ARITHMETICAL REDUCIBILITIES, I

by

Alan L. Selman

Report 71-6

February, 1971

/ps

ARITHMETICAL REDUCIBILITIES, I

by

Alan L. Selman

Abstract

A Σ_n-reducibility relation is defined to be a transitive and reflexive relation $\mathcal{R}$ on sets of natural numbers, so that for every two sets A and B, $A\mathcal{R}B$ implies $A \in \Sigma_n^B$. Two hierarchies of such reducibilities are studied, $\mathcal{R}_n$, $n < \omega$, and $\mathcal{S}_n$, $n < \omega$. The reducibilities of each hierarchy have natural properties not possessed by the other. Each $\mathcal{R}_n$ generalizes relative recursion; each $\mathcal{S}_n$ has the property that the class of Σ_n sets is the $\underset{\sim}{0}$ degree for the $\mathcal{S}_n$-degree structure. Various theorems concerning the structure of these reducibilities are proved.

Arithmetical Reducibilities, I[1]

by

Alan L. Selman

Introduction. Let two sets of natural numbers A, and B, be given. As is well known, A is arithmetical in B if and only if there is some n so that $A \in \Sigma_n^B$. The relation "Σ_n in" defined on the set of all subsets of ω is not transitive. On the other hand, the nice properties of relative recursion (Turing reducibility), $\leq_r$, are reflexivity and transitivity. Thus $r \cap r^{-1}$ is an equivalence relation. We are interested in this paper in easily definable subrelations of "Σ_n in" which share these properties. Thus we are led to the following definition.

Definition 0.1. If $\mathfrak{R}$ and $\mathfrak{T}$ are binary relations defined on the set of all subsets of ω, then $\mathfrak{R}$ is an $\mathfrak{T}$-reducibility relation, if $\mathfrak{R}$ is reflexive, $\mathfrak{R}$ is transitive, and for all sets A and B, if $A\mathfrak{R}B$, then $A\mathfrak{T}B$.

Let $\mathfrak{A}$ denote the relation "arithmetical in", defined by $A\mathfrak{A}B$ if and only if A is arithmetical in B. $\mathfrak{A}$ is itself reflexive and transitive. By an arithmetical reducibility we mean an $\mathfrak{A}$-reducibility relation.

If $\mathcal{I}$ is the relation "Σ_n in", then the term "Σ_n-reducibility" will be used rather than the more cumbersome "'Σ_n in'-reducibility". Similarly, the terms "Π_n-reducibility" and "Δ_n-reducibility" will be used though the relations involved are "Π_n in" and "Δ_n in".

Let $\mathcal{R}$ be a Σ_n-reducibility relation. We single out three desirable properties which we would like $\mathcal{R}$ to possess: $\mathcal{R}$ should generalize relative recursion, the class of Σ_n sets should be the $\underset{\sim}{0}$ degree for the degree structure of $\mathcal{R}$, and, finally, $\mathcal{R}$ should be as "large" in "Σ_n in" as possible, i.e., $\mathcal{R}$ should be a maximal transitive subrelation of "Σ_n in". Actually, since every set is recursive in its complement no transitive subrelation of "Σ_n in" can both generalize relative recursion and have the class of Σ_n sets as its $\underset{\sim}{0}$ degree (Theorem 1.4, below). We define two <u>sequences</u> of relations, $\mathcal{R}_n$, $n < \omega$, and $\mathcal{S}_n$, $n < \omega$, so that the reducibilities of each sequence have natural properties not possessed by the other.

<u>Definition</u> 0.2. $A \; \mathcal{R}_n \; B \leftrightarrow \forall X[X \in \Sigma_n^A \rightarrow X \in \Sigma_n^B]$, $n \geq 1$.
$A \; \mathcal{S}_n \; B \leftrightarrow \forall X[B \in \Sigma_n^X \rightarrow A \in \Sigma_n^X]$, $n \geq 1$.

Both $\mathcal{R}_n$ and $\mathcal{S}_n$ are Σ_n-reducibility relations. We will see that $\mathcal{R}_n$ generalizes relative recursion, and that the class of Σ_n sets is the $\underset{\sim}{0}$ degree for the $\mathcal{S}_n$-degree structure.

Various theorems concerning the structure of these reducibilities are proved. This paper consists of two chapters. Chapter $\underset{\sim}{1}$ of this paper, in particular, contains proofs of the following theorems:

(1) $\mathfrak{R}_n \subset \mathfrak{R}_{n+1}$, $\mathfrak{R}_n \neq \mathfrak{R}_{n+1}$;

(2) $\mathfrak{R}_n$ is a maximal transitive subrelation of "Σ_n in"; and

(3) the $\mathfrak{R}_n$-ordering is an upper semilattice.

The sequence of relations $\mathfrak{S}_n$ is studied in Chapter $\underset{\sim}{2}$. This chapter contains proofs of the following theorems:

(1) $\mathfrak{S}_n \subset \mathfrak{S}_{n+1}$, $\mathfrak{S}_n \neq \mathfrak{S}_{n+1}$;

(2) $\mathfrak{S}_1$ is a maximal transitive subrelation of "Σ_1 in"; and

(3) the $\mathfrak{S}_n$ ordering is an upper semilattice.

The question of maximality for $n > 1$ is open.

We conclude this introductory section now with some additional notation. If $\mathfrak{R}$ is any reflexive and transitive relation, we will often write "$A \leq_{\mathfrak{R}} B$" for "$A \mathfrak{R} B$". Also, we will use the notation "$A \leq_r B$" for "A recursive in B", "$A \leq_m B$" for "A is many-one reducible to B", and "$A \leq_1 B$" for "A is one-one reducible to B". For any set A, $\overline{A}$ will denote the complement of A. (The universe of discourse is ω throughout.) References [9] and [10] are cited as standard references to the Kleene-Post degrees of unsolvability. Kleene-Post degrees are denoted by $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$, etc.

Chapter $\underset{\sim}{1}$. The Sequence of Relations $\mathfrak{R}_n$

1. Prerequisites.

The deeper results in this chapter about the sequence $\mathfrak{R}_n$, $n < \omega$, are based on the observation (Theorem 1.8) that $A \ \mathfrak{R}_{n+1} \ B \leftrightarrow A^{(n)} \leq_r B^{(n)}$. This observation enables us to apply theorems concerning the degrees of unsolvability to obtain our results. In particular, the following extensions of well known theorems concerning the existence of sets of natural numbers are used.

Theorem 1.1. $\forall \underset{\sim}{a} \exists \underset{\sim}{b} \exists \underset{\sim}{c} [\underset{\sim}{a}^{(n)} = \underset{\sim}{b}^{(n)} = \underset{\sim}{c}^{(n)} = \underset{\sim}{b} \vee \underset{\sim}{c} \ \& \ \underset{\sim}{b} \notin \Sigma_n^{\underset{\sim}{a}} \ \& \ \underset{\sim}{c} \notin \Sigma_n^{\underset{\sim}{a}}]$.

Theorem 1.2. $\forall \underset{\sim}{a} \forall \underset{\sim}{b} \exists \underset{\sim}{c} [\underset{\sim}{c}^{(n)} = \underset{\sim}{c} \vee \underset{\sim}{a}^{(n)} = \underset{\sim}{b} \vee \underset{\sim}{a}^{(n)}]$.

Theorem 1.3. $\exists A \exists B [A \notin \Sigma_n^B \ \& \ B \notin \Sigma_n^A \ \& \ \underset{\sim}{d}(A)^{(n)} = \underset{\sim}{0}^{(n)} = \underset{\sim}{d}(B)^{(n)}]$.

Theorem 1.1 for the case $n = 1$ without the additional properties $\underset{\sim}{b} \notin \Sigma_1^{\underset{\sim}{a}}$ and $\underset{\sim}{a} \notin \Sigma_1^{\underset{\sim}{b}}$ is due to Spector [17]. The technique used to prove $b \notin \Sigma_n^{\underset{\sim}{a}}$ and $a \notin \Sigma_n^{\underset{\sim}{b}}$ is due to Shoenfield [16]. Theorem 1.2 for the case $n = 1$ is a relativized version of Friedberg's characterization of the complete degrees [1]. Corresponding to the original Kleene-Post construction [6] of Σ_1-incomparable sets in Δ_2, Theorem 1.3 establishes the existence of Σ_n-incomparable sets in Δ_{n+1}. Peter Hinman [3] has proved, corresponding to the Friedberg-Muchnik theorem ([5], and [11]), that there exist Δ_{n+1}-incomparable sets in Σ_{n+1}.

The proofs of Theorems 1.1, 1.2, and 1.3 are not presented here. They are proved by forcing arguments which mimic the original proofs for the case $n = 1$. The interested reader may consult either [12] or [14] for a description of the forcing method and the proofs of these theorems.

<u>Corollary</u> 1.1. $\forall \underset{\sim}{a} \forall \underset{\sim}{b} [\underset{\sim}{a}^{(n)} < \underset{\sim}{b} \rightarrow \exists \underset{\sim}{c} [\underset{\sim}{c}^{(n)} = \underset{\sim}{b} \ \& \ \underset{\sim}{c} | \underset{\sim}{a}^{(n)}]]$.

The proof is an immediate consequence of Theorem 1.2.

<u>Corollary</u> 1.2. $\forall \underset{\sim}{a} \forall \underset{\sim}{b} \exists \underset{\sim}{c} [\underset{\sim}{c}^{(n+1)} = \underset{\sim}{c}^{(n)} \vee \underset{\sim}{a}^{(n+1)} = \underset{\sim}{b} \vee \underset{\sim}{a}^{(n+1)}]$.

<u>Proof</u>. By Theorem 1.2, $\underset{\sim}{c}^{(n+1)} \leq \underset{\sim}{c} \vee \underset{\sim}{a}^{(n+1)} \leq \underset{\sim}{c}^{(n)} \vee \underset{\sim}{a}^{(n+1)}$ $= (\underset{\sim}{c} \vee \underset{\sim}{a}^{(n+1)}) \vee \underset{\sim}{c}^{(n)} \leq \underset{\sim}{c}^{(n+1)}$.

2. <u>Basic Properties</u>.

<u>Theorem</u> 1.4. No relation $\mathfrak{R}$ satisfies both the property

(P1) $\forall A,B [A \leq_r B \rightarrow A \mathfrak{R} B]$,

and the property

(P2) $\forall A,B [B \in \Sigma_n \ \& \ A \mathfrak{R} B \rightarrow A \in \Sigma_n]$.

<u>Proof</u>. Suppose $\mathfrak{R}$ satisfies both (P1) and (P2). Choose $A \in \Pi_n$ so that $A \notin \Sigma_n$. $A \leq_r \overline{A}$. Thus $A \mathfrak{R} \overline{A}$. $\overline{A} \in \Sigma_n$. Therefore, by (P2), $A \in \Sigma_n$. Contradiction.

Theorem 1.5. If $\mathfrak{R}$ is a Σ_n-reducibility relation, and if $\mathfrak{R}$ satisfies property (P1), then $\mathfrak{R}$ is a Δ_n-reducibility relation.

