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FUNCTIONALS ON BANACH SPACES

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Research Report 71-22

MAY, 1971

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If M is a Banach space of real valued measurable functions on a measure space and if $x, y \in M$ then x is said to be orthogonal to y in the lattice theoretic sense $(x \perp_L y)$ if the set $\{t | x(t)y(t) \neq 0\}$ is of measure zero. A real valued function F on M is said to be L-additive if it is continuous and $F(x+y) = F(x)+F(y)$ whenever $x \perp_L y$. Integral representations of L-additive functionals has been the subject of extensive study in recent years. For these and related results we refer to Drewnowskii and Orlicz [1] and Sundaresan [2] and the bibliography cited therein. The concept of orthogonality in the definition of L-additive functionals on M is very

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A.M.S. Subject Classification: Primary 4610, 4615.

Key words and phrases: Banach spaces, Hilbert spaces, Orthogonality, Additive functions.

natural in these spaces. However there are several other concepts of orthogonality which have been studied in detail in arbitrary Banach spaces, see for example James [3,4]. These concepts are generalizations of orthogonality in Eucledian spaces and are of intrinsic geometric interest. One such concept of orthogonality is as follows. If B is a real Banach space and $x,y \in B$ then x is said to be orthogonal to y, in short $x \perp y$, if $\|x+\lambda y\| \geq \|x\|$ for all real numbers λ. The purpose of the present note is to characterize continuous real valued functions F on B such that $F(x+y) = F(x) + F(y)$ whenever $x \perp y$. For a motivation of the study of such functionals it is enough to note that for $x,y \in L_p(\mu)$, $x \perp_L y$ implies $x \perp y$ while in general the implication cannot be reversed.

We discuss the results for the two cases when (1) the domain of F is a Hilbert space and (2) the domain of F is a Banach space of dimension at least 3, separately. We generalize these results to the case when F is an additive function on B taking values in a locally convex space. In the rest of the paper B stands for an arbitrary real Banach space. Before proceeding to the main results we summarize useful known results concerning the concept of orthogonality ($\perp$) described above. (1) Orthogonality is homogenous i.e., $x \perp y \Longrightarrow \lambda x \perp \mu y$ for all real numbers λ and μ. (2) Orthogonality is not symmetric in general i.e. $x \perp y$ does not necessarily imply $y \perp x$. However it is known, Day [5], that if dimension $B \geq 3$, then orthogonality is symmetric if and only if B is isometric to a Hilbert space. (3) If $x \neq 0 \neq y$ then $x \perp y$ if and only if there exists a nonzero continuous linear functional f such that $f(x) = \|f\| \, \|x\|$ and $f(y) = 0$. (4) If L is a subspace of B and $x \in B$ then let us say that $x \perp L$ if $x \perp y$ for all $y \in L$. If B is a finite dimensional or more generally if B is a reflexive Banach space and L is a proper closed subspace then there exists a vector $x \neq 0$ such that $x \perp L$. For a proof of this assertion we refer to James [4].

DEFINITION. If B is a real Banach space then a function F on B into a locally convex space E is said to be additive if $F(x+y) = F(x) + F(y)$ whenever $x \perp y$ and F is a continuous function.

THEOREM 1. If H is a Hilbert space of dimension ≥ 2, then F is an additive functional on H if and only if there is a constant c and a vector $y \in H$ such that

$$F(x) = c \, \|x\|^2 + (y, x)$$

where $(\cdot\,,\cdot)$ denotes the inner product in H.

<u>Proof</u>. Since the verification that any function F as in the theorem is an additive functional is straight forward we supply the proof for the "only if" part.

Let F be an additive functional on H. Let F_1, F_2 be respectively the symmetric and anti-symmetric parts of F i.e., F_1, F_2 are the functions on H defined by

$F_1(x) = \frac{1}{2}\,[F(x) + F(-x)]$ and $F_2(x) = \frac{1}{2}\,[F(x) - F(-x)]$.

