow \infty} \|\varphi_n\|_{\mathfrak{S}}$. The reverse inequality follows from (F), since $\mathfrak{S} \in \mathfrak{F}$.

THEOREM 3.14. If K satisfies (P), then $\mathfrak{T} = \mathfrak{T}_K$ is locally closed.

PROOF. Let (φ_n) be a \mathfrak{T} -bounded increasing sequence of positive elements of \mathfrak{T} and φ its \mathfrak{L} -limit. By Theorems 3.12 and 1.6, $\varphi \in \mathfrak{T}$. For every $\varepsilon > 0$, by Theorems 3.8 and 1.2 there is $\varphi' \in \mathfrak{S}$ such that $\|\varphi - \varphi'\|_{\mathfrak{T}} \leq \varepsilon$. Set $\psi = \inf\{\varphi, |\varphi'|\}$. Since $0 \leq \psi \leq |\varphi'|$, $\psi \in \mathfrak{S}$. Since $|\varphi - \psi| = \sup\{0, \varphi - |\varphi'|\} \leq |\varphi - \varphi'|$, $\|\varphi\|_{\mathfrak{T}} - \varepsilon \leq \|\varphi\|_{\mathfrak{T}} - \|\varphi - \varphi'\|_{\mathfrak{T}} \leq \|\varphi\|_{\mathfrak{T}} - \|\varphi - \psi\|_{\mathfrak{T}} \leq \|\psi\|_{\mathfrak{T}} = \|\psi\|_{\mathfrak{S}}$.

Define $\psi_n = \inf\{\psi, \varphi_n\}$. Then $0 \leq \psi_n \leq \psi$; therefore $\psi_n \in \mathfrak{S}$.

Also $0 \leq \psi_n \leq \varphi_n$; therefore $\|\psi_n\|_{\mathfrak{S}} = \|\psi_n\|_{\mathfrak{T}} \leq \|\varphi_n\|_{\mathfrak{T}}$. Finally $\psi - \psi_n = \sup\{0, \psi - \varphi_n\} \leq \varphi - \varphi_n$. Thus (ψ_n) is an \mathfrak{S} -bounded increasing sequence of positive terms in \mathfrak{S} with $\psi_n \in \mathfrak{S}$. Since K satisfies (P), $\|\psi\|_{\mathfrak{S}} = \lim_{n \rightarrow \infty} \|\psi_n\|_{\mathfrak{S}}$. Then $\|\varphi\|_{\mathfrak{T}} - \varepsilon \leq \|\psi\|_{\mathfrak{S}} = \lim_{n \rightarrow \infty} \|\psi_n\|_{\mathfrak{S}} \leq \lim_{n \rightarrow \infty} \|\varphi_n\|_{\mathfrak{T}}$.

Since ε was arbitrary, Corollary 1.5 allows us to conclude that \mathfrak{T} is locally closed.

We devote the remaining of the section to an important particular case in which K satisfies (P) and, hence, \tilde{T} is locally closed; namely, when $G = \mathbb{R}^d \times H$, d finite, H a compact group, G with the product group structure, $K = K_0 \times H$ where K_0 is a compact symmetric convex polytope in \mathbb{R}^d with 0 as an interior point. We have not been able to obtain a similar result when K_0 is an arbitrary (compact symmetric) convex set in \mathbb{R}^d .

The importance of this particular case derives from the fact that every connected locally compact Abelian group is topologically isomorphic to a group $\mathbb{R}^d \times H$, d finite, with H a connected compact Abelian group. We also remark that for any locally compact Abelian group, the connected component of e in the group is a subgroup, topologically isomorphic to a group of this type, and the group can be written as a disjoint union of translates of this component (cf. [3], Theorem 9.14); because of these facts, the results we are going to prove may be applied to any locally compact Abelian group. We omit the details of the argument.