Proof. Suppose $A \mathfrak{R} B$. Then $A \in \Sigma_n^B$. $\overline{A} \leq_r A$, therefore $\overline{A} \mathfrak{R} B$. Thus $\overline{A} \in \Sigma_n^B$. Hence $A \in \Delta_n^B$. Therefore, $\mathfrak{R}$ is a Δ_n-reducibility relation.

Theorem 1.6. For all $n \geq 1$, $\mathfrak{R}_n$ is a Σ_n-reducibility relation, and $\mathfrak{R}_n$ satisfies property (P1) of Theorem 0.1.

The proof of Theorem 1.6 is immediate.

Corollary 1.3. For all $n \geq 1$, $\mathfrak{R}_n$ is a Δ_n-reducibility relation.

Corollary 1.4. $A \mathfrak{R}_1 B \leftrightarrow A \leq_r B$.

Proof. $A \leq_r B \rightarrow A \mathfrak{R}_1 B$, because $\mathfrak{R}_1$ satisfies property (P1). By Corollary 1.3, $A \mathfrak{R}_1 B \rightarrow A \leq_r B$.

Corollary 1.5. $\leq_r$ and $\leq_{\mathfrak{R}_1}$ are the only Σ_1-reducibility relations which satisfy property (P1).

Theorem 1.7. For all $n \geq 1$, $A \mathfrak{R}_n B \leftrightarrow A^{(n)} \leq_1 B^{(n)}$.

Proof. $X \in \Sigma_n^A \leftrightarrow X \leq_1 A^{(n)}$. Assume $A^{(n)} \leq_1 B^{(n)}$. Then, $X \in \Sigma_n^A \rightarrow X \leq_1 A^{(n)} \rightarrow X \leq_1 B^{(n)} \rightarrow X \in \Sigma_n^B$. Conversely, suppose $A \mathfrak{R}_n B$. $A^{(n)} \in \Sigma_n^A$. Therefore, $A^{(n)} \in \Sigma_n^B$. Thus, $A^{(n)} \leq_1 B^{(n)}$.

<u>Corollary</u> 1.6. $A' \leq_1 B' \leftrightarrow A \leq_r B$.

Corollary 1.6 appears in [9, p.255].

<u>Theorem</u> 1.8. For all $n \geq 0$, $A \,\Re_{n+1}\, B \leftrightarrow A^{(n)} \leq_r B^{(n)}$.

<u>Proof</u>. By Theorem 1.7, $A \,\Re_{n+1}\, B \leftrightarrow A^{(n+1)} \leq_1 B^{(n+1)}$. By Corollary 1.6, $A^{(n+1)} \leq_1 B^{(n+1)} \leftrightarrow A^{(n)} \leq_r B^{(n)}$.

<u>Theorem</u> 1.9. For all $n \geq 0$, $A \,\Re_{n+1}\, B \leftrightarrow A^{(n)} \in \Delta^B_{n+1}$.

<u>Proof</u>. By Post's theorem [8], $A^{(n)} \leq_r B^{(n)} \leftrightarrow A^{(n)} \in \Delta^B_{n+1}$. Then use Theorem 1.8.

<u>Theorem</u> 1.10. For all $n \geq 1$, $A \,\Re_n\, B \leftrightarrow \forall X[X \in \Delta^A_n \rightarrow X \in \Delta^B_n]$.

<u>Proof</u>. Suppose $A \,\Re_n\, B$. Then, $X \in \Pi^A_n \rightarrow \overline{X} \in \Sigma^A_n \rightarrow \overline{X} \in \Sigma^B_n \rightarrow X \in \Pi^B_n$. Thus, $X \in \Delta^A_n \rightarrow X \in \Delta^B_n$.

Conversely, suppose $\forall X[X \in \Delta^A_n \rightarrow X \in \Delta^B_n]$. $A^{(n-1)} \in \Delta^A_n$. Therefore, $A^{(n-1)} \in \Delta^B_n$. Then use Theorem 1.9.

<u>Theorem</u> 1.11. (The hierarchy theorem) For every $n \geq 1$, $\Re_n \subsetneqq \Re_{n+1}$. In fact $\forall n \exists A \exists B [A \in \Sigma_1 \ \& \ B \in \Sigma_1 \ \& \ A \,\Re_{n+1}\, B \ \& \ \neg A \,\Re_n\, B]$.

<u>Proof</u>. For all A and B, $A \leq_r B \rightarrow A' \leq_r B'$. Thus $\Re_n \subseteq \Re_{n+1}$ follows from Theorem 1.8.

Consider the following theorem proved by Friedberg [10, p.85]:

$$\forall \underset{\sim}{a} \forall \underset{\sim}{b} [\underset{\sim}{a} < \underset{\sim}{b} \text{ and } \underset{\sim}{b} \text{ r.e. } \underset{\sim}{a} \rightarrow \exists \underset{\sim}{c}, \underset{\sim}{d} (\underset{\sim}{c} \vee \underset{\sim}{d} = \underset{\sim}{b} \;\&\; \underset{\sim}{a}' = \underset{\sim}{c}' = \underset{\sim}{d}' \;\&\; \underset{\sim}{c} \text{ r.e. } \underset{\sim}{a} \;\&\; \underset{\sim}{d} \text{ r.e. } \underset{\sim}{a} \;\&\; \underset{\sim}{a} \leq \underset{\sim}{c} \;\&\; \underset{\sim}{a} \leq \underset{\sim}{d})].$$

Let $\underset{\sim}{b} = \underset{\sim}{0}^{(n+1)}$ and $\underset{\sim}{a} = \underset{\sim}{0}^{(n)}$. Then $\exists \underset{\sim}{c}, \underset{\sim}{d} [\underset{\sim}{c} | \underset{\sim}{d} \;\&\; \underset{\sim}{0}^{(n+1)} = \underset{\sim}{c}' = \underset{\sim}{d}' \;\&\; \underset{\sim}{0}^{(n)} \leq \underset{\sim}{c} \;\&\; \underset{\sim}{0}^{(n)} \leq \underset{\sim}{d} \;\&\; \underset{\sim}{c} \text{ r.e. } \underset{\sim}{0}^{(n)} \;\&\; \underset{\sim}{d} \text{ r.e. } \underset{\sim}{0}^{(n)}]$. If $n = 0$, choose A and B so that $\underset{\sim}{d}(A) = \underset{\sim}{c}$ and $\underset{\sim}{d}(B) = \underset{\sim}{d}$. Then $A \in \Sigma_1 \;\&\; B \in \Sigma_1 \;\&\; A' = B' \;\&\; A|B$. Thus $A \,\mathfrak{R}_2\, B$, but not $A \,\mathfrak{R}_1\, B$, by Theorem 1.8 and Corollary 1.4. Suppose $n \geq 1$. Shoenfield and Sacks (see [10, p.105] and [16]) have proved that

$$\forall \underset{\sim}{a}, \underset{\sim}{b} [(\underset{\sim}{a}' \leq \underset{\sim}{b} \leq \underset{\sim}{a}'' \;\&\; \underset{\sim}{b} \text{ r.e. } \underset{\sim}{a}) \leftrightarrow \exists \underset{\sim}{h} [\underset{\sim}{a} \leq \underset{\sim}{h} \leq \underset{\sim}{a}' \;\&\; \underset{\sim}{h} \text{ r.e. } \underset{\sim}{a} \;\&\; \underset{\sim}{h}' = \underset{\sim}{b}]].$$

n successive applications of this theorem to the degrees $\underset{\sim}{c}$ and $\underset{\sim}{d}$ yield r.e. degrees $\underset{\sim}{x}$ and $\underset{\sim}{y}$ so that $\underset{\sim}{c} = \underset{\sim}{x}^{(n)}$ and $\underset{\sim}{d} = \underset{\sim}{y}^{(n)}$. Choose A and B so that $\underset{\sim}{d}(A) = \underset{\sim}{x}$ and $\underset{\sim}{d}(B) = \underset{\sim}{y}$. Then $A \in \Sigma_1 \;\&\; B \in \Sigma_1 \;\&\; A^{(n+1)} = B^{(n+1)}$, but $A^{(n)} | B^{(n)}$. Thus, by Theorem 1.8, $A \,\mathfrak{R}_{n+2}\, B$, but not $A \,\mathfrak{R}_{n+1}\, B$. This proves Theorem 1.11 for all $n \geq 1$.

3. <u>Maximality</u>.

In this section it is shown that each $\mathfrak{R}_n$ is a maximal Σ_n-reducibility relation. That is, for each n and for each relation $\mathfrak{S}$, if $\mathfrak{R}_n$ is a proper subrelation of $\mathfrak{S}$, and if $\mathfrak{S}$ is included in the relation "Σ_n in", then $\mathfrak{S}$ is <u>not</u> transitive.

Observe that, by Corollary 1.5, this result is immediate for the case $n = 1$.

Lemma 1.1. $\forall X[X^{(n)} \leq_r A^{(n)} \rightarrow X \leq_r B] \rightarrow [A^{(n)} \leq_r B]$.

Proof. Let $\underset{\sim}{a} = \underset{\sim}{d}(A)$. By Theorem 1.1, $\exists \underset{\sim}{c},\underset{\sim}{d}[\underset{\sim}{c} \vee \underset{\sim}{d} = \underset{\sim}{a}^{(n)} = \underset{\sim}{d}^{(n)} = \underset{\sim}{c}^{(n)}]$. Choose sets C and D so that $\underset{\sim}{d}(C) = \underset{\sim}{c}$ and $\underset{\sim}{d}(D) = \underset{\sim}{d}$. $C^{(n)} \leq_r A^{(n)}$ and $D^{(n)} \leq_r A^{(n)}$. Thus $C \leq_r B$ and $D \leq_r B$. Therefore $\underset{\sim}{c} \vee \underset{\sim}{d} \leq \underset{\sim}{b}$. Hence, since $\underset{\sim}{c} \vee \underset{\sim}{d} = \underset{\sim}{a}^{(n)}$, $A^{(n)} \leq_r B$.

Notice that the hypothesis of Lemma 1.1 can be rewritten $\forall X[X \,\mathcal{R}_{n+1}\, A \rightarrow X \leq_r A]$, and that from the conclusion we can infer $\forall X[X \in \Delta^A_{n+1} \rightarrow X \leq_r B]$. This is interesting because $\mathcal{R}_{n+1}$ is not equal to the relation "Δ_{n+1} in".

Also, $\forall X[X' \leq_r A \rightarrow X \leq_r B] \rightarrow A \leq_r B$ is <u>false</u>. In fact, choose A so that $\underset{\sim}{d}(A) < \underset{\sim}{0}'$. Then, for all sets B, $\forall X[X' \leq_r A \rightarrow X \leq_r B]$. Choose B so that $A \not\leq_r B$.

Theorem 1.12. For each $n \geq 1$, $\mathcal{R}_{n+1}$ is a maximal Δ_{n+1}-reducibility relation.

