FLAT SPACES OF CONTINUOUS FUNCTIONS

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1. Introduction.

In [7], Harrell and Karlovitz call a Banach space _flat_ if there exists on the surface of its unit ball a curve of length 2 with antipodal endpoints. They observe that $L^1(\mu)$, where μ is Lebesgue measure on the unit interval, is flat, but that $\ell^1(\aleph_0)$ is not. They had shown earlier [6] that a flat space is not reflexive, and that $C([0,1])$ is flat. In [12], Schäffer showed that $L^1(\mu)$ for a general measure space is flat if and only if μ is not purely atomic.

Continuing the investigation of the flatness of "classical" spaces, we are led to consider the space $C(K)$ for a compact Hausdorff space K, and, more generally, the subspace $C_\sigma(K)$ of those functions that are skew with respect to an involutory automorphism σ of K. The purpose of this paper is to give a complete account of which $C_\sigma(K)$ are flat: in terms of the topology of K, they are exactly those for which there exists a non-empty

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dense-in-itself set in K not containing fixed points of σ. The flatness of $C_\sigma(K)$ can also be characterized in terms of the geometry of its dual: in particular, $C_\sigma(K)$ is flat if and only if its dual is flat.

These various characterizations yield a similar account for $C(K)$ itself and for $C_0(T)$, the space of continuous functions vanishing at infinity on the locally compact Hausdorff space T. Among other results concerning spaces congruent to some $C_\sigma(K)$ we note the fact that every infinite-dimensional space $L^\infty(\mu)$ is flat.

The spaces $C_\sigma(K)$ are discussed and characterized by their metric properties in [2; pp. 87-96], an account of work due in the main to Jerison. Lindenstrauss [8] proposes an interesting definition of "classical Banach spaces in the isometric sense"; he points out that they turn out to be exactly the Banach spaces congruent to $L^p(\mu)$ for $1 \leqq p < \infty$, together with those whose dual is congruent to some $L^1(\mu)$. Now the $L^p(\mu)$ are reflexive, and therefore not flat, for $1 < p < \infty$; and the $L^1(\mu)$ were classified as to their flatness in [12]. The spaces $C_\sigma(K)$ are important instances of spaces with duals congruent to L^1-spaces, but do not exhaust this class by far (see [9] and references given there for a complete description). It would be interesting to decide which of the remaining such spaces are flat--thus completing the survey of all "classical" spaces--or at least which M-spaces or G-spaces are flat (terminology as in [9]). The fragmentary results available on this point are not included here.

The question of flatness of Banach spaces belongs to an area of investigation begun in [11] and continued in other papers, dealing with certain metric parameters of the unit spheres of normed spaces. In another paper [13] one of us shall discuss the values of these, viz., the <u>inner diameter</u>, the <u>perimeter</u>, and the <u>girth</u>, for all the spaces treated here.

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2. Preliminaries.

If X is a normed space, a <u>subspace</u> of X is a linear manifold of X (not necessarily closed), provided with the norm of X. A <u>congruence</u> is an isometric isomorphism of one normed space onto another.

A curve in X is a "rectifiable geometric curve" as defined in [1; pp. 23-26]; for terminological details see [11; p. 61]. The length of a curve c is $\ell(c)$, and its standard representation in terms of arc-length is $g_c : [0,\ell(c)] \to X$.

X is <u>flat</u> if there is a curve of length 2 in the boundary of the unit ball of X such that its endpoints are antipodal; i.e., a curve c with $\ell(c) = 2$, $\|g_c(s)\| = 1$ for $s \in [0,2]$, and $g_c(0) + g_c(2) = 0$. If a subspace is flat, it obviously follows that X itself is flat.