It is at once verified that F_1, F_2 are additive functionals on H and $F_1(x) = F_1(-x)$ and $F_2(x) = -F_2(-x)$. Let $\{e^\alpha\}_{\alpha\in A}$ be a basis for H. For each $\alpha \in A$ let f_α be the continuous function on $R \to R$ defined by $f_\alpha(\lambda) = F_1(\lambda e^\alpha)$. From the additivity and continuity of F it follows that if $x = \sum_{\alpha\in A} x_\alpha\, e^\alpha$ is a vector in H then $F(x) = \sum_{\alpha\in A} f_\alpha(x_\alpha)$. We proceed to show that f_α is independent of α and $f_\alpha(\lambda) = \lambda^2\, f_\alpha(1)$ for all real numbers λ. Let $\alpha,\beta \in A$, $\alpha \neq \beta$. Let $x = \lambda(e^\alpha + e^\beta)$ and $y = \lambda(e^\alpha - e^\beta)$ where λ is an arbitrary real number. It is verified that $x \perp y$. Thus $F_1(x+y) = F_1(x) + F_1(y)$ and $F_1(x-y) = F_1(x) + F_1(-y)$. Noting that $f_\alpha(0) = 0 = F_1(0)$ and $f_\alpha(\lambda) = f_\alpha(-\lambda)$ it is seen from the preceding equations that $f_\alpha(2\lambda) = 2[f_\alpha(\lambda)+f_\beta(\lambda)] = f_\beta(2\lambda)$. Thus $f_\alpha = f_\beta$. Let us denote the functions f_α by f. From the preceding equations it follows that $f(2\lambda) = 4f(\lambda)$ for all real numbers λ. Consider now for arbitrary real numbers λ vectors x,y defined by $x = \lambda e^\alpha + \lambda^2 e^\beta$ and $y = \lambda e^\alpha - e^\beta$ where $\alpha,\beta \in A$, $\alpha \neq \beta$. Since $x \perp y$ $F_1(x+y) = F_1(x) + F_1(y)$ and $F_1(x-y) = F_1(x) + F_1(-y)$. Once again using $f(\lambda) = f(-\lambda)$, $f(2\lambda) = 4\,f(\lambda)$ and $f(0) = 0$ it follows from the preceding equations $f(\lambda^2+1) + f(\lambda^2-1) = 2[f(\lambda^2) + f(1)]$. By straightforward induction it is verified that $f(m) = m^2 f(1)$ for all integers m. Further since $f(2\lambda) = 4\,f(\lambda)$ it is verified that for integers m, n that $f(\frac{m}{2^n}) = (\frac{m}{2^n})^2\,f(1)$. Since f is a continuous function it follows that $f(\lambda) = \lambda^2 f(1)$ for all real numbers λ. Thus if $f(1) = c$ and $x = \Sigma\, x_\alpha\, e^\alpha$ is a vector in H then $F_1(x) = c \sum_{\alpha\in A} x_\alpha{}^2 = c\,\|x\|^2$. Next consider the additive functional F_2 on H. Let for real numbers λ, $F_2(\lambda e^\alpha) = g_\alpha(\lambda)$. Consider now for arbitrary non negative real numbers λ, μ vectors x,y defined by $x = \lambda e^\alpha + a e^\beta$, $y = \mu e^\alpha - a e^\beta$ where $a^2 = \lambda\mu$. It is verified that $x \perp y$. Thus $F_2(x+y) = F_2(x) + F_2(y)$. Since $g_\alpha(\lambda) = -g_\alpha(-\lambda)$ for all

$\alpha \in A$ and for real numbers λ, it follows from the preceding equation that $g_\alpha(\lambda + \mu) = g_\alpha(\lambda) + g_\alpha(\mu)$. Since $g_\alpha(\lambda) = -g_\alpha(-\lambda)$ it follows that $g_\alpha : R \to R$ is a linear function for all $\alpha \in A$. Let $x = \sum_{\alpha \in A} x_\alpha e^\alpha$ be a vector in H. Since $F_2(x) = \sum_{\alpha \in A} g_\alpha(x_\alpha)$ it is verified that F_2 is a linear functional. Since F_2 is also continuous it follows that there is a unique vector $y \in H$ such that $F_2(x) = (y,x)$ for all $x \in H$. Thus $F(x) = F_1(x) + F_2(x) = c\,\|x\|^2 + (y,x)$ where c and y are uniquely determined by F. This completes the proof of the theorem.

Before proceeding to the case of Banach spaces $B(\dim B \geq 3)$ which are not isometric with a Hilbert space let us note a couple of useful lemmas.

LEMMA 1. Let B be a two dimensional Banach space such that orthogonality is not symmetric in B. Then a functional F on B is additive if and only if F is linear.

<u>Proof</u>. Suppose $x, y \in B$ such that $x \perp y$ but y is not $\perp x$. From (3) in the introductory remarks it follows we can choose a real number $a \neq 0$ such that $y \perp ay + x$. Since for any two real numbers A, B, $\frac{B}{a}x + (A+B)y = Ay + \frac{B}{a}(ay+x)$

we have

$$F\left(\frac{B}{a}x\right) + F[(A+B)y] = F(Ay) + F\left[\frac{B}{a}(ay+x)\right]$$

$$= F(Ay) + F(By) + F\left(\frac{B}{a}x\right) .$$

Thus (*) $F[(A+B)y] = F(Ay) + F(By)$.

Since y is not orthogonal to x there is a real number $b \neq 0$ such that $bx + y \perp x$. Suppose x is not orthogonal to $bx + y$. Then it follows from the preceding argument that (**) $F[(A+B)x] = F(Ax) + F(Bx)$. If $x \perp bx+y$, since $x \perp y$ as well, it follows from the identity $(A + B)x + \frac{A}{b}y = Bx + \frac{A}{b}(bx+y)$ that $F[(A+B)x] + F(\frac{A}{b}y)$ $= F(Bx) + F[\frac{A}{b}(bx+y)] = F(Bx) + F(Ax) + F(\frac{A}{b}y)$ that $F[(A+B)x] = F(Ax) + F(Bx)$. Thus (**) is satisfied again. It follows from (*) and (**) and continuity of F that $F(\lambda x + \mu y) = \lambda F(x) + \mu F(y)$ for all real numbers λ and μ. Hence F is linear. Since a linear functional on B is additive the proof of the lemma is complete.