Proof. Suppose $\mathcal{S}$ is a binary relation so that $\mathcal{R}_n \subsetneq \mathcal{S}$ and so that $A\mathcal{S}B \rightarrow A \in \Delta^B_{n+1}$, for all A and B. Then, there exist sets A and B so that $\neg A \,\mathcal{R}_n\, B$, but $A\mathcal{S}B$. By Lemma 1.1 and Theorem 1.8, $\exists X[X^{(n)} \leq_r A^{(n)} \;\&\; X \not\leq_r B^{(n)}]$. That is, $\exists X[X \,\mathcal{R}_{n+1}\, A \;\&\; X \notin \Delta^B_{n+1}]$. Since $\mathcal{R}_{n+1} \subset \mathcal{S}$, $\exists X[X\mathcal{S}A \;\&\; X \notin \Delta^B_{n+1}]$. If $\mathcal{S}$ is transitive, then

$X\mathcal{S}A$ and $A\mathcal{S}B$ yields $X\mathcal{S}B$. But this is impossible, since $X\in\Delta_{n+1}^{B}$. Thus $\mathcal{S}$ is not transitive.

<u>Theorem</u> 1.13. For each $n \geq 1$, $\mathcal{R}_{n+1}$ is a maximal Σ_{n+1}-reducibility relation.

<u>Proof</u>. If $\mathcal{R}_{n+1} \subset \mathcal{S}$, then $\mathcal{S}$ satisfies property (P1) of Theorem 1.4. The proof follows from Theorem 1.5 and Theorem 1.12.

4. <u>Set Inclusions</u>.

The purpose of this section is to describe completely the set inclusion relationships among the relations $\mathcal{R}_{n+1}$, "Σ_n in", "Π_n in", "Σ_{n+1} in", and "Π_{n+1} in".

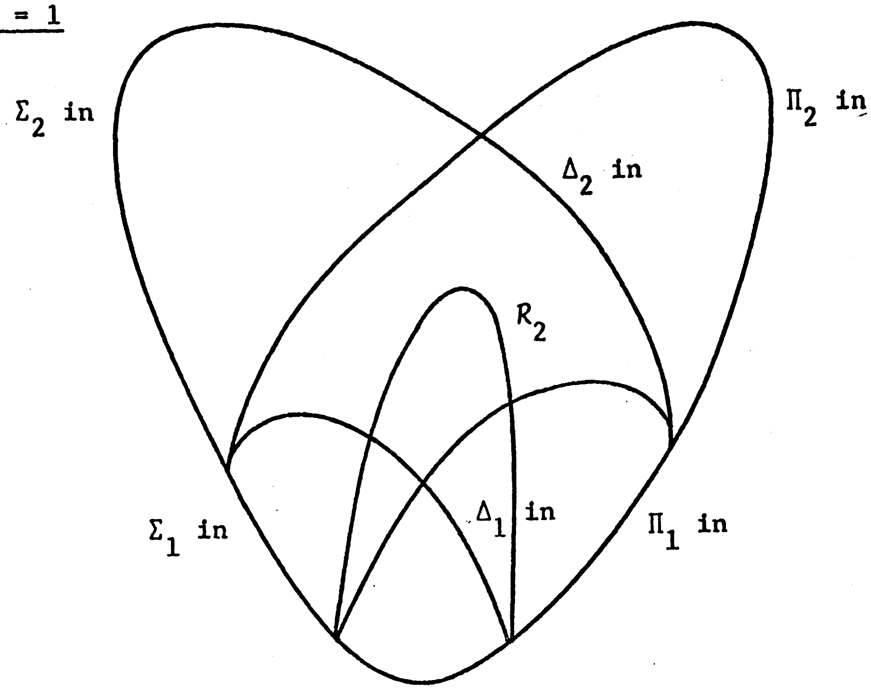
Our result is that the following figure is correct.

It is already known by Theorem 1.6 and Corollary 1.3, that, for all $n \geq 1$, the relation "Δ_1 in" is included in the relation $\mathcal{R}_{n+1}$, and $\mathcal{R}_{n+1}$ is included in the relation "Δ_{n+1} in". Of course, if $n \neq 1$, then "Δ_n in" is not included in $\mathcal{R}_{n+1}$. (Look at $\langle B', B\rangle$ for any set B).

<u>Theorem</u> 1.14. If $\mathcal{R}$ is an $\mathcal{A}$-reducibility relation which includes the relation "Σ_1 in", then $\mathcal{R} = \mathcal{A}$.

<u>Proof</u>. Suppose $A\mathcal{A}B$. Then, for some n, $A\in\Sigma_n^{B}$. Thus, $A\in\Sigma_1^{B^{(n\dot{-}1)}}$. Now, for all k, $B^{(k+1)}\in\Sigma_1^{B^{(k)}}$. Since $\mathcal{R}$ includes "Σ_1 in", $A\mathcal{R}B^{(n\dot{-}1)}$, $B^{(n\dot{-}1)}\mathcal{R}B^{(n\dot{-}2)}, \ldots, B^{(2)}\mathcal{R}B'$, $B'\mathcal{R}B$. Thus $A\mathcal{R}B$. Hence $\mathcal{A} = \mathcal{R}$.

Case n = 1



Case n > 1

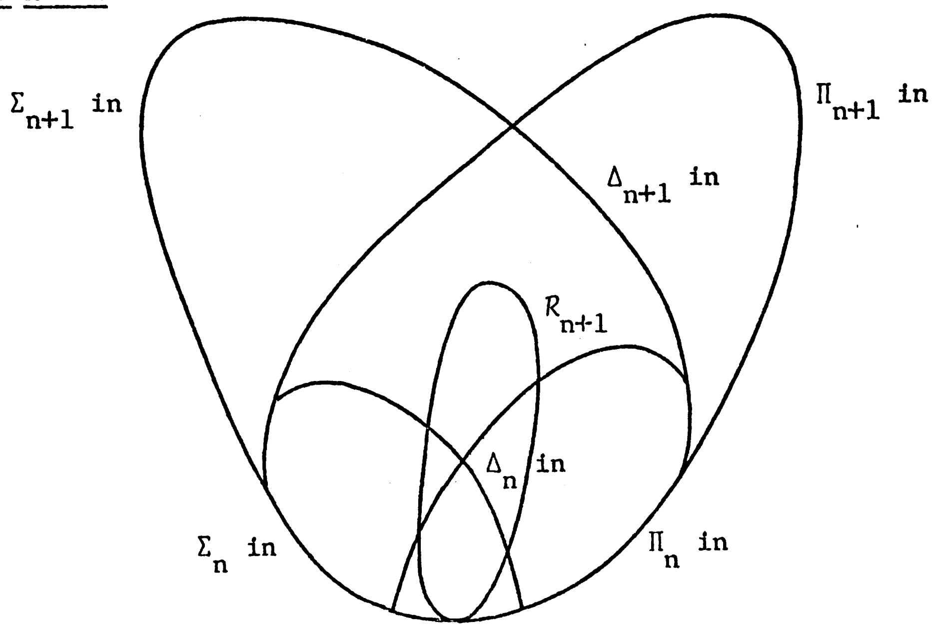


Figure R_n set inclusion relationships

Corollary 1.7. For all $n \geq 1$, $\exists A \exists B[A \in \Sigma_n^B \ \& \ \neg A \ \mathfrak{R}_{n+1} \ B]$.

Theorem 1.15. For all $n \geq 1$, $\exists A \exists B[A \in \Sigma_n^B \ \& \ A \ \mathfrak{R}_{n+1} \ B \ \& \ A \notin \Delta_n^B]$.

Proof. By Corollary 1.2, with $\underset{\sim}{a} = \underset{\sim}{0}$, and $\underset{\sim}{b} = \underset{\sim}{0}^{(2n+2)}$, there exists $\underset{\sim}{c}$ so that $\underset{\sim}{c}^{(n+1)} = \underset{\sim}{c}^{(n)} \vee \underset{\sim}{0}^{(n+1)} = \underset{\sim}{0}^{(2n+2)}$. Choose A and B so that $\underset{\sim}{d}(A) = \underset{\sim}{0}^{(n+1)}$, $A \in \Sigma_{n+1}$, and $\underset{\sim}{d}(B) = \underset{\sim}{c}$. Then, $A \in \Sigma_{n+1}^B$, $\underset{\sim}{d}(A) \mid \underset{\sim}{d}(B)^{(n)}$, and $A^{(n+1)} = B^{(n+1)}$. That is, $A \in \Sigma_{n+1}^B$, $A \notin \Delta_{n+1}^B$, and $A \ \mathfrak{R}_{n+2} \ B$, for all $n \geq 0$.

Corollary 1.8. For all $n \geq 1$, $\exists A \exists B[A \in \Pi_n^B \ \& \ A \ \mathfrak{R}_{n+1} \ B \ \& \ A \notin \Delta_n^B]$.

Theorem 1.16. For all $n \geq 1$, $\exists A \exists B[A \ \mathfrak{R}_{n+1} \ B \ \& \ A \notin \Sigma_n^B \ \& \ A \notin \Pi_n^B]$.

Proof. Choose A recursive. By Theorem 1.1, there exists $\underset{\sim}{b}$ so that $\underset{\sim}{d}(A)^{(n)} = \underset{\sim}{b}^{(n)} \ \& \ \underset{\sim}{b} \notin \Sigma_n^{\underset{\sim}{d}(A)}$. Then, choose B so that $\underset{\sim}{d}(B) = \underset{\sim}{b}$.

5. $\mathfrak{R}_n$-degrees.

As has been mentioned in the introduction, if $\mathfrak{R}$ is any reflexive and transitive relation, then $\mathfrak{R} \cap \mathfrak{R}^{-1}$ is an equivalence relation. We call the equivalence classes of such a relation the $\mathfrak{R}$-degrees. $\mathfrak{R}$-degrees shall be denoted with boldface lower case letters subscripted by $\mathfrak{R}$, to distinguish them from Kleene-Post degrees. If A is any set, then $\underset{\sim}{d}_{\mathfrak{R}}(A)$ is the $\mathfrak{R}$-degree to which A belongs. If $\underset{\sim}{a}_{\mathfrak{R}}$ and $\underset{\sim}{b}_{\mathfrak{R}}$ are any two $\mathfrak{R}$-degrees, then we define $\underset{\sim}{a}_{\mathfrak{R}} \leq \underset{\sim}{b}_{\mathfrak{R}}$ if and only if $A \leq_{\mathfrak{R}} B$, where $\underset{\sim}{d}_{\mathfrak{R}}(A) = \underset{\sim}{a}_{\mathfrak{R}}$ and $\underset{\sim}{d}_{\mathfrak{R}}(B) = \underset{\sim}{b}_{\mathfrak{R}}$.

This definition is well-defined. Thus, the set of all $\Re$-degrees is always a partially ordered set.

This section is devoted to a brief development of the structure of the $\Re_n$-degrees. Observe that, by Theorem 1.8, $\underset{\sim}{d}_{\Re_{n+1}}(A) \leq \underset{\sim}{d}_{\Re_{n+1}}(B)$ if and only if $\underset{\sim}{d}(A)^{(n)} \leq \underset{\sim}{d}(B)^{(n)}$.

<u>Theorem</u> 1.17. The $\Re_n$-degree ordering is an upper semilattice.

<u>Proof</u>. If $n = 1$, there is nothing to prove (since $A \leq_{\Re_1} B \leftrightarrow A \leq_r B$).