Let T be a Hausdorff space; then $C(T)$ is the Banach space of all bounded real-valued continuous functions on T with the supremum norm. Let σ be an involutory automorphism of T, i.e., a homomorphism of T onto T with $\sigma \circ \sigma = \mathrm{id}$. Then $C_\sigma(T)$ denotes the closed subspace $\{f \in C(T) : f(t) + f(\sigma t) = 0,\ t \in T\}$ of $C(T)$; it is also a Banach space. We set $T^\sigma = \{t \in T : \sigma t \neq t\}$ the open set of points not fixed by σ, and observe once and for all that

(1) $$f \in C_\sigma(T) \text{ implies } f(T\setminus T^\sigma) \subset \{0\}.$$

If T is locally compact, $C_0(T)$ denotes the closed subspace of $C(T)$ consisting of the real-valued continuous functions on T that vanish at infinity. If σ is as before, we set $C_{0\sigma}(T) = C_0(T) \cap C_\sigma(T)$.

We summarize a useful remark for the study of $C_\sigma(K)$, K compact, in the following lemma.

1. *Lemma*. *Let K be a compact Hausdorff space and σ an involutory automorphism of K. Let $K' = K^\sigma \cup \{\infty\}$ be the one-point compactification of the locally compact space K^σ, and $\sigma' : K' \to K'$ defined by $\sigma' t = \sigma t$, $t \in K^\sigma$ and $\sigma'\infty = \infty$. Then σ' is an involutory automorphism of the compact Hausdorff space K', $K'^{\sigma'} = K^\sigma$, and the mapping $f \mapsto f' : C_\sigma(K) \to C_{\sigma'}(K')$ defined by $f'(t) = f(t)$, $t \in K^\sigma$ and $f'(\infty) = 0$ is a congruence*.

Proof. Immediate from the definitions and (1).

A Hausdorff space T contains a largest dense-in-itself subset; this set is closed and is called the <u>perfect core</u> of T. A space is <u>scattered</u> if its perfect core is empty. Pełczyński and Semadeni [10] have given a great number of equivalent conditions for a compact space K to be scattered, and especially some involving $C(K)$ and $(C(K))^*$. We reformulate for our use three of these conditions. If K is a compact Hausdorff space and $t \in K$, the <u>evaluation functional</u> $e_t \in (C(K))^*$ is defined by $\langle f, e_t \rangle = f(t)$, $f \in C(K)$.

2. <u>Theorem</u> (Pełczyński and Semadeni). <u>Let</u> K <u>be a compact Hausdorff space. The following statements are equivalent</u>:

(a): K <u>is not scattered</u>;

(b): <u>there exists</u> $h \in C(K)$ <u>such that</u> $h(K) = [0,1]$;

(c): <u>the linear mapping</u> $\Gamma : \ell^1(K) \to (C(K))^*$ <u>defined by</u> $\Gamma y = \sum_{t \in K} y(t) e_t$, $y \in \ell^1(K)$, <u>is not surjective</u>.

<u>Proof</u>. [10; Main Theorem, (0), (3), (11)].

3. <u>The main result</u>.

We examine the following properties that a normed space X may have:

(F1): X <u>is flat</u>;

(F2): X^* <u>is not the closed linear span of the extreme points of its unit ball</u>;

(F3): X^* <u>is not congruent to</u> $\ell^1(A)$ <u>for any set</u> A;

(F4): X^* <u>is flat</u>.

We observe that $\ell^1(A)$ is the closed linear span of the extreme points of its unit ball, so that (F2) always implies (F3).

Before we discuss these conditions as applicable to a space $C_\sigma(K)$, we look at a special case. We define $\pi : [-1,1] \to [-1,1]$ by $\pi t = -t$, an involutory automorphism of $[-1,1]$. The proof of the following lemma is an adaptation of a construction in [6].

3. <u>Lemma</u>. <u>The space</u> $C_\pi([-1,1])$ <u>is flat</u>.

<u>Proof</u>. We define $g : [0,2] \to C_\pi([-1,1])$ by

$$(g(s))(t) = -(g(s))(-t) = \begin{cases} 2(1-s)t & 0 \leqq t \leqq \frac{1}{2} \\ |4-4t-s|-1 & \frac{1}{2} \leqq t \leqq 1 \end{cases} \qquad 0 \leqq s \leqq 2 .$$

Then $\|g(s)\| = |(g(s))(1-\frac{1}{4}s)| = 1$, and $\|g(s')-g(s)\| = |s'-s|$, as is easily verified directly. Therefore g is Lipschitzian, and is the standard representation in terms of arc-length of a curve of length 2 in the boundary of the unit ball of $C_{\pi}([-1,1])$. But $g(2) = -g(0)$, so the endpoints of the curve are antipodal, and the space is flat.

