SPACES OF CONTINUOUS FUNCTIONS INTO R^2

by

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Report 71-7

February, 1971

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If X is a compact Hausdorff space and B is a Banach space, let us denote the Banach space of continuous functions on X into B with the usual supremum norm by $C(X;B)$. The well-known Banach-Stone theorem, Day [1] asserts that if R is the real line and X,Y are compact Hausdorff spaces then the spaces $C(X;R)$ and $C(Y;R)$ are isometric if and only if X is homeomorphic with Y. The purpose of the present note is to show that the Banach-Stone type of theorem is not true if R is replaced by R^2, the two dimensional Banach space with the supremum norm.

If X and Y are two topological spaces we denote the topological sum associated with disjoint copies of X and Y by $X + Y$. For a definition of $X + Y$ and for other undefined topological terms in this note we refer to Dugundji [2]. We proceed to establish two useful lemmas,

Lemma 1, If X,Y are two compact Hausdorff spaces then $C(X;R^2)$ is isometric with $C(Y;R^2)$ if and only if $X + X$ is homeomorphic with $Y + Y_o$

Proof. It is verified that $C(X;R^2)$ is isometric with $C(YrR^2)$ if and only if $C(Xx2;R)$ is isometric with $C(YX2;R)$. Hence it follows from the Banach-Stone theorem that $C(X;R^2)$ is isometric with $C(Y?R^2)$ if and only if $X \times 2$ is homeomorphic with $Y \times 2$ i$_o$e. $X + X$ is homeomorphic with $Y + Y$.

Lemma 2_o There exist compact Hausdorff spaces X such that X is homeomorphic with $X + \{p\} + \{q\}$ while X fails to be homeomorphic with $X + \{p\}$.

Proofo Let N_i ,N_2 be the two disjoint denumerable discrete spaces {a_i|ij>l} and {bji^l}. Let f be the function defined on Nj^ U N_2 onto itself by setting f(a^ = b_i and f(b_i) = a_i for all i J> 1. Let 0N_i. be the Stone-Cech compactifications of N_i, i = 1,2. From the properties of j3N_i it follows that f|Nn extends to a homeomorphism F on /JN-. onto j8N_2* Let Y = j3N_1 ~ N^. Let X be the space ^Ni U$_F$ J$_Y$ ^N2 obtained by attaching j8N_1 to 0N_2 by F|Y. We proceed to show X has the properties stated in the lemma#

From the definition of the space X it follows that N_1 U N_2 is embedded in X as an open set. We continue to denote the copy of N_x U N_2 in X by ^ U N_2. Noting that FJY is a homeomorphism it is verified that X is Hausdorff. Further since X is the quotient of a compact space itself is compact.

Consider now the spaces X + {p} + {q} and X, Consider the map g : N_1 U N_2 + {p} + {q}-—>N_1 U N_2 by assigning g(p) = a_x, g(q) = h^ g($a_\pm$) = a_{i+1}, g{$h_\pm$) = b_{i+1} . It is verified that g extends to a homeomorphism on X + {p} + {q} onto X.

Next we proceed to show that X is not homeomorphic with X + {p}. We note that if a : X + {p} X + {p) is the map defined by a(p) = p_3 a|N_1 U N_2 = f, and a(x) = x for xeX ~ (N_1 U N_2) then a is a homeomorphism on X + {p} onto X + {p} such that a^2 is the identity map. Further from the properties of jSN^ it follows that a point XGX is isolated if and only if xeN_1 U N_2* Thus a is an involuntary automorphism on X + {p} onto X + {p} with exactly one invariant isolated point. Thus in order to verify that X and X + {p} are not homeomorphic it is sufficient to verify that there exists no involutory automorphism on X with exactly one invariant isolated point.

If possible let h : X—> X be such a involution. Without loss of generality we can assume that a_1 is the isolated point invariant under h_o Let $t^c{}_n$^nVL 'be t*le se<3uence i-n x obtained as follows. c^ = a^ $c_2 = f(c_1) = h_1$ and $c_3 = h(c_2)$. If c_1, 1 <£ i <£ n^ are already defined let $c_{n+1} = f(c_n)$ if n is odd and $c_{n+1} = h(c_n)$ if n is even. From the properties of f and h it is inferred that the range of the sequence c denoted by C is a subset of $N_1 \cup N_2$ and the sequence is injective. For convenience let us relable members of C by setting $c_{2n} = r\_n$ and $^C2_{n+i}$ ^ $£_n$+l for n = 1>2J3J**<> • Note that $f(4_n) = n_n3$ $f(rj_n) = £_n$, for all n and if n } 2 then $h(rj_n) = £_{n+1}$ and $h(\hat{}_n) = r?_{n-x}$. Thus C^^tNj) if and only if n^Njt^) and for each n, $(4_n.T?_n\} = (a_i,b_i\}$ for a suitable i depending on n. Let now P be the subset of c defined by P= {4_{3k+2}l^k^^{1}3 u $\hat{}_3$k+2l^k^^{1}}- Note that f(P) = P. Further since the sequence c is injective it follows that there is an infinite set B c N_1 n P. Let Si = U3k+2^3k+2 GB} and ^2 = fr?$_{3k+2}$|rJ$_{3k+2}$ €B}. The sets % 0 N_{15} J^ n N_2, *T2* H N_{15} L2 n N_2 partition B. Since B is an infinite set one of the sets $J_1 \cap N_j$ is infinite. As a typical case let us assume that $£_1 \cap N_1$ is infinite. Denoting the closure of a subset M in the space X by $\overline{M}$