Let $\underset{\sim}{a}_{\Re_{n+1}}$ and $\underset{\sim}{b}_{\Re_{n+1}}$ be any two $\Re_{n+1}$-degrees. Choose sets A and B so that $\underset{\sim}{d}_{\Re_{n+1}}(A) = \underset{\sim}{a}_{\Re_{n+1}}$ and $\underset{\sim}{d}_{\Re_{n+1}}(B) = \underset{\sim}{b}_{\Re_{n+1}}$. Then, by Theorem 1.2, $\exists \underset{\sim}{c}[\underset{\sim}{c}^{(n)} = \underset{\sim}{c} \vee \underset{\sim}{a}^{(n)} = \underset{\sim}{b}^{(n)} \vee \underset{\sim}{a}^{(n)}]$, where $\underset{\sim}{d}(A) = \underset{\sim}{a}$ and $\underset{\sim}{d}(B) = \underset{\sim}{b}$. Choose C so that $\underset{\sim}{d}(C) = \underset{\sim}{c}$, and let $\underset{\sim}{c}_{\Re_{n+1}} = \underset{\sim}{d}_{\Re_{n+1}}(C)$. Since $\underset{\sim}{c}^{(n)} = \underset{\sim}{a}^{(n)} \vee \underset{\sim}{b}^{(n)}$. $\underset{\sim}{d}_{\Re_{n+1}}(C)$ is a least upper bound of the $\Re_{n+1}$-degrees $\underset{\sim}{a}_{\Re_{n+1}}$ and $\underset{\sim}{b}_{\Re_{n+1}}$.

<u>Theorem</u> 1.18. For each $\Re_{n+1}$-degree $\underset{\sim}{a}_{\Re_{n+1}}$ there is a greater one. In fact, define $\underset{\sim}{a}'_{\Re_{n+1}}$ to be $\underset{\sim}{d}_{\Re_{n+1}}(A')$, where $\underset{\sim}{d}_{\Re_{n+1}}(A) = \underset{\sim}{a}_{\Re_{n+1}}$. Then $\underset{\sim}{a}'_{\Re_{n+1}}$ is well-defined and $\underset{\sim}{a}_{\Re_{n+1}} < \underset{\sim}{a}'_{\Re_{n+1}}$.

<u>Proof</u>. Suppose $\underset{\sim}{d}_{\Re_{n+1}}(A) = \underset{\sim}{d}_{\Re_{n+1}}(B)$. Then, $\underset{\sim}{d}(A)^{(n)} = \underset{\sim}{d}(B)^{(n)}$. $\underset{\sim}{d}_{\Re_{n+1}}(A') = \{C | \underset{\sim}{d}(C)^{(n)} = \underset{\sim}{d}(A')^{(n)} = \underset{\sim}{d}(A)^{(n+1)}\}$.

$\underset{\sim}{d}(B')^{(n)} = \underset{\sim}{d}(B)^{(n+1)} = \underset{\sim}{d}(A)^{(n+1)}$. Thus $\underset{\sim}{d}_{\mathcal{R}_{n+1}}(A') = \underset{\sim}{d}_{\mathcal{R}_{n+1}}(B')$. Therefore $\underset{\sim}{a}'_{\mathcal{R}_{n+1}}$ is well-defined. $\underset{\sim}{d}(A)^{(n)} < \underset{\sim}{d}(A')^{(n)}$. Therefore $\underset{\sim}{a}_{\mathcal{R}_{n+1}} < \underset{\sim}{a}'_{\mathcal{R}_{n+1}}$.

The upper semilattice of $\mathcal{R}_{n+1}$-degrees has a least element. In fact, define $\underset{\sim}{0}_{\mathcal{R}_{n+1}} = \underset{\sim}{d}_{\mathcal{R}_{n+1}}(A)$, where $\underset{\sim}{d}(A) = \underset{\sim}{0}$.

Kleene and Post have proved [6] that there exist degrees $\underset{\sim}{b}$ and $\underset{\sim}{c}$ such that

(i) $\forall m[\underset{\sim}{0}^{(m)} \leq \underset{\sim}{b}$ and $\underset{\sim}{0}^{(m)} \leq \underset{\sim}{c}]$; and

(ii) $\forall \underset{\sim}{d}[[\underset{\sim}{d} \leq \underset{\sim}{b} \ \& \ \underset{\sim}{d} \leq \underset{\sim}{c}] \rightarrow \exists m[\underset{\sim}{d} \leq \underset{\sim}{0}^{(m)}]]$.

A corollary of this theorem is that the ordering of the Kleene-Post degrees is not a lattice. We will apply this theorem to show that the ordering of the $\mathcal{R}_{n+1}$-degrees is not a lattice.

<u>Theorem</u> 1.19. The $\mathcal{R}_{n+1}$-degree ordering is not a lattice.

<u>Proof</u>. Consider the degrees $\underset{\sim}{b}$ and $\underset{\sim}{c}$ which are given by the theorem quoted in the previous paragraph. By Corollary 1.1, there exist degrees $\underset{\sim}{a}$ and $\underset{\sim}{d}$ so that $\underset{\sim}{a}^{(n)} = \underset{\sim}{b}$ and $\underset{\sim}{d}^{(n)} = \underset{\sim}{c}$. By (i), we have,

(iii) $\forall m[\underset{\sim}{0}^{(m+n)} \leq \underset{\sim}{a}^{(n)} \ \& \ \underset{\sim}{0}^{(m+n)} \leq \underset{\sim}{d}^{(n)}]$.

By (ii), we have

(iv) $\forall \underset{\sim}{e}[[\underset{\sim}{e}^{(n)} \leq \underset{\sim}{a}^{(n)} \ \& \ \underset{\sim}{e}^{(n)} \leq \underset{\sim}{d}^{(n)}] \rightarrow \exists m[\underset{\sim}{e}^{(n)} \leq \underset{\sim}{0}^{(m+n)}]]$.

Let $\underset{\sim}{a}_{\mathfrak{R}_{n+1}}$ and $\underset{\sim}{d}_{\mathfrak{R}_{n+1}}$ be the $\mathfrak{R}_{n+1}$ degrees of the members of $\underset{\sim}{a}$ and $\underset{\sim}{d}$, respectively. Then,

(iii') for every m, $\underset{\sim}{0}^{(m)}_{\mathfrak{R}_{n+1}} \leq \underset{\sim}{a}_{\mathfrak{R}_{n+1}}$ and $\underset{\sim}{0}^{(m)}_{\mathfrak{R}_{n+1}} \leq \underset{\sim}{d}_{\mathfrak{R}_{n+1}}$; and

(iv') $\forall e_{\mathfrak{R}_{n+1}} [[\underset{\sim}{e}_{\mathfrak{R}_{n+1}} \leq \underset{\sim}{a}_{\mathfrak{R}_{n+1}} \;\&\; \underset{\sim}{e}_{\mathfrak{R}_{n+1}} \leq \underset{\sim}{d}_{\mathfrak{R}_{n+1}}] \rightarrow \exists m [e \leq \underset{\sim}{0}^{(m)}_{\mathfrak{R}_{n+1}}]]$.

Thus, $\underset{\sim}{a}_{\mathfrak{R}_{n+1}}$ and $\underset{\sim}{d}_{\mathfrak{R}_{n+1}}$ can have no greater lower bound.

By Theorem 1.1, there exist sets A and B so that $A^{(n+1)} = B^{(n+1)} = A^{(n)} \vee B^{(n)} = 0^{(n+1)}$. Thus,

<u>Theorem</u> 1.20. $\exists \underset{\sim}{a}_{\mathfrak{R}_{n+1}} \exists \underset{\sim}{b}_{\mathfrak{R}_{n+1}} [\underset{\sim}{a}'_{\mathfrak{R}_{n+1}} = \underset{\sim}{b}'_{\mathfrak{R}_{n+1}} = \underset{\sim}{a}_{\mathfrak{R}_{n+1}} \vee \underset{\sim}{b}_{\mathfrak{R}_{n+1}} = \underset{\sim}{0}'_{\mathfrak{R}_{n+1}}]$.

Extend the definition of $\mathfrak{R}_n$ to number theoretic functions and predicates in the usual way (see [6]). Then we may make the following definition.

<u>Definition</u> 1.1. A is $\mathfrak{R}_n$-enumerable in B if and only if $A = \emptyset$ or there exists a function f so that $f \leq_{\mathfrak{R}_n} B$ and A is the range of f.

<u>Theorem</u> 1.21. A is $\mathfrak{R}_n$-enumerable in B if and only if there exists a predicate $R^B(x,y)$ so that $R^B \leq_{\mathfrak{R}_n} B$ and $\forall x(x \in A \leftrightarrow \exists y R^B(x,y))$.

<u>Proof</u>. Assume A is $\mathfrak{R}_n$-enumerable in B. If $a = \emptyset$, then $x \in A \leftrightarrow \exists y[x \neq x \;\&\; y \neq y]$. If $A \neq \emptyset$, it is the range of a function f, $f \leq_{\mathfrak{R}_n} B$. Then, $x \in A \leftrightarrow \exists y[f(y) = x]$. Conversely, assume

$x \in A \leftrightarrow \exists y R^B(x,y)$, where $R^B \leq_{\Re_n} B$. If $A = \emptyset$, it is $\Re_n$-enumerable in B. If $A \neq \emptyset$, let $k \in A$. Define a function f by:

$$f(x) = \begin{cases} k, & \neg R^B((x)_0, (x)_1) \\ (x)_0, & R^B((x)_0, (x)_1). \end{cases}$$

Clearly, A is the range of f. $f \leq_r R^B$ and $R^B \leq_{\Re_n} B$. Thus, $f \leq_{\Re_n} B$.

Observe that since $\leq_r$ and $\leq_{\Re_1}$ are identical, A is $\Re_1$-enumerable in B if and only if $A \in \Sigma_1^B$.

Theorem 1.22. For all $n \geq 1$, if A is $\Re_n$-enumerable in B, then $A \in \Sigma_n^B$.

Proof. Suppose $x \in A \leftrightarrow \exists y R^B(x,y)$, where $R^B \leq_{\Re_n} B$. Then $R^B \in \Sigma_n^B$. Thus $A \in \Sigma_n^B$.

The converse of Theorem 1.22 is false for $n \neq 1$. For any $n \geq 1$ and any set B, let $A = B^{(n+1)}$. $A \in \Sigma_{n+1}^B$. If $R^B(x,y)$ is a predicate for which $\forall x[x \in A \leftrightarrow \exists y R^B(x,y))$, then $B^{(n)} \leq_r R^B$. On the other hand, $B^{(n)} \not\leq_{\Re_{n+1}} B$. Therefore, $R^B \not\leq_{\Re_{n+1}} B$.

Corollary 1.9.

(1) There exist sets A and B so that A is not $\Re_n$-enumerable in B, B is not $\Re_n$-enumerable in A, $\underset{\sim}{d}_{\Re_{n+1}}(A) \leq \underset{\sim}{0}'_{\Re_{n+1}}$, and $\underset{\sim}{d}_{\Re_{n+1}}(B) \leq \underset{\sim}{0}'_{\Re_{n+1}}$.

(2) $\exists \mathbf{a}_{\mathfrak{R}_n} \exists \mathbf{b}_{\mathfrak{R}_n} [\mathbf{0}_{\mathfrak{R}_n} < \mathbf{a}_{\mathfrak{R}_n} < \mathbf{0}'_{\mathfrak{R}_n} \ \& \ \mathbf{0}_{\mathfrak{R}_n} < \mathbf{b}_{\mathfrak{R}_n} < \mathbf{0}'_{\mathfrak{R}_n} \ \& \ \mathbf{a}_{\mathfrak{R}_n} | \mathbf{b}_{\mathfrak{R}_n}]$.