LEMMA 3.15. Let K_0 be a compact convex symmetric polytope in \mathbb{R}^d , supported by linear forms f^1, \dots, f^n , i.e.

$$K_0 = \{x = (x^1, \dots, x^d) \in \mathbb{R}^d \mid -1 \leq f^i(x) \leq 1, i=1, \dots, n\}.$$

Let $\varphi = \sum_{\xi \in \Xi} a_\xi \chi_{\xi + K_0}$, Ξ finite, contained in \mathbb{R}^d , $a_\xi \geq 0$.

For any positive integer r , φ may be majorized by a linear combination (with non-negative coefficients) of $(2r)^n + p$

characteristic functions of translates of K_0 , whose coefficients add up to at most $(\sum_{\xi \in \Xi} a_\xi)(1 + \frac{2np}{r})$; here p is the least positive integer such that $\text{supp } \varphi$ is covered, except for perhaps a null set, by p translates of K_0 .

PROOF. Let r be an arbitrary positive integer. Consider now the functions $\psi^i : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$(3.9) \quad \psi^i(t) = \sum_{\xi \in \Xi} \{a_\xi | f^i(\xi) < t\}; \quad i = 1, \dots, n.$$

These step functions ψ^i are nonnegative, left continuous, nondecreasing, 0 for large negative t , and constant $(= \sum_{\xi \in \Xi} a_\xi)$ for large positive t .

We now define the following numbers

$$(3.10) \quad \begin{aligned} t_j^i &= \max\{t | \psi^i(t) \leq \frac{j}{r} \sum_{\xi \in \Xi} a_\xi\} & i = 1, \dots, n \\ & & j = 0, \dots, r-1 \\ t_r^i &= \infty & i = 1, \dots, n. \end{aligned}$$

The definition of t_r^i is consistent with those of t_j^i , $0 \leq j \leq r-1$, by the above mentioned properties of ψ^i .

Let us also remark that for every $i = 1, \dots, n$; $j = 0, \dots, r-1$,

$$(3.11) \quad \psi^i(t_{j+1}^i) - \psi^i(t_j^i + 0) \leq \frac{1}{r} \sum_{\xi \in \Xi} a_\xi \quad \text{if } t_j^i < t_{j+1}^i.$$

With the use of these numbers, we define the following, possibly empty, subsets of \mathbb{R}^d :

$$\begin{aligned} B_{2j}^i &= \{x \in \mathbb{R}^d | f^i(x) = t_j^i\} \\ B_{2j+1}^i &= \{x \in \mathbb{R}^d | t_j^i < f^i(x) < t_{j+1}^i\} \quad i = 1, \dots, n; \quad j = 0, \dots, r-1. \end{aligned}$$

Consider now the class $\mathcal{B} = \{B_{k_1}^1 \cap \dots \cap B_{k_n}^n \mid 0 \leq k_1, \dots, k_n \leq 2r-1\}$ of (possibly empty) subsets of \mathbb{R}^d . This class of sets is disjoint and has at most $(2r)^n$ elements; and every $\xi \in \Xi$ is contained in one, and only one, set of \mathcal{B} . We select an arbitrary point in each one of the at most $(2r)^n$ nonempty sets of \mathcal{B} and define a map $\tau : \Xi \rightarrow \mathbb{R}^d$ by letting $\tau(\xi)$ be the chosen point of the set in \mathcal{B} to which ξ belongs. We remark that the set $\{\tau(\xi) \mid \xi \in \Xi\}$ has cardinality at most $(2r)^n$.

We consider now a new function $\varphi' = \sum_{\xi \in \Xi} a_\xi \chi_{\tau(\xi) + K_0}$, a linear combination of at most $(2r)^n$ distinct characteristic functions of translates of K_0 .

We want to obtain a bound for $|\varphi' - \varphi|$ on \mathbb{R}^d . Let $x \in \mathbb{R}^d$ be given. Since $\varphi'(x) - \varphi(x) = \sum_{\xi \in \Xi} a_\xi (\chi_{\tau(\xi) + K_0}(x) - \chi_{\xi + K_0}(x))$ we have to determine those ξ for which $\chi_{\tau(\xi) + K_0}(x)$ and

$\chi_{\xi + K_0}(x)$ differ (and therefore differ by 1).