LEMMA 2. Suppose F is an additive functional on a real Banach space B such that F is linear on a subspace L of dimension at least 2. If $Z \in B$ such that $Z \perp L$ then F is linear on the span of Z and L.

<u>Proof</u>. We can assume that $Z \notin L$. Let u be a non-zero vector in L. Choose a such that $Z + au$ is not orthogonal to u. Since $Z + au$ is orthogonal to a maximal closed subspace of B that does not contain u, there is a real number b and a vector $v \in L$ such that $Z + au \perp Z + bu + v$. Since $u \perp cu + v$ for some c we can assume that v is so chosen that $u \perp v$. Thus

$$\begin{aligned} F[A(Z+au) + B(Z+bu+v)] &= F[A(Z+au)] + F[B(Z+bu+v)] \\ &= F(AZ) + F(BZ) + F(aAu) + F(bBu+Bv) \\ &= F(AZ) + F(BZ) + F[(aA+bB)u + Bv]. \end{aligned}$$

Also,

$$\begin{aligned} F[A(Z+au) + B(Z+bu+v)] &= F[(A+B)Z + (aA+bB)u + Bv] \\ &= F[(A+B)Z] + F[(aA+bB)u + Bv] \end{aligned}$$

so that $F[(A+B)Z] = F(AZ) + F(BZ)$ and F is linear on the span of Z and L.

THEOREM 2. Let B be a real Banach space of dimension at least 3. If B is not isometric to a Hilbert space and if F is an additive functional on B then F is a continuous linear functional.

<u>Proof</u>. Since $\dim B \geq 3$, and B is not isometric with a Hilbert space there are nonzero vectors x, y such that $x \perp y$ but y is not orthogonal to x. Let L be the two dimensional subspace spanned by x and y. From Lemma 1 it is inferred that F is linear on L. Now suppose $Z \notin L$. Then there is a vector $\xi \neq 0$, in the span of Z and L such that $\xi \perp L$. It follows from Lemma 2, F is linear on the span of Z and L. Now if $Z \in B$, and Z' is not in the span of Z and L then repeating the above argument it is verified that F is linear on the span of Z, Z' and L. Thus for arbitrary vectors Z, Z', F is linear on the span of Z, Z'. Hence, F is linear. Since by hypothesis F is continuous, F is a continuous linear functional. This completes the proof of Theorem 2.

THEOREM 3. Let F be a continuous function on a Banach space B into a locally convex space E. If B is a Hilbert space of dimension at least 2 then F is additive if and only if there exists a vector $\xi \in E$ and a continuous linear operator $T : B \to E$ such that $F(x) = \|x\|^2 \xi + T(x)$ for all $x \in E$. Further if dimension $B \geq 3$ and B is not isometric with a Hilbert space then F is a continuous linear operator on B to E.

Proof. Let B be a Hilbert space. Let E^* be the topological dual of E. Let F be a additive function on $B \to E$. Let F_1, F_2 be the symmetric and antisymmetric parts of F as defined in the proof of Theorem 1. Now if $f \in E^*$ then $f \circ F_1$, and $f \circ F_2$ are symmetric and anti-symmetric additive functionals on B. Thus there is a constant c_f and a continuous linear functional ℓ_f on B such that $f \circ F_1(x) = c_f \|x\|^2$ and $f \circ F_2(x) = \ell_f(x)$. It is verified that the mapping $f \to c_f$ is a ω^*-continuous linear functional on E^*. Hence there exists a fixed vector $\xi \in E$ such that $f(\xi) = c_f$ for all $f \in E^*$. Since E^* separates points in E it follows that $F_1(x) = \|x\|^2\xi$. Again since for each $f \in E^*$, $f \circ F_2$ is a linear functional on B it follows that F_2 is a linear operator on $B \to E$. Further since F is continuous F_2 is a continuous linear operator on $B \to E$. Considering symmetric and anti-symmetric parts of F it is verified that ξ and T are uniquely determined by F. Thus $F(x) = \|x\|^2 \xi + T(x)$ as stated in the theorem. Since any function F with the above representation is an additive function the proof of the part of the theorem dealing with the case when B is a Hilbert space is complete. The case when B is not a Hilbert space is similarly dealt and the proof is omitted.

In conclusion it might be mentioned that our investigation of additive functionals on two dimensional normed linear space is not complete. The results do not provide a characterization of additive functionals on two dimensional normed linear spaces in which orthogonality is symmetric but which is not isometric with a Hilbert space. For a general method of constructing such two dimensional norms, we refer to Day [5]. However we conclude this note stating the following conjecture. If B is a two dimensional normed linear space not isometric with the Eucledian space then every additive functional on B is linear.

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