In the rest of this section, we shall be dealing with a given compact Hausdorff space K and an involutory automorphism σ of K. The following construction is useful. Let V be a closed set in K with $V \cap \sigma V = \emptyset$, and let $f_0 \in C(V)$ be given. By the Tietze Extension Theorem there exists $f_1 \in C(K)$ with $\|f_1\| = \|f_0\|$ and $f_1(t) = -f_1(\sigma t) = f_0(t)$, $t \in V$. We define $f : K \to R$ by $f(t) = \frac{1}{2}(f_1(t) - f_1(\sigma t))$, $t \in K$, and find $f \in C_{\sigma}(K)$, $\|f\| = \|f_0\|$, and $f(t) = f_0(t)$, $t \in V$. Such a function f shall be called a <u>skew Tietze extension of</u> f_0.

For every $t \in K$, we consider the evaluation functional $e_t^{\sigma} \in (C_{\sigma}(K))^*$, (the restriction of e_t to $C_{\sigma}(K)$) defined by $\langle f, e_t^{\sigma} \rangle = f(t)$, $f \in C_{\sigma}(K)$. <u>The set</u> $\{e_t^{\sigma} : t \in K^{\sigma}\}$ <u>is exactly the set of extreme points of the unit ball of</u> $(C_{\sigma}(K))^*$ [2; p. 89].

4. Lemma. With K, σ as specified, let a non-empty set $P \subset K$ satisfy $P \cap \sigma P = \emptyset$. Then the linear mapping $\Gamma_P : \ell^1(P) \to (C_\sigma(K))^*$ defined by $\Gamma_P y = \sum_{t \in P} y(t) e_t^\sigma$, $y \in \ell^1(P)$, is isometric.

Proof. Obviously, $P \subset K^\sigma$. Now $\|e_t^\sigma\| = 1$, $t \in P$, so Γ_P is well defined, linear, and bounded, and $\|\Gamma_P\| \leqq 1$. It remains to prove that $\|\Gamma_P y\| \geqq \|y\|$ for all $y \in \ell^1(P)$, or at least for all those with finite support. If $Q \subset P$ is finite and $y(t) = 0$, $t \in P \setminus Q$, we can find, by means of a skew Tietze extension, $f \in C_\sigma(K)$ with $\|f\| = 1$ and $f(t) = \operatorname{sgn} y(t)$, $t \in Q$. Then

$$\|\Gamma_P y\| \geqq \|f\| \|\Gamma_P y\| \geqq \left| \left\langle f, \sum_{t \in Q} y(t) e_t^\sigma \right\rangle \right|$$

$$= \sum_{t \in Q} y(t) \operatorname{sgn} y(t) = \sum_{t \in Q} |y(t)| = \|y\| .$$

We are now ready to characterize those K and σ for which $C_\sigma(K)$ satisfies (F1)-(F4).

5. Theorem. Let K be a compact Hausdorff space and σ an involutory automorphism of K. Then (F1), (F2), (F3), (F4) are equivalent for $X = C_\sigma(K)$, and also equivalent to each of the following statements:

(a): K^σ *is not scattered;*

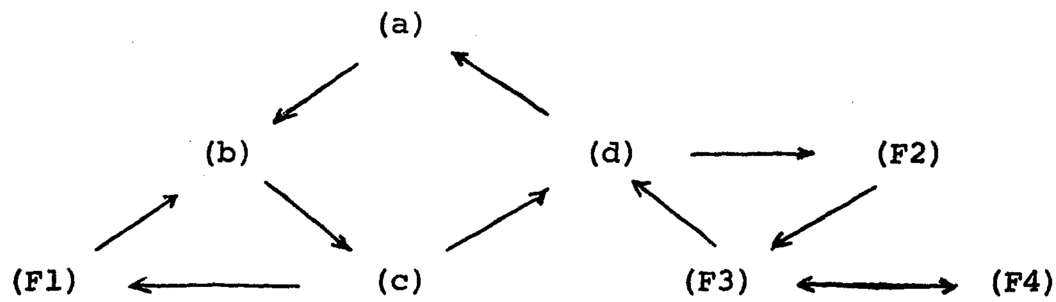
(b): *there exists* $h \in C_\sigma(K)$ *with* $h(K^\sigma) = [-1,1]$;

(c): *there exists* $h \in C_\sigma(K)$ *with* $h(K) = [-1,1]$.