The only such ξ are at most those for which, for some i , $f^i(\xi) \neq f^i(\tau(\xi))$, and $f^i(\xi)$ and $f^i(x) + 1$, or $f^i(\xi)$ and $f^i(x) - 1$, lie in the same open interval (t_j^i, t_{j+1}^i) for some j , $0 \leq j \leq r-1$. But if, for fixed i , $t_j^i < f^i(x) + 1 < t_{j+1}^i$, or $t_j^i < f^i(x) - 1 < t_{j+1}^i$, then

$$\begin{aligned} \sum_{\xi \in \Xi} \{a_\xi \mid t_j^i < f^i(\xi) < t_{j+1}^i\} &= \psi^i(t_{j+1}^i) - \psi^i(t_j^i + 0) \leq \\ &\leq \frac{1}{r} \sum_{\xi \in \Xi} a_\xi \quad \text{by (3.11)}. \end{aligned}$$

Hence taking into account all possible contributions for

$i = 1, \dots, n$, we conclude that

$$0 \leq |\varphi'(x) - \varphi(x)| \leq \frac{2n}{r} \sum_{\xi \in \Xi} a_{\xi}.$$

There is a least positive integer p such that $\text{supp } \varphi \subset \bigcup_{i=1}^p s_i + K_0$, for some $s_i \in \mathbb{R}^d$.

Thus

$$\begin{aligned} 0 \leq \varphi \leq \varphi' + \left(\frac{2n}{r} \sum_{\xi \in \Xi} a_{\xi}\right) \sum_{i=1}^p \chi_{s_i + K_0} &= \\ &= \sum_{\xi \in \Xi} a_{\xi} \chi_{\tau(\xi) + K_0} + \left(\frac{2n}{r} \sum_{\xi \in \Xi} a_{\xi}\right) \sum_{i=1}^p \chi_{s_i + K_0}. \end{aligned}$$

The last member is the majorant required to prove this Lemma.

COROLLARY 3.16. Assume $G = \mathbb{R}^d \times H$, d finite, H a compact group, G with the product group structure. Let K_0 be as in Lemma 3.13, and $K = K_0 \times H$. Let $\varphi = \sum_{\xi \in \Xi} a_{\xi} \chi_{(\xi + K_0) \times H}$, $a_{\xi} \geq 0$, $\Xi \subset \mathbb{R}^d$, finite. For any positive integer r , φ may be majorized by a linear combination of $(2r)^n + p$ characteristic functions of translates of K , whose coefficients add up to at most $(\sum_{\xi \in \Xi} a_{\xi})(1 + \frac{2np}{r})$; where p is the least positive integer such that $\text{supp } \varphi$ is covered, except for perhaps a null set, by p translates of K .

PROOF. We remark that for $(\tau, h) \in \mathbb{R}^d \times H$ its action on

$\chi_K = \chi_{K_0 \times H}$ is $(\tau, h)\chi_K = \chi_{(\tau + K_0) \times H}$, since $hH = H$. Then, any translate of $K = K_0 \times H$ is of the form $(\tau + K_0) \times H$ for some $\tau \in \mathbb{R}^d$. Now

$$\begin{aligned} \varphi(x, h) &= \sum_{\xi \in \Xi} a_{\xi} \chi_{(\xi + K_0) \times H}(x, h) = \sum_{\xi \in \Xi} a_{\xi} \chi_{\xi + K_0}(x) \chi_H(h) = \\ &= \sum_{\xi \in \Xi} a_{\xi} \chi_{\xi + K_0}(x) =: \tilde{\varphi}(x). \end{aligned}$$

By Lemma 3.15, $\tilde{\varphi}(x)$ may be majorized by a suitable linear combination of characteristic functions of translates of K_0 , and the conclusion follows immediately from the previous inequalities and remark.