<u>Proof</u>. (1) is an immediate consequence of Theorem 1.3, and (2) is an immediate consequence of Theorem 1.20.

<u>Remarks</u>.

(1) Let $\mathcal{D}_{\mathfrak{R}_n}$ be the set of all $\mathfrak{R}_n$-degrees. It is not known whether the structure $\langle \mathcal{D}_{\mathfrak{R}_n}, \leq, ' \rangle$ is elementarily equivalent to the structure $\langle \mathcal{D}_{\mathfrak{R}_m}, \leq, ' \rangle$, for $n \neq m$.

(2) For any recursive degree $\mathbf{a}$, let $\mathbf{a}_{\mathfrak{R}_n}$ be the $\mathfrak{R}_n$-degree of the members of $\mathbf{a}$. The function $\mathbf{a} \rightarrow \mathbf{a}_{\mathfrak{R}_n}$ is not 1 - 1. $\mathbf{a} \rightarrow \mathbf{a}_{\mathfrak{R}_n}$ is not even a homomorphism. In fact, if $\mathbf{a}_{\mathfrak{R}_{n+1}} \vee \mathbf{b}_{\mathfrak{R}_{n+1}} = \mathbf{c}_{\mathfrak{R}_{n+1}}$, then there are sets A,B, and C so that $\mathbf{d}_{\mathfrak{R}_{n+1}}(A) = \mathbf{a}_{\mathfrak{R}_{n+1}}$, $\mathbf{d}_{\mathfrak{R}_{n+1}}(B) = \mathbf{b}_{\mathfrak{R}_{n+1}}$, and $\mathbf{d}_{\mathfrak{R}_{n+1}}(C) = \mathbf{c}_{\mathfrak{R}_{n+1}}$. But in general $A^{(n)} \vee B^{(n)} \neq (A \vee B)^{(n)}$.

Chapter 2. The Sequence of Relations $\mathcal{S}_n$

1. Preliminaries.

Much of the contents of this section are standard and refer mainly to [4] and [9].

We make use of the primitive recursive functions p_i, $(a)_i$, and $\ell h(a)$, defined by Kleene in [4]. A sequence number is a number $\alpha = p_o^{a_o} \cdot \ldots \cdot p_s^{a_s}$ so that for all $i \leq s$, $a_i > 0$. For any two sequence numbers α and β, define $\alpha \succ \beta$ if and only if $\ell h(\alpha) \geq \ell h(\beta)$ and $(\beta)_i = (\alpha)_i$, for all $i < \ell h(\beta)$.

If f is any partial function whose domain includes the set $\{0,1,2,\ldots,n\}$, then $\overline{f}(n+1) = \prod_{i \leq n} p_i^{f(i)+1}$ is a sequence number. Moreover, if α is any sequence number, and if a partial function f is defined by $f(i) = (\alpha)_i - 1$, for all $i < \ell h(\alpha)$, then $\alpha = \overline{f}(\ell h(\alpha))$.

A two variable sequence number is a number

$$\alpha = \prod_{i \leq n} p_i^{\prod_{j \leq n} p_j^{a_{ij}}},$$

so that for all $i,j \leq n$, $a_{ij} > 0$. For two variable sequence numbers, we define

$$\alpha \succ_2 \beta \leftrightarrow \ell h(\alpha) \geq \ell h(\beta) \ \& \ \forall i < \ell h(\beta) \quad (\alpha)_i \succ (\beta)_i \ .$$

If h is a function of two variables, then define

$$\overline{h}(n+1,n+1) = \prod_{i\leq n} p_i^{\prod_{j\leq n} p_j^{h(i,j)+1}} .$$

Thus, $h(i,j) + 1 = (\overline{h}(n+1,n+1))_{i,j}$, if $i \leq n$ and $j \leq n$. (This discussion may be carried out for functions of n variables, where $n > 2$, but for our purposes it is not necessary to do so).

Definition 2.1. Characteristic sequence numbers.
Define $Ch(\alpha) \equiv \alpha$ is a sequence number so that $(\alpha)_i \in \{1,2\}$, for all $i < \ell h(\alpha)$. Define $Ch_2(\alpha) \equiv \alpha$ is a two variable sequence number so that $(\alpha)_{i,j} \in \{1,2\}$, for all $i,j < \ell h(\alpha)$.

The predicates $\succ$, $\succ_2$, $Ch(\alpha)$, and $Ch_2(\alpha)$ are primitive recursive.

A recursive predicate $R(w,x_1,\ldots,x_n)$ will be called <u>monotonic increasing</u> if, for all sequence numbers α and β, $\alpha \succ \beta$ and $R(\beta,x_1,\ldots,x_n)$ implies $R(\alpha,x_1,\ldots,x_n)$. Given a recursive predicate $R(w,x_1,\ldots,x_n)$, define $R^*(w,x_1,\ldots,x_n) \equiv$ $\exists v[\ell h(v) \leq \ell h(w)\ \&\ \forall i < \ell h(v)((v)_i = (w)_i\ \&\ R(v,x_1,\ldots,x_k)]$. It is immediate that $R^*(w,x_1,\ldots,x_n)$ is a monotonic increasing recursive predicate and that, for any function f, $\exists y R^*(\overline{f}(y),x_1,\ldots,x_n) \equiv \exists y R(\overline{f}(y),x_1,\ldots,x_n)$.

Corresponding to the predicate T_n^1 as defined in [5], there is a predicate $T_n^2(w,e,x_1,\ldots,x_n)$ so that

$T_n^h(e,x_1,\ldots,x_n,y) \equiv T_n^2(\overline{h}(y,y),e,x_1,\ldots,x_n)$. (See [4, p.291]). T_n^1 as defined in [5], enables the normal form and enumeration theorems to be written using $\overline{f}$ instead of $\tilde{f}$. By the previous paragraph, we will assume, without loss of generality, that T_n^2 is monotonic increasing.

2. Basic Properties.

Whereas the relations $\mathfrak{R}_n$ are Δ_n-reducibilities, the relations $\mathcal{S}_n$ are Σ_n-reducibilities but not Δ_n-reducibilities. Consequently, we shall simultaneously consider the sequence $\mathcal{S}_n$, $n < \omega$, defined in Definition 0.2, and those sequences of relations, defined as follows.

Definition 2.2. For each $n \geq 1$, $A \ \mathcal{P}_n \ B \leftrightarrow \forall X[B \in \Pi_n^X \rightarrow A \in \Pi_n^X]$; $A \ \mathcal{D}_n \ B \leftrightarrow \forall X(B \in \Delta_n^X \rightarrow A \in \Delta_n^X]$.

Theorem 2.1. For each $n \geq 1$, $\mathcal{S}_n(\mathcal{P}_n, \mathcal{D}_n)$ is a Σ_n-$(\Pi_n$-$,\Delta_n$-$)$ reducibility relation.

The proof is obvious.

Theorem 2.2.

(1) $A \ \mathcal{S}_n \ B \leftrightarrow \overline{A} \ \mathcal{P}_n \ \overline{B}$;

(2) $A \ \mathcal{D}_1 \ B \leftrightarrow A \leq_r B$;

(3) $A \leq_r B \rightarrow A \ \mathcal{D}_n \ B$, for all n;

(4) $B \in \Delta_n \ \& \ A \ \mathcal{D}_n \ B \rightarrow A \in \Delta_n$;

(5) $B \in \Sigma_n \ \& \ A \ \mathcal{S}_n \ B \rightarrow A \in \Sigma_n$;

(6) $B\in\Pi_n$ & $A\ \mathcal{P}_n\ B \rightarrow A\in\Pi_n$;

(7) $A\in\Delta_n \rightarrow A\ \mathcal{D}_n\ B$, for all B;

(8) $A\in\Sigma_n \rightarrow A\ \mathcal{S}_n\ B$, for all B;

(9) $A\in\Pi_n \rightarrow A\ \mathcal{P}_n\ B$, for all B;

(10) $\leq_r \not\subseteq \mathcal{S}_n$ and $\leq_r \not\subseteq \mathcal{P}_n$.

__Proof__. The proof of each of the statements (1)-(9) is obvious. (10) follows from (5) and (6) together with Theorem 1.4.

__Theorem__ 2.3. (The hierarchy theorem) For all $n \geq 1$, $\mathcal{S}_n \subsetneqq \mathcal{S}_{n+1}$, $\mathcal{P}_n \subsetneqq \mathcal{P}_{n+1}$, and $\mathcal{D}_n \subsetneqq \mathcal{D}_{n+1}$.

__Proof__. Suppose $A\ \mathcal{S}_n\ B$, and $B\in\Sigma^X_{n+1}$. Then $B \leq_1 X^{(n+1)}$. $X^{(n+1)} = (X')^{(n)}$. Since $A\ \mathcal{S}_n\ B$, $A \leq_1 (X')^{(n)}$. Thus $A\in\Sigma^X_{n+1}$. That is, $A\ \mathcal{S}_{n+1}\ B$. Hence $\mathcal{S}_n$ is included in $\mathcal{S}_{n+1}$.

Similarly, $B \leq_1 \overline{A^{(n+1)}} \rightarrow B \leq_1 \overline{(A')^{(n)}}$. Thus $A\ \mathcal{P}_n\ B \rightarrow A\ \mathcal{P}_{n+1}\ B$.

Suppose $A\ \mathcal{D}_n\ B$, and $B\in\Delta^X_{n+1}$. Then $B \leq_r X^{(n)}$. $X^{(n)} = (X')^{n-1}$. Thus $B \leq_r (X')^{(n-1)}$. Since $A\ \mathcal{D}_n\ B$, $A \leq_r (X')^{(n-1)}$. Thus $A\in\Delta^X_{n+1}$. Therefore, $\mathcal{D}_n$ is included in $\mathcal{D}_{n+1}$.

To see that $\mathcal{S}_n \neq \mathcal{S}_{n+1}$, choose sets A and B so that $A\in\Sigma_{n+1}$, $A\notin\Sigma_n$ and $B\in\Sigma_n$. Then, $A\ \mathcal{S}_{n+1}\ B$, but $A \not\leq_{\mathcal{S}_n} B$, by Theorem 2.2, (8) and (5).

Similarly, it can be shown that $\mathcal{P}_n \neq \mathcal{P}_{n+1}$, and $\mathcal{D}_n \neq \mathcal{D}_{n+1}$.

__Theorem__ 2.4.

(1) "Σ_1 in" $\not\subseteq \mathcal{S}_n$, "$\Sigma_1$ in" $\not\subseteq \mathcal{P}_n$, and "$\Sigma_1$ in" $\not\subseteq \mathcal{D}_n$.

(2) "Σ_n in" $\not\subseteq \mathcal{S}_{n+1}$ and "Π_n in" $\not\subseteq \mathcal{P}_{n+1}$, for $n \geq 1$. "Δ_n in" $\not\subseteq \mathcal{D}_{n+1}$, for $n > 1$.

(3) $\mathcal{S}_{n+1} \not\subseteq$ "Σ_n in", $\mathcal{P}_{n+1} \not\subseteq$ "Π_n in", and $\mathcal{D}_{n+1} \not\subseteq$ "Δ_n in".