Proof. We add one more statement to the list:

(d): *if* $P \subset K$ *satisfies* $P \cap \sigma P = \emptyset$, $P \cup \sigma P = K^\sigma$, *the isometric linear mapping* $\Gamma_P : \ell^1(P) \to (C_\sigma(K))^*$ *defined in Lemma* 4 *is not surjective;*

and prove the implications



In view of the formulation of statements (a), (b), (c), (d) it is possible to apply Lemma 1 (observing (1)) and assume without loss, as we shall in this proof, that $K \backslash K^\sigma$ is a singleton, say $\{\infty\}$. If $K^\sigma = \emptyset$, the theorem is trivial. We therefore assume without loss that $K^\sigma \neq \emptyset$.

The implication (F2) $\longrightarrow$ (F3) was noted above, and the implications (b) $\longrightarrow$ (c) and (F3) $\longrightarrow$ (d) are trivial.

(a) <u>implies</u> (b). The perfect core S of K^σ is not empty; choose $t_0 \in S$. Since $t_0 \neq \sigma t_0$, there exists an open neighborhood U of t_0 such that $\mathrm{cl}\, U \cap \mathrm{cl}(\sigma U) = \emptyset$; in particular, $\mathrm{cl}\ U \subset K^\sigma$. Then $U \cap S$ is non-empty and dense-in-itself, hence $\mathrm{cl}\ U$ is compact and not scattered. By Theorem 2 there exists $h_0 \in C(\mathrm{cl}\ U)$ with $h_0(\mathrm{cl}\ U) = [0,1]$. A skew Tietze extension h of h_0 satisfies $h \in C_\sigma(K)$, $[-1,1] \supset h(K^\sigma) \supset h(\mathrm{cl}\ U) \cup h(\mathrm{cl}(\sigma U)) = [0,1] \cup [-1,0] = [-1,1]$, as required by (b).

(c) <u>implies</u> (F1). With h as in (c), the mapping $\varphi \mapsto \varphi \circ h$ is a congruence of $C_\pi([-1,1])$ onto a closed subspace of $C_\sigma(K)$. By Lemma 3, this subspace is flat; hence $C_\sigma(K)$ itself is flat.

(F1) <u>implies</u> (b). Let c be a curve of length 2 in the boundary of the unit ball of $C_\sigma(K)$, with antipodal endpoints. Let $r \in [-1,1]$ be given. Since $g_c(1-r) \in C_\sigma(K)$, $\|g_c(1-r)\| = 1$, there exists $t_r \in K^\sigma$ such that $(g_c(1-r))(t_r) = 1$. Then

$$r = 1-(1-r) \leqq 1-\|g_c(1-r)-g_c(0)\| \leqq 1-(1-(g_c(0))(t_r))$$

$$= (g_c(0))(t_r) = (1+(g_c(0))(t_r))-1 \leqq \|g_c(1-r)+g_c(0)\|-1$$

$$= \|g_c(2) - g_c(1-r)\|-1 \leqq 2-(1-r)-1 = r .$$

Therefore $r = (g_c(0))(t_r) \in (g_c(0))(K^\sigma)$; since $r \in [-1,1]$ is arbitrary and $\|g_c(0)\| = 1$, we conclude that (b) is satisfied with $h = g_c(0)$.

(c) <u>implies</u> (d). With h as in (c), consider once more the congruence $\varphi \mapsto \varphi \circ h$ of $C_\pi([-1,1])$ onto a closed subspace of $C_\sigma(K)$. If, contrary to (d), Γ_P were surjective for some $P \subset K$, $P \cap \sigma P = \emptyset$, then every element of $(C_\pi([-1,1]))^*$ would, by the Hahn-Banach Theorem, be of the form $\sum_{t\in P} y(t)e^\pi_{h(t)}$, $y \in \ell^1(P)$; however, the linear functional $\varphi \mapsto \int_0^1 \varphi(r)dr$ on $C_\pi([-1,1])$ is bounded, but not of this form.