Finally we arrive at the desired result for the particular case.

THEOREM 13.17. Assume $G = \mathbb{R}^d \times H$, d finite, H a compact group, G with the product group structure. Assume $K = K_0 \times H$, K_0 a compact symmetric polytope in \mathbb{R}^d with 0 as an interior point. Then K satisfies (P) and, consequently, $\tilde{T} = \tilde{T}_K$ is locally closed.

PROOF. Let $\varphi \in \tilde{S} = \tilde{S}_K$ and $\rho > 1$ be given. Let ψ be any function, necessarily in \tilde{S} , such that $0 \leq \psi \leq |\varphi|$. By definition of \tilde{S} , there is a finite set $T \subset \mathbb{R}^d$ and a positive real valued function $(a_{\tau})_{\tau \in T}$ such that

$$(3.12) \quad 0 \leq \psi \leq \sum_{\tau \in T} a_{\tau} \chi_{(\tau + K_0) \times H}$$

$$(3.13) \quad \sum_{\tau \in T} a_{\tau} \leq \rho^{1/2} \|\varphi\|_{\tilde{S}}.$$

Let p be the least positive integer such that $\text{supp } \psi \subset \bigcup_{i=1}^p s_i K$ for some $s_i \in G$; then p depends on φ but

not on ψ . Choose now r large enough so that $1 + \frac{2np}{r} \leq \rho^{1/2}$, where n is, as in Lemma 3.15, the number of linear forms that support K_0 . We obtain the required $N = N(\rho, \varphi)$ by setting $N = (2r)^n + p$. Indeed, by Corollary 3.16, $\sum_{\tau \in T} a_\tau \chi_{(\tau + K_0) \times H}$, and hence ψ , may be majorized by a linear combination of N translates of K such that the sum of the coefficients is $\leq (\sum_{\tau \in T} a_\tau) (1 + \frac{2np}{r}) \leq \rho \|\psi\|_{\Sigma}$ by (3.13) and the choice of r . Due to the arbitrariness of ρ, φ and ψ , it follows that K satisfies (P). By Theorem 3.14, we have in this case that $\tilde{T} = \tilde{T}_K$ is locally closed.

4. Associate spaces in \mathfrak{F}

Let $\mathfrak{F} \in \mathfrak{F}$ be given. Consider the set \mathfrak{F}' of all measurable and valued functions ψ on G such that

$$(4.1) \quad \left| \int_G \varphi \psi d\mu \right| \leq k \quad \text{for all } \varphi \in \Sigma(\mathfrak{F}), \quad 0 \leq k \text{ depending on } \psi \text{ alone.}$$

\mathfrak{F}' is clearly a vector space. Since $\varphi \in \Sigma(\mathfrak{F})$ implies $|\varphi| \operatorname{sgn} \psi \in \Sigma(\mathfrak{F})$, by property (F), (4.1) implies and hence is equivalent to the (apparently stronger) condition

$$(4.2) \quad \int_G |\varphi \psi| d\mu \leq k \quad \text{for every } \varphi \in \Sigma(\mathfrak{F}).$$

We may define

$$(4.3) \quad \|\psi\|_{\mathfrak{F}'} = \sup \left\{ \left| \int_G \varphi \psi d\mu \right| \mid \varphi \in \Sigma(\mathfrak{F}) \right\} = \sup \left\{ \int_G |\varphi \psi| d\mu \mid \varphi \in \Sigma(\mathfrak{F}) \right\} < \infty.$$