(4) $\mathcal{S}_n \not\subseteq$ "Δ_n in" and $\mathcal{P}_n \not\subseteq$ "Δ_n in".

Proof. Theorem 1.14 proves (1), and (2) follows from (1). To prove (3), choose B recursive, and choose A so that either $A \in \Sigma_{n+1}$ and $A \notin \Sigma_n$, $A \in \Pi_{n+1}$ and $A \notin \Pi_n$, or $A \in \Delta_{n+1}$ and $A \notin \Delta_n$. Then A $\mathcal{S}_{n+1}$ B and $A \notin \Sigma_n^B$, A $\mathcal{P}_{n+1}$ B and $A \notin \Pi_n^B$, or A $\mathcal{D}_{n+1}$ B and $A \notin \Delta_n^B$ respectively. To prove (4), choose B recursive, and choose A so that $A \in \Sigma_n$ and $A \notin \Delta_n$, or $A \in \Pi_n$ and $A \notin \Delta_n$. Then $A \leq_{\mathcal{S}_n} B$ and $A \notin \Delta_n^B$ or $A \leq_{\mathcal{P}_n} B$ and $A \notin \Delta_n^B$, respectively.

Theorem 2.5. $\mathcal{S}_n \cap \mathcal{P}_n \subseteq \mathcal{D}_n$.

Proof. Suppose A $\mathcal{S}_n$ B and A $\mathcal{P}_n$ B. $B \in \Delta_n^X \longrightarrow B \in \Sigma_n^X$ & $B \in \Pi_n^X \longrightarrow A \in \Sigma_n^X$ & $A \in \Pi_n^X \longrightarrow A \in \Delta_n^X$. Thus, A $\mathcal{D}_n$ B.

Theorem 2.6. $\mathcal{S}_n \cap \mathcal{P}_n \neq \mathcal{D}_n$.

Proof. $\mathcal{S}_n \cap \mathcal{P}_n = \mathcal{D}_n$ implies $\mathcal{D}_n \subset \mathcal{S}_n$. This is impossible, because $\mathcal{D}_n$ generalizes relative recursiveness and $\mathcal{S}_n$ does not generalize relative recursiveness.

3. The Relation $\mathcal{S}_1$.

Theorem 2.7. $\mathcal{S}_1$ is a maximal Σ_1-reducibility relation.

Proof. Suppose $\mathcal{S}_1 \subset \mathfrak{R} \subseteq$ "Σ_1 in". Then $\exists A \exists B[A \not\leq_{\mathcal{S}_1} B \ \& \ A \mathfrak{R} B]$. $A \not\leq_{\mathcal{S}_1} B$, therefore $\exists X[B \in \Sigma_1^X \ \& \ A \notin \Sigma_1^X]$. For some predicate R^X, R^X recursive in X, $\forall x(x \in B \leftrightarrow \exists y R^X(x,y))$. For any set C, $R^X \in \Sigma_1^C \rightarrow \exists y R^X \in \Sigma_1^C \rightarrow B \in \Sigma_1^C$. Thus, $B \ \mathcal{S}_1 \ R^X$. Therefore, $B \mathfrak{R} R^X$. If $\mathfrak{R}$ is transitive, then $A \mathfrak{R} R^X$. But $A \notin \Sigma_1^{R^X}$, since $A \notin \Sigma_1^X$. Thus $\mathfrak{R}$ is not transitive.

Corollary 2.1. $\forall X[B \in \Sigma_1^X \rightarrow A \in \Sigma_1^X] \leftrightarrow \forall X[B \ \mathcal{S}_1 \ X \rightarrow A \in \Sigma_1^X]$.

Lemma 2.1. For any two sets A and B, if there exist recursive functions f and g so that $\forall x[x \in A \leftrightarrow \exists y \forall z_{z<f(y)} g(x,y,z) \in B)$, then $A \ \mathcal{S}_1 \ B$.

Proof. The proof follows easily from the definition of $\mathcal{S}_1$.

The converse of Lemma 2.1 is also true if $B \neq \emptyset$, $B \neq \omega$. Define $A \ \mathcal{J}_1 \ B \leftrightarrow$ there exist recursive functions f and g so that $\forall x(x \in A \leftrightarrow \exists y \forall z_{z<f(y)} g(x,y,z) \in B)$. It will be shown that $A \ \mathcal{S}_1 \ B \leftrightarrow A \ \mathcal{J}_1 \ B$, if $B \neq \emptyset$, $B \neq \omega$.

Lemma 2.2. If there exist recursive functions f and g so that $\forall x[x \in A \leftrightarrow \exists y \forall i_{i<f(y)} \forall j_{j<f(y)} g(x,y,i,j) \in B)$, then there exist recursive functions f_1 and g_1 so that

$$\forall x(x \in A \leftrightarrow \exists y \forall z_{z<f_1(y)} g_1(x,y,z) \in B).$$

Proof. Define a recursive function $\tau: \omega \times \omega \xrightarrow[\text{onto}]{1-1} \omega$, so that $\tau(i,j) < \tau(n,n) \leftrightarrow i < n \ \& \ j < n$, $\tau(n,n) < \tau(n+1,n+1)$, and so that between $\tau(n,n)$ and $\tau(n+1,n+1)$ are precisely the values $\tau(i,n)$ and $\tau(n,j)$, for each $i < n$ & $j < n$, so that the ordered pairs $\langle i,n\rangle$ and $\langle n,j\rangle$, for $i < n$ & $j < n$, are ordered lexicographically.

[To illustrate, τ orders $\omega \times \omega$ as follows: $\langle 0,0\rangle$, $\langle 1,1\rangle$, $\langle 0,1\rangle$, $\langle 1,0\rangle$, $\langle 2,2\rangle$, $\langle 0,2\rangle$, $\langle 1,2\rangle$, $\langle 2,0\rangle$, $\langle 2,1\rangle$, $\langle 3,3\rangle$, $\langle 0,3\rangle$, $\langle 1,3\rangle$, $\langle 2,3\rangle$, $\langle 3,0\rangle$, $\langle 3,1\rangle$, $\langle 3,2\rangle$, $\langle 4,4\rangle$, ...

τ may be defined by the following recursion scheme:

$$f(0,0) = 0;$$

$$f(0,i+1) = f(i+1,i+1) + 1 = f(i,i\dot{-}1) + 2;$$

$$f(i+1,0) = f(i,i+1) + 1;$$

$$f(i+1,j+1) = \begin{cases} f(i,i\dot{-}1) + 1, & i = j \\ f(i,j+1) + 1, & i < j \\ f(i+1,j) + 1, & i > j.] \end{cases}$$

Then, there are recursive functions Π_1 and Π_2 so that $\tau(\Pi_1(i),\Pi_2(i)) = i$, for all i. For any n, since $\tau(i,j) < \tau(n,n) \leftrightarrow i < n \;\&\; j < n$, $z < \tau(n,n) \leftrightarrow \Pi_1(z) < n \;\&\; \Pi_2(z) < n$.

Now, assume A and B satisfy the hypothesis of Lemma 2.2, it follows that $\forall x(x\epsilon A \leftrightarrow \exists y \forall z_{z<\tau(f(y),f(y))}\, g(x,y,\Pi_1(z),\Pi_2(z))\epsilon B)$.

Choose $f_1(y) = \tau(f(y),f(y))$, and choose $g_1(x,y,z) = g(x,y,\Pi_1(z),\Pi_2(z))$.

Lemma 2.3. For any set A and any set B, B non-empty and $B \neq \omega$, if there exist no recursive functions f and g so that $\forall x(x\in A \longleftrightarrow \exists y \forall z_{z<f(y)} g(x,y,z)\in B)$, then there exists a set C so that $\forall x(x\in B \longleftrightarrow \exists y 2^x \cdot 3^y \in C)$ and $A \notin \Sigma_1^C$.

Proof. Let A and B be given so that the hypotheses of Lemma 2.3 are satisfied. We will construct a function h of two variables, as the union of a chain of two variable characteristic sequence numbers, so that $\forall x(x\in B \leftrightarrow \exists y h(x,y)=0)$, and so that $A \notin \Sigma_1^h$.

Define $\mathrm{adm}(\alpha) \longleftrightarrow \mathrm{Ch}_2(\alpha)$ & $\forall x,y < \ell h(\alpha)(((\alpha)_{x,y} = 1 \rightarrow x\in B)$.

Then, $\alpha >_{\mathrm{adm}} B \leftrightarrow \alpha >_2 \beta$ & $\mathrm{adm}(\alpha)$ & $\mathrm{adm}(\beta)$.

Stage 0. Define $h_0 = 1$. $\mathrm{adm}(h_0)$.

Stage e + 1. By induction hypothesis h_e is defined and $\mathrm{adm}(h_e)$. It is also assumed that $\forall x < \ell h(h_e)(x\in B \leftrightarrow \exists y < \ell h(h_e)((h_e)_{x,y}=1))$.

There are three cases to be considered:

Case 1. $\exists x \exists \alpha >_{\mathrm{adm}} h_e[x\notin A$ & $T_1^2(\alpha,e,x)]$;

Case 2. Case 1 fails and $\exists x \forall \alpha >_{\mathrm{adm}} h_e[x\in A$ & $\overline{T}_1^2(\alpha,e,x)]$;

Case 3. Cases 1 and 2 both fail. That is,

$\forall x[x\in A \leftrightarrow \exists \alpha >_{\mathrm{adm}} h_e T_1^2(\alpha,e,x)]$.

We argue first that in fact case 3 fails. It follows that case 1 occurs or case 2 occurs. Suppose that

$\forall x[x\in A \longleftrightarrow \exists \alpha >_{\mathrm{adm}} h_e T_1^2(\alpha,e,x)]$.

$$\exists\alpha >_{adm} h_e T_1^2(\alpha,e,x)$$

$$\leftrightarrow \exists y[Ch_2(y) \ \& \ y \succ h_e \ \& \ adm(y) \ \& \ T_1^2(y,e,x)]$$

$$\leftrightarrow \exists y[Ch_2(y) \ \& \ y \succ h_e \ \& \ T_1^2(y,e,x)$$

$$\& \ \forall i,j < \ell h(y)((y)_{i,j} = 1 \rightarrow x \in B)].$$

Define $R(x,y) \equiv [Ch_2(y) \ \& \ y \succ h_e \ \& \ T_1^2(y,e,x)]$. Then, $\forall x[x \in A \leftrightarrow \exists y(R(x,y) \ \& \ \forall i,j < \ell h(y)((y)_{i,j} = 1 \rightarrow i \in B))]$, and the predicate $R(x,y)$ is recursive.

Choose natural numbers a and b so that $a \in B \ \& \ b \notin B$. Define

$$g_1(y,i,j) = \begin{cases} i, & (y)_{i,j} = 1; \\ a, & (y)_{i,j} \neq 1. \end{cases}$$

Suppose $\forall i,j < \ell h(y)((y)_{i,j} = 1 \rightarrow i \in B)$. Then $\forall i,j < \ell h(y) g_1(y,i,j) \in B$.

Conversely, if $\forall i,j < \ell h(y) g_1(y,i,j) \in B$, if $i < \ell h(y)$, $j < \ell h(y)$, and $(y)_{i,j} = 1$, then $g_1(y,i,j) = i$. Thus, $\forall i,j < \ell h(y)((y)_{i,j} = 1 \rightarrow i \in B)$.