(d) <u>implies</u> (a). Assume, contrary to (a), that K^σ is scattered. Since $K\backslash K^\sigma$ is a singleton, K itself is scattered. Let $x^* \in (C_\sigma(K))^*$ be given. By the Hahn-Banach Theorem, x^* can be extended to an element of $(C(K))^*$. By Theorem 2, there exists $y_0 \in \ell^1(K)$ such that

$$\langle f,x^*\rangle = \langle f,\Gamma y_0\rangle = \langle f, \sum_{t\in K} y_0(t)e_t\rangle = \langle f, \sum_{t\in K^\sigma} y_0(t)e^\sigma_t\rangle,$$

$$f \in C_\sigma(K),$$

since $\langle f, e_\infty \rangle = f(\infty) = 0$. Thus

(2) $$x^* = \sum_{t \in K^\sigma} y_0(t) e_t^\sigma .$$

Let P be any set in K that is maximal with respect to the condition $P \cap \sigma P = \emptyset$ (such exist, by Zorn's Lemma); then $P \cup \sigma P = K^\sigma$. We define $y \in \ell^1(P)$ by $y(t) = y_0(t) - y_0(\sigma t)$, $t \in P$ (so that $\|y\| \leqq \|y_0\|$). Then (2) implies--since $e^\sigma_{\sigma t} = -e_t^\sigma$, $t \in P$--

$$x^* = \sum_{t \in P} y_0(t) e_t^\sigma + \sum_{t \in \sigma P} y_0(t) e_t^\sigma = \sum_{t \in P} (y_0(t) e_t^\sigma + y_0(\sigma t) e_{\sigma t}^\sigma)$$

$$= \sum_{t \in P} y(t) e_t^\sigma = \Gamma_P y .$$

Since $x^* \in (C_\sigma(K))^*$ was arbitrary, Γ_P is surjective, in contradiction to (d).

(d) <u>implies</u> (F2). Let $P \subset K$ satisfy $P \cap \sigma P = \emptyset$, $P \cup \sigma P = K^\sigma$; we have just shown that such a set exists. As noted earlier in this section, the set of extreme points of the unit ball of $(C_\sigma(K))^*$ is $\{e_t^\sigma : t \in K^\sigma\} = \{\pm e_t^\sigma : t \in P\}$. But the image of Γ_P contains this set, and hence also contains (actually, coincides with) the closed linear span of this set of extreme points. The required implication follows.

(F3) _is equivalent to_ (F4). Since $(C_\sigma(K))^*$ is an abstract L-space (cf. [9]), it is flat if and only if it is not congruent to $\ell^1(A)$ for any set A [12].

Remark 1. Using statement (a), it is possible to apply the equivalences of Pełczyński and Semadeni [10] to derive many other conditions equivalent to (F1)-(F4) for $X = C_\sigma(K)$; e.g., _there exists a non-atomic regular finite Borel measure_ ν _on_ K _such that_ $\nu(K^\sigma) > 0$.

Remark 2. In [13] we shall give further conditions on the metric structure of the unit balls of X, X^* that are equivalent to (F1)-(F4) for $X = C_\sigma(K)$.

4. _Applications to other spaces_.

Theorem 5 provides criteria for the flatness of Banach spaces congruent to $C_\sigma(K)$. The following theorems summarize some of these criteria.

6. Theorem. *If T is a locally compact Hausdorff space, (F1), (F2), (F3), (F4) are equivalent for $X = C_0(T)$, and also equivalent to each of the following statements:*

(a): *T is not scattered;*

(b): *there exists $h \in C_0(T)$ with $h(T) = [0,1]$.*

More in particular, if K is a compact Hausdorff space, (F1), (F2), (F3), (F4) are equivalent for $X = C(K)$, and also equivalent to each of the following statements:

(a): *K is not scattered;*