$\|\cdot\|_{\mathfrak{F}'}$ is obviously a seminorm in \mathfrak{F}' . To prove that it is a norm, let $\psi \in \mathfrak{F}'$ and be $\neq 0$. By the argument of Lemma 2.4, there is a nonnull measurable set E and $\sigma > 0$ such that $\sigma \chi_E \leq |\psi|$. By the same Lemma 2.4 there is a set $F \subset G$, measurable, bounded and of positive measure such that $\chi_F \in \mathfrak{F}$. By Lemma 2.3 there is $x \in G$ such that $\mu(E \cap xF) > 0$. Since \mathfrak{F} satisfies (T) $x\chi_F \in \mathfrak{F}$ and $\|x\chi_F\|_{\mathfrak{F}} = \|\chi_F\|_{\mathfrak{F}}$. Then

$$\|\psi\|_{\mathfrak{F}'} \|x\chi_F\|_{\mathfrak{F}} \geq \int_G (x\chi_F) |\psi| d\mu \geq \sigma \int_G (x\chi_F) \chi_E d\mu \geq \sigma \mu(E \cap xF) > 0.$$

Thus \mathfrak{F}' is a normed space with norm $\|\cdot\|_{\mathfrak{F}'}$. It is called the associate space of \mathfrak{F} .

By the above definition, \mathbb{F}' is isometrically isomorphic to a linear manifold of the dual \mathbb{F}^* (continuous linear functionals) of \mathbb{F} (it will follow from Theorems 4.5 and 4.7 that it is a closed subspace), under the map from \mathbb{F}' into \mathbb{F}^* , $\psi \rightarrow \psi^*$, where $\psi^*(\varphi) = \int_G \varphi \psi d\mu$.

LEMMA 4.1. If $\mathbb{F} \in \mathfrak{J}$ is a Banach space and ψ is a measurable function such that $\int_G \varphi \psi d\mu$ exists and is finite for all $\varphi \in \mathbb{F}$, then $\psi \in \mathbb{F}'$.

PROOF. [7] Lemma 4.7. Cf. also [4] p. 8.

LEMMA 4.2. If $\mathbb{F} \in \mathfrak{J}$, then for every bounded set E we have $\chi_E \in \mathbb{F}'$.

PROOF. Since \mathbb{F} satisfies (N), it is obvious by definition of \mathbb{F}' .

LEMMA 4.3. If $\mathbb{F}, \mathbb{G} \in \mathfrak{J}$ and $\mathbb{F} \leq \mathbb{G}$, then $\mathbb{G}' \leq \mathbb{F}'$. If \mathbb{F}, \mathbb{G} are norm-equivalent so are \mathbb{F}', \mathbb{G}' .

PROOF. The first assertion is obvious from the definitions.

The second follows from the first and from $(\alpha\mathbb{F})' = \alpha^{-1}\mathbb{F}'$ for every $\alpha > 0$.

THEOREM 4.4. For every $\mathbb{F} \in \mathfrak{J}$, $\mathbb{F}' = (\text{lc } \mathbb{F})'$.

PROOF. As in [7] Theorem 4.15 with obvious changes in notation.

THEOREM 4.5. For any $\mathbb{F} \in \mathfrak{J}$, $\mathbb{F}' \in \mathfrak{J}$.

PROOF. First we prove that $\tilde{\mathbb{F}}'$ satisfies (T). Let $\tau \in G$, $\varphi \in \tilde{\mathbb{F}}$, $\psi \in \tilde{\mathbb{F}}'$ be given. Then

$$\begin{aligned} \int_G |\varphi(t)(\tau\psi)(t)| d\mu(t) &= \int_G |\varphi(t)\psi(\tau^{-1}t)| d\mu(t) = \int_G |\varphi(\tau t)\psi(t)| d\mu(t) = \\ &= \int_G |\tau^{-1}\varphi(t)\psi(t)| d\mu(t) \leq \|\psi\|_{\tilde{\mathbb{F}}'} \|\tau^{-1}\varphi\|_{\tilde{\mathbb{F}}} = \|\psi\|_{\tilde{\mathbb{F}}'} \|\varphi\|_{\tilde{\mathbb{F}}}. \end{aligned}$$

This implies $\tau\psi \in \tilde{\mathbb{F}}'$ and $\|\tau\psi\|_{\tilde{\mathbb{F}}'} \leq \|\psi\|_{\tilde{\mathbb{F}}'}$, as we see by taking supremum over $\varphi \in \Sigma(\tilde{\mathbb{F}})$. Thus $\tilde{\mathbb{F}}'$ satisfies (T).