Therefore,

$$\forall x[x \in A \leftrightarrow \exists y(R(x,y) \ \& \ \forall i,j < \ell h(y) g_1(y,i,j) \in B)].$$

Define

$$g_2(x,y,i,j) = \begin{cases} g_1(y,i,j), R(x,y); \\ \\ b, \qquad \overline{R}(x,y). \end{cases}$$

Suppose $R(x,y)$ & $\forall i,j < \ell h(y) g_1(y,i,j) \in B$. Then, for each i and j, $g_2(x,y,i,j) = g_1(y,i,j)$. Thus $\forall i,j < \ell h(y) g_2(x,y,i,j) \in B$.

Conversely, suppose $\forall i,j < \ell h(y) g_2(x,y,i,j) \in B$. Then $\forall i,j < \ell h(y) g_2(x,y,i,j) = g_1(y,i,j)$, since $b \notin B$. Thus, $R(x,y)$ & $\forall i,j < \ell h(y) g_1(y,i,j) \in B$.

Therefore,

$$\forall x [x \in A \leftrightarrow \exists y \forall i < \ell h(y) \forall j < \ell h(y) g_2(x,y,i,j) \in B].$$

Then, by Lemma 2.2, there exist recursive functions f and g so that $\forall x (x \in A \leftrightarrow \exists y \forall z_{z<f(y)} g(x,y,x) \in B)$. But, this contradicts the hypothesis of our Lemma 2.2. Thus, case 3 does not occur.

We return now to cases 1 and 2 of stage $e + 1$:

Case 1. $\exists x \exists \alpha >_{adm} h_e [x \notin A \ \& \ T_1^2(\alpha,e,x)]$. Choose x and let $\alpha' = \mu\alpha >_{adm} h_e [x \notin A \ \& \ T_1^2(\alpha,e,x)]$.

$$\alpha' = \prod_{i<\ell h(\alpha')} p_i^{\prod_{j<\ell h(\alpha')} p_j^{a_{ij}}},$$

where, for each $i,j < \ell h(\alpha')$, $a_{ij} \in \{1,2\}$. For each $i < \ell h(\alpha')$, define

$$\kappa(i) = \left(\prod_{j<\ell h(\alpha')} p_j^{a_{ij}} \right) \cdot p_{\ell h(\alpha')}^{a_{i\ell h(\alpha')}} ,$$

where $a_{i\ell h(\alpha')} = 1$, if $i \in B$, and $a_{i\ell h(\alpha')} = 2$, if $i \notin B$.

Define

$$\kappa(\ell h(\alpha')) = \prod_{j<\ell h(\alpha')} p_j,$$

if $\ell h(\alpha') \in B$, and

$$\kappa(\ell h(\alpha')) = \prod_{j<\ell h(\alpha')} p_j^2,$$

if $\ell h(\alpha') \notin B$. Define

$$h_{e+1} = \prod_{i \leq \ell h(\alpha')} p_i^{\kappa(i)}.$$

Then, $adm(h_{e+1})$, and for each $x < \ell h(h_{e+1})$, if $x \in B$, then $(h_{e+1})_{x,\ell h(h_{e+1})} = 1$.

Case 2. $\exists x \forall \alpha >_{adm} h_e [x \in A \;\&\; \overline{T}_1^2(\alpha,e,x)]$. Choose x, and let $\alpha' = h_e$. Extend α' to h_{e+1} exactly as in case 1.

Finally, define h by

$$h(x,y) = \left(h_{\mu e[x,y<h_e]} \right)_{x,y} \dot{-} 1.$$

Suppose, for some x and y, that $h(x,y) = 0$. $adm(h_{\mu e[x,y<h_e]})$. Thus, $(h_{\mu e[x,y<h_e]})_{x,y} = 1$. Hence $x \in B$.

Now, suppose $x \in B$. Let $e' = \mu e[x < \ell h(e))$. Then $\exists y < \ell h(h_{e'})((h_{e'})_{x,y} = 1)$. Thus, $\exists y h(x,y) = 0$.

Therefore

$$x \in B \leftrightarrow \exists y h(x,y) = 0.$$

We show that $A \notin \Sigma_1^h$. $A \in \Sigma_1^h$ if and only if there exists an e so that $\forall x(x \in A \leftrightarrow \exists y T_1^2(\overline{h}(y,y), e, x))$. For each e, either case 1 or case 2 of stage $e + 1$ holds. If case 1 holds, then there is an x so that $x \notin A$ and so that $T_1^2(\alpha', e, x)$. Since T_1^2 is monotonic increasing and $h_{e+1} \succ \alpha'$, $T_1^2(h_{e+1}, e, x)$. Thus, $\exists y T_1^2(\overline{h}(y,y), e, x)$. On the other hand, if case 2 holds, then there is an x so that $x \in A$ and so that $\forall \alpha \succ \alpha' \overline{T}_1^2(\alpha, e, x)$. Thus, $\forall y > \ell h(\alpha') \overline{T}_1^2(\overline{h}(y,y), e, x)$. But, $\overline{T}_1^2$ is monotonic decreasing. Thus, $\forall y \overline{T}_1^2(\overline{h}(y,y), e, x)$. Therefore $A \notin \Sigma_1^h$.

Define $C = \{z \mid h((z)_0, (z)_1) = 0\}$. Then $A \notin \Sigma_1^C$, and $\forall x(x \in B \leftrightarrow \exists y 2^x \cdot 3^y \in C)$. This completes the proof of Lemma 2.3.

We are now ready to state the equivalence which was promised in the discussion following the proof of Lemma 2.1. The proof follows directly from Lemmas 2.1 and 2.3.

<u>Theorem</u> 2.8. For all A and B, $B \neq \emptyset$ and $B \neq \omega$, $A \mathcal{S}_1 B \leftrightarrow A \mathcal{T}_1 B$.

<u>Remark</u>. $A \mathcal{T}_1 \emptyset \rightarrow A = \emptyset$, and $A \mathcal{T}_1 \omega \rightarrow A = \omega$. On the other hand, $A \mathcal{S}_1 \emptyset \leftrightarrow A \in \Sigma_1$ and $A \mathcal{S}_1 \omega \leftrightarrow A \in \Sigma_1$.

By Theorem 2.2(10), $A \leq_r B$ does not imply $A \leq_{\mathcal{S}_1} B$. We establish next the existence of a set A so that A and $\overline{A}$ are $\mathcal{S}_1$-incomparable.

Lemma 2.4. There exists a function α so that $\{x|\alpha(x) = 0\} \neq \emptyset$, $\{x|\alpha(x) = 0\} \neq \omega$, and

(1) $\forall x(\alpha(x) = 0$ or $\alpha(x) = 1)$;

(2) there are no partial recursive functions g_1 and g_2 so that $\alpha(x) = 0 \leftrightarrow \exists y \forall i(g_1(y)$ defined & $(i<g_1(y) \rightarrow (g_2(x,y,i)$ defined & $\alpha(g_2(x,y,i)) = 1)))$;

(3) there are no partial recursive functions g_1 and g_2 so that $\alpha(x) = 1 \leftrightarrow \exists y \forall i(g_1(y)$ defined & $(i<g_1(y) \rightarrow (g_2(x,y,i)$ defined & $\alpha(g_2(x,y,i)) = 0)))$.

Proof. α will be constructed by induction.

Condition (2) is equivalent to the following (2'):

(2') $\forall e \forall \ell \exists x\{[\alpha(x) = 0$ & $\forall y \exists i(\{e\}(y)$ defined $\rightarrow (i<\{e\}(y)$ & $(\{\ell\}(x,y,i)$ defined $\rightarrow \alpha(\{\ell\}(x,y,i)) = 0)))]$ or $[\alpha(x) = 1$ & $\exists y \forall i(\{e\}(y)$ defined & $(i<\{e\}(y) \rightarrow (\{\ell\}(x,y,i)$ defined & $\alpha(\{\ell\}(x,y,i)) = 1)))]\}$.

Condition (3) is equivalent to the following (3'):

(3') $\forall e \forall \ell \exists x\{[\alpha(x) = 1$ & $\forall y \exists i(\{e\}(y)$ defined $\rightarrow (i<\{e\}(y)$ & $(\{\ell\}(x,y,i)$ defined $\rightarrow \alpha(\{\ell\}(x,y,i)) = 1)))]$ or $[\alpha(x) = 0$ & $\exists y \forall i(\{e\}(y)$ defined & $(i<\{e\}(y) \rightarrow (\{\ell\}(x,y,i)$ defined & $\alpha(\{\ell\}(x,y,i)) = 0)))]\}$.

Let τ, Π_1, and Π_2 be as defined in the proof of Lemma 2.2.

Construction of α:

<u>Stage</u> 0. Define $\alpha_0 = 1$.

<u>Stage</u> 2s + 1. By induction hypothesis α_{2s} is defined. α_{2s+1} shall be defined at this stage so that the part of (2') within the quantifiers on e and ℓ holds at $e = \Pi_1(s)$ and $\ell = \Pi_2(s)$ for all extensions of α_{2s+1}. Let $\Pi_1(s) = e$ and $\Pi_2(s) = \ell$.

<u>Case</u> 1. $\exists x[(x \geq \ell h(\alpha_{2s})$ or $(x < \ell h(\alpha_{2s})$ & $(\alpha_{2s})_x = 1))$ & $\forall y[\{e\}(y)$ undefined or $\exists i(i < \{e\}(y)$ & $(\{\ell\}(x,y,i)$ undefined or $(\{\ell\}(x,y,i) < \ell h(\alpha_{2s})$ & $(\alpha_{2s})_{\{\ell\}(x,y,i)} = 1)))]]$.

Let a be the least x satisfying the hypothesis of case 1.

Suppose $a < \ell h(\alpha_{2s})$ & $(\alpha_{2s})_x = 1$. Then (2') is already satisfied at e and ℓ. Define

$$\alpha_{2s+1} = \alpha_{2s} \cdot p^{1}_{\ell h(\alpha_{2s})} \cdot p^{2}_{\ell h(\alpha_{2s})+1}.$$

(Thus α is not the characteristic function of either $\emptyset$ or ω.)

Suppose $a > \ell h(\alpha_{2s})$. Define

$$\alpha_{2s+1} = \alpha_{2s} \cdot \prod_{\ell h(\alpha_{2s}) \leq x \leq a} p^{1} \cdot p^{2}_{a+1}.$$

(Then, $\alpha(a) = 0$ and α is not a constant function.)

Case 2. $\forall x[(x \geq \ell h(\alpha_{2s}) \text{ or } (x < \ell h(\alpha_{2s}) \ \& \ (\alpha_{2s})_x = 1)) \rightarrow \exists y[\{e\}(y)$ defined $\& \ \forall i(i < \{e\}(y) \rightarrow (\{\ell\}(x,y,i)$ defined $\& \ (\{\ell\}(x,y,i) > \ell h(\alpha_{2s})$ or $(\alpha_{2s})_{\{\ell\}(x,y,i)} = 2)))]]$.

Let $a = \ell h(\alpha_{2s})$. Let b be the least y satisfying the consequent of case 2. Let $c = \max_{i<\{e\}(b)} \{\ell\}(x,y,i)$.