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Repiac^ irv Irttas 1.0 and II on page 4:

"Let vis., .if $\eta_{3k+1} N_2$" by:

Let -us define a sequ^mce . t ^ i.,-.t' by assigning

$\xi_{3k} = \iota_{3k}$ if $\xi_{3k} N_1$ $\iota_{3k} = f^{-1}(\xi_{3k})$ if $\xi_{3k} N_2$

$\eta_{3k+1} = \iota_{3k+1}$ if $\eta_{3k+1} N_1$ and $\iota_{3k+1} = f^{-1}(\eta_{3k+1})$

if $\eta_{3k+1} N_2$

Ca

it follows from the definition of X that $\overline{J^\wedge \text{fl } N^\wedge} \sim (Lj\_ \text{ n } Nj) ? j6$ and $\overline{\{^\wedge \text{n } N_x)} \sim (x^\wedge \text{ n } N_x) = \overline{f(i^\wedge \text{ n } N_L)} \sim f(j^\wedge \text{ n } N_1) =$ $= \overline{(L2 \text{ n } N_2)} \sim (L2 \text{ n } N_2)$. Let htJJi n $N^\wedge = D_i$ for $i = 1,2$. Since h is a homeomorphism it follows that $\overline{D}_1 \sim D_1 = \overline{D}_2 \sim D_2 \wedge fb$. We proceed to show that this is untenable by verifying $\overline{D}_1$ $(1\overline{D}_2$- = *ft).* We note that if $x \in D_1 (D_2)$ then x is of the form $T)_{3k+1} U_{3k})$ for some k since $M4_{3k+2}) = n_{3k+1}$ and $h(T]_{3k+2}) = 4_{3(k+1)}$. Thus

$$\overline{D}_X \text{ n } \overline{D}_2 \text{ c } \overline{u_{3k} | k \wedge 1\}} \text{ n } \overline{\{T7_{3k+1} | k \geq 1\}}.$$

Let us define a sequence $\{t_{3i} | i \geq 1\}$ by assigning $£_{3k} = t_{3k}$ if $\wedge 3k+1 \in N_1$ and $f(T»3kfl) = fc_{3k+1}$ if $i3k+1 \in N_2$. Let $T_1 = \{t_{3k} | k \wedge 1\}$ and $T_2 = \{t_{3k+1} | \wedge l5\#$ Noting that the sequence c is injective, $f(C_n) = V_n$ an($\wedge$ $\wedge$ $^2=$ $\wedge$ it is at once verified that T_1 and T_2 are disjoint subsets of $N^\wedge$. Thus from the properties of βN_1 it follows that $cl_{\beta N_1} T_1 \text{ n } cl_{\beta N^\wedge} T_2 = fb$. However from the construction of the space X it is verified that $\overline{\{4_{3k} | kJ \geq l\}} = cl_{\beta N_1} T_1$ and $\overline{CT»_{3k+1} | k \wedge l\}} = cl \wedge T_2$. Hence $D]L \text{ n } D_2 \text{ c } cl \wedge T_L \text{ 0 } cl_{\beta N_1} T_2 = \theta$.

Hence X does not admit an automorphism of the type h and X is not homeomorphic with X + {p}.

Theorem. Let X be a compact Hausdorff space of the type described in lemma *2.* Let Y = X + {p}. Then $C(X?R^2)$ is isometric with $C(Y;R^2)$; however X is not homeomorphic with Y.

Since X + X = X + X + {p} + {q} = Y + Y it follows from lemma 1 that $C(X;R^2)$ is isometric with $C(Y?R^2)$. Now from lemma 2 it follows that X is not homeomorphic with Y, completing the proof of the theorem.

In conclusion it might be mentioned that the possibility of generalizing Banach-Stone theorem to certain categories of Banach spaces are discussed in Jerison [3] and Sundaresan [4]. The counter example in this note complements the theorem 5 in [4] since the unit cell of R^2 is an example of an S-cylinder.

References

1. Day, M. M., *Normed Linear Spaces*, Springer-Verlag, 1962.

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