We prove now that $\tilde{\mathbb{F}}'$ satisfies (N). We begin by showing that there is α_K such that for every $\psi \in \tilde{\mathbb{F}}'$, ψ is integrable on K and $\int_K |\psi| d\mu \leq \alpha_K \|\psi\|_{\tilde{\mathbb{F}}'}$.

By Lemma 2.4, there is a measurable set $E \subset K$ such that $\mu(E) > 0$ and $\chi_E \in \tilde{\mathbb{F}}$. For every $s \in G$, $s\chi_E \in \tilde{\mathbb{F}}$ and then

$$\int_K (s\chi_E) |\psi| d\mu \leq \int_G (s\chi_E) |\psi| d\mu \leq \|s\chi_E\|_{\tilde{\mathbb{F}}} \|\psi\|_{\tilde{\mathbb{F}}'} = \|\chi_E\|_{\tilde{\mathbb{F}}} \|\psi\|_{\tilde{\mathbb{F}}'}.$$

The set K^2 is compact, $\mu(K^2) < \infty$, hence, by Fubini's Theorem,

$$\begin{aligned} \infty > \mu(K^2) \|\chi_E\|_{\tilde{\mathbb{F}}} \|\psi\|_{\tilde{\mathbb{F}}'} &\geq \int_{K^2} d\mu(\tau) \int_K (\tau^{-1}\chi_E)(t) |\psi(t)| d\mu(t) = \\ &= \int_{K^2} d\mu(\tau) \int_K \chi_E(\tau t) |\psi(t)| d\mu(t) = \int_K |\psi(t)| d\mu(t) \left(\int_{K^2} \chi_E(\tau t) d\mu(\tau) \right) = \dots \end{aligned}$$

Since $\chi_E t^{-1}$ vanishes outside K^2 for all $t \in K$ ($(\chi_E t^{-1})(\tau) = \chi_E(\tau t) \neq 0$ iff $\tau t \in E \subset K$, and thus only if $\tau \in E t^{-1} \subset K^2$, $t \in K$), we may continue our chain of equalities:

$$\dots = \int_K |\psi(t)| d\mu(t) \left(\int_G \chi_E(\tau t) d\mu(\tau) \right) = \int_K |\psi(t)| \Delta_{\mathbb{R}}(t^{-1}) d\mu(t) \int_G \chi_E(\tau) d\mu(\tau) =$$

$$= \mu(E) \int_K \Delta_r(t^{-1}) |\psi(t)| d\mu(t) \geq \delta_K \mu(E) \int_K |\psi(t)| d\mu(t)$$

where $\delta_K = \min\{\Delta_r(t^{-1}) \mid t \in K\} > 0$, for the right-hand modular function $\Delta_r(t)$ is continuous and positive and K is compact.

Therefore we may choose $\alpha_K = \delta_K^{-1} \mu(E)^{-1} \mu(K^2) \|\chi_E\|_{\tilde{F}}$. Now for any compact C , $C \subset \bigcup_{i=1}^n \tau_i K$, for some n and some $\tau_i \in G$, $i = 1, \dots, n$. Then

$$\int_C |\psi| d\mu \leq \sum_{i=1}^n \int_{\tau_i K} |\psi| d\mu = \sum_{i=1}^n \int_K |\tau_i^{-1} \psi| d\mu \leq n \alpha_K \|\psi\|_{\tilde{F}'},$$

since \tilde{F}' satisfies (T). Thus \tilde{F}' satisfies (N).

From (4.2) and (4.3) it is obvious that \tilde{F}' satisfies (F).

From Lemma 4.2 it is clear that $\tilde{F}' \neq \{0\}$.

THEOREM 4.6. For any $F \in \mathfrak{J}$, F' is locally closed.

PROOF. [7] Theorem 4.17 (cf. [5] p. 113).