Suppose $c < \ell h(\alpha_{2s})$. Then, define

$$\alpha_{2s+1} = \alpha_{2s} \cdot p^2_{\ell h(\alpha_{2s})} \cdot p^1_{\ell h(\alpha_{2s})+1}.$$

Then $\alpha(a)$ is defined so that $\alpha(a) = 1$. Thus (2') is satisfied at e and ℓ.

Suppose $c > \ell h(\alpha_{2s})$. Define

$$\alpha_{2s+1} = \alpha_{2s} \cdot \prod_{\ell h(\alpha_{2s}) \leq x \leq c} p^2_x \cdot p^1_{c+1}.$$

Then $\alpha(a) = 1$ and $\forall i < \{e\}(b)$, $\alpha(\{\ell\}(a,b,i)) = 1$. Thus, (2') is satisfied at e and ℓ.

Stage $2s + 2$. By induction hypothesis α_{2s+1} is defined. α_{2s+2} is defined at this stage so that the part of (3') within the quantifiers on e and ℓ holds at $e = \Pi_1(s)$ for all extensions of α_{2s+2}.

The construction of α_{2s+2} is the same, *mutatis mutandis*, as α_{2s+1}. Finally, let α be defined by $\alpha(x) = (\alpha_{\mu s[x < \ell h(\alpha_s)]})_x \dot{-} 1$.

<u>Theorem</u> 2.9. There exists a set A so that $A \not\leq_{\mathcal{S}_1} \overline{A}$ and $\overline{A} \not\leq_{\mathcal{S}_1} A$.

<u>Proof</u>. Apply Lemma 2.4 to obtain the function α. Let $A = \{x|\alpha(x) = 0\}$. Then $A \not\leq_{\mathcal{T}_1} \overline{A}$ and $\overline{A} \not\leq_{\mathcal{T}_1} A$. $A \neq \emptyset$ and $A \neq \omega$. Thus, by Theorem 2.8, $A \leq_{\mathcal{S}_1} \overline{A}$ and $\overline{A} \leq_{\mathcal{S}_1} A$.

4. $\mathcal{S}_n$-<u>degrees</u>.

We conclude this chapter with a brief development of the $\mathcal{S}_n$-degrees. Concepts and notation are analogous to those of the first paragraph of Chapter 1, section 5.

<u>Theorem</u> 2.10. For each $\mathcal{S}_n$-degree there is a larger one.

<u>Proof</u>. For each set A, $\underset{\sim}{d}_{\mathcal{S}_n}(A) < \underset{\sim}{d}_{\mathcal{S}_n}(A^{(n+1)})$, since $A^{(n+1)} \not\leq_{\mathcal{S}_n} A$.

<u>Remark</u>. $\underset{\sim}{d}_{\mathcal{S}_1}(A')$ is not always greater than $\underset{\sim}{d}_{\mathcal{S}_1}(A)$. For the proof we cite [13, Theorem 4 and Corollary 3, pp.6-7].

<u>Definition</u> 2.3. Define the recursive sup. of the two sets A and B by $A \vee B = \{x|(x)_0 \in A \ \& \ (x)_1 \in B\}$.

<u>Lemma</u> 2.5. $A \leq_{\mathcal{S}_1} A \vee B$ and $B \leq_{\mathcal{S}_1} A \vee B$.

<u>Proof</u>. $x \in A \leftrightarrow \exists y 2^x \cdot 3^y \in A \vee B$. $x \in B \leftrightarrow \exists y 2^y \cdot 3^x \in A \vee B$. Thus, $\forall n(A \leq_{\mathcal{S}_n} A \vee B \ \& \ B \leq_{\mathcal{S}_n} A \vee B)$.

<u>Theorem</u> 2.11. $\forall X[A \vee B \in \Sigma_n^X \leftrightarrow A \in \Sigma_n^X \ \& \ B \in \Sigma_n^X]$.

<u>Proof</u>. The implication from left to right follows from Lemma 2.5.

Suppose $A \in \Sigma_n^X$ and $B \in \Sigma_n^X$. For some S^X and R^X, recursive in X, $x \in A \leftrightarrow \exists y_1 \ldots Qy_n S^X(x, y_1, \ldots, y_n)$, and $x \in B \leftrightarrow \exists y_1 \ldots Qy_n R^X(x, y_1, \ldots, y_n)$. Thus $x \in A \vee B \leftrightarrow [\exists y_1 \ldots Qy_n S^X((x)_0, y_1, \ldots, y_n)$ & $\exists y_1 \ldots Qy_n R^X((x)_1, y_1, \ldots, y_n)]$. Thus, $A \vee B \in \Sigma_n^X$.

<u>Theorem</u> 2.12. $\underset{\sim}{d}_{\mathfrak{s}_n}(A \vee B)$ is the least upper bound of $\underset{\sim}{d}_{\mathfrak{s}_n}(A)$ and $\underset{\sim}{d}_{\mathfrak{s}_n}(B)$. Hence, the Σ_n-degree ordering is an upper semilattice.

<u>Proof</u>. By Lemma 2.5 and the remark following, $d_{\mathfrak{s}_n}(A \vee B)$ is an upper bound. Suppose $A \leq_{\mathfrak{s}_n} C$ and $B \leq_{\mathfrak{s}_n} C$. Then, $\forall X[C \in \Sigma_n^X \rightarrow A \in \Sigma_n^X \ \& \ B \in \Sigma_n^X]$. Thus, by Theorem 2.11, $\forall X[C \in \Sigma_n^X \rightarrow A \vee B \in \Sigma_n^X]$. Hence, $A \vee B \leq_{\mathfrak{s}_n} C$. Thus, $\underset{\sim}{d}_{\mathfrak{s}_n}(A \vee B)$ is a least upper bound.

<u>Theorem</u> 2.13. $\underset{\sim}{O}_{\mathfrak{s}_n}$ is the class of all Σ_n sets.

<u>Proof</u>. The proof follows immediately from Theorem 2.2 (5) and (8).

The example used to prove the following theorem was suggested by Thomas Grilliot.

<u>Theorem</u> 2.14. The $\mathfrak{s}_1$-degree ordering is not a lattice.

<u>Proof</u>. Let C be a complete Σ_1^1 set. (We will without further mention use the theorems of [5] that are by now well-known.)

It will be shown that there is no greatest lower bound of $\underset{\sim}{d}_{\mathcal{S}_1}(C)$ and $\underset{\sim}{d}_{\mathcal{S}_1}(\overline{C})$. Suppose $A \leq_{\mathcal{S}_1} C$ and $A \leq_{\mathcal{S}_1} \overline{C}$. By Theorem 2.8, there exist recursive functions f and g so that $\forall u(u \in A \leftrightarrow \exists y \forall z < f(y) g(x,y,z) \in C)$. Since $C \in \Sigma_1^1$, there is a recursive predicate P so that $\forall u(u \in C \leftrightarrow \exists \alpha \forall x P(\overline{\alpha}(x), u))$, where α is a function variable. Thus, $\forall u(u \in A \leftrightarrow \exists y \forall z < f(y) \exists \alpha \forall x P(\overline{\alpha}(x), g(u,x,z)))$. Therefore, $A \in \Sigma_1^1$. Similarly it can be shown that $A \in \Pi_1^1$. From $A \in \Sigma_1^1 \cap \Pi_1^1$ it follows that $A'' \in \Sigma_1^1 \cap \Pi_1^1$. Hence $A'' \leq_1 C$ and $A'' \leq_1 \overline{C}$. Therefore, $A'' \leq_{\mathcal{S}_1} C$ and $A'' \leq_{\mathcal{S}_1} \overline{C}$. By Theorem 2.10, $\underset{\sim}{d}_{\mathcal{S}_1}(A) \lneq \underset{\sim}{d}_{\mathcal{S}_1}(A'')$. Thus A is not a greatest lower bound.

Open Questions.

As has already been mentioned in the Introduction, the principal open question about the hierarchy of relations $\mathcal{S}_n$, is whether $\mathcal{S}_n$, for $n > 1$, is a maximal Σ_n-reducibility. It is also not known whether there is something analogous to Theorem 2.8 for $n > 1$. Our paper [15] will contain a more detailed discussion of these questions. Without such a characterization for $\mathcal{S}_n$, $n > 1$, it is not known, for C a complete Σ_1^1 set, whether $A \leq_{\mathcal{S}_n} C \rightarrow A \in \Sigma_1^1$. Thus it is not known whether the above argument is applicable if $n > 1$. Indeed, we can only conjecture that the $\mathcal{S}_n$-degree ordering is not a lattice.

FOOTNOTES

1. This paper is part of the author's doctoral dissertation directed by Professor Paul Axt and partially supported by NSF Grant GP 7077. The results in this paper were first announced in [11].

BIBLIOGRAPHY

[1] Friedberg, Richard, "A Criterion for Completeness of Degrees of Unsolvability", J.S.L. 22(1957), 159.

[2] ________________, "Two Recursively Enumerable Sets of Incomparable Degrees of Unsolvability", Proc. Nat. Acad. of Sci. U.S.A. 43(1957), 236-238.

[3] Hinman, P. G., "Some Applications of Forcing to Hierarchy Problems in Arithmetic", Z. Math. Logik Grundl. Math. 15 (1969), 341-352.

[4] Kleene, S., Introduction to metamathematics, 5th ed., D. Van Nostrand, Princeton, N. J., 1950.

[5] __________, "Arithmetical Predicates and Function Quantifiers", Trans. A.M.S. 79(1955), 312.

[6] Kleene, S. and Emil L. Post, "The Upper Semi-Lattice of Degrees of Recursive Unsolvability", Annals of Math. 59(1954), 379-407.

[7] Muchnik, A. A., "Negative Answer to the Problem of Reducibility of the Theory of Algorithms", (in Russian), Dokl. Akad. Nauk SSSR 108(1956), 194-197.

[8] Post, Emil L., "Degrees of Recursive Unsolvability", preliminary report, Bull. A.M.S. 54(1948), 641-642.

[9] Rogers, Hartley, Jr., Theory of Recursive Functions and Effective Computability, McGraw-Hill Book Co., New York, 1967.

[10] Sacks, Gerald E., "Degrees of Unsolvability", Annals of Mathematics Studies, no.55, Princeton University Press, Princeton, N. J., 1963.

[11] Selman, Alan L., "Two Hierarchies of Reducibilities", Notices A.M.S. 16(1969), 951; Abstract 668-15.

[12] ______________, Arithmetical Reducibilities and Sets of Formulas Valid in Finite Structures, Ph.D. Thesis, Pennsylvania State University, 1970.

[13] Selman, Alan L., "Sets of Formulas Valid in Finite Structures", Department of Mathematics, Carnegie-Mellon University RR 70-41, 1970.

[14] ______________., "Applications of Forcing to Definability Problems in the Arithmetical Hierarchy", Department of Mathematics, Carnegie-Mellon University RR 70-52, 1970.

[15] ______________., "Arithmetical Reducibilities, II", in preparation.

[16] Shoenfield, J. R., "On Degrees of Unsolvability", Annals of Mathematics 69(1959), 644-653.

[17] Spector, Clifford, "On Degrees of Recursive Unsolvability", Annals of Mathematics 64(1956), 581.