THEOREM 4.7. If $F \in \mathfrak{J}$ is locally closed, $F'' = F$.

PROOF. [7] Theorem 4.18 and Remark. It follows from a theorem of Luxemburg [4] and Luxemburg and Zaanen [5] p. 113.

COROLLARY 4.8. If $F \in \mathfrak{J}$, then $F'' = \text{lc } F$.

PROOF. Theorems 4.4 and 4.7.

THEOREM 4.9. A quasi locally closed space $F \in \mathfrak{J}$ contains the characteristic function of every bounded measurable set $E \subset G$.

PROOF. It is sufficient to assume \tilde{F} locally closed. Then by Theorems 4.5 and 4.7 and Lemma 4.2, $\chi_{E \in \tilde{F}''} = \tilde{F}$.

COROLLARY 4.10. If $F \in \mathfrak{J}$ is quasi locally closed, then \tilde{F} is weaker than \tilde{T} .

PROOF. Theorems 4.7 and 3.

Combining Theorem 3.12 and Corollary 4.10, we conclude that T is, up to norm-equivalence, the strongest quasi locally closed space in \mathfrak{J} .

We prove now:

THEOREM 4.11. $\tilde{T}' = \tilde{M}$ and $\tilde{M}' = \text{lc } \tilde{T}$. If \tilde{T} is locally closed, then $\tilde{M}' = \tilde{T}$; this is the case, in particular, if K satisfies (P), and, more in particular, if $G = \mathbb{R}^d \times H$, d finite, H a compact group, G with the product group structure, $K = K_0 \times H$, where K_0 is a compact symmetric convex polytope in \mathbb{R}^d with 0 as an interior point.

PROOF. Let $\psi \in \tilde{M}$ be given. For every $\varphi \in \tilde{T}$, we have $|\varphi| \leq \sum_{i=1}^{\infty} a_i \chi_{\tau_i K}$ for some sequences $(a_i), (\tau_i)$ with $\sum_{i=1}^{\infty} a_i < \infty$. Using (T) for \tilde{M} , we have

$$\begin{aligned} \int_G |\varphi \psi| d\mu &\leq \sum_{i=1}^{\infty} a_i \int_G (\tau_i \chi_K) |\psi| d\mu \leq \sum_{i=1}^{\infty} a_i \int_K |\tau_i^{-1} \psi| d\mu \leq \\ &\leq \sum_{i=1}^{\infty} a_i \|\tau_i^{-1} \psi\|_{\tilde{M}} \leq \left(\sum_{i=1}^{\infty} a_i \right) \|\psi\|_{\tilde{M}}. \end{aligned}$$

Taking infimum of the last member over all suitable sequences, we obtain $\int_G |\varphi \psi| d\mu \leq \|\varphi\|_{\tilde{T}} \|\psi\|_{\tilde{M}}$, so that $\psi \in \tilde{T}'$ and

$\|\psi\|_{\tilde{T}'} \leq \|\psi\|_{\tilde{M}}$. Thus $\tilde{M} \leq \tilde{T}'$.

Conversely let $\psi \in \tilde{T}'$. For every $s \in G$

$$\int_{sK} |\psi| d\mu = \int_G (\chi_{sK}) |\psi| d\mu \leq \|\chi_{sK}\|_{\tilde{T}} \|\psi\|_{\tilde{T}'} = \|\psi\|_{\tilde{T}'}$$

whence $\psi \in \tilde{M}$ and, taking supremum of the first member over all s ,

we have $\|\psi\|_{\tilde{M}} \leq \|\psi\|_{\tilde{T}'}$, which implies $\tilde{T}' \leq \tilde{M}$. Thus $\tilde{T}' = \tilde{M}$.

From Corollary 4.8, $\tilde{M}' = \tilde{T}'' = \text{lc } \tilde{T}$. In case \tilde{T} is locally closed we indeed have $\tilde{M}' = \text{lc } \tilde{T} = \tilde{T}$. The last statement of the theorem follows from Theorems 3.14 and 3.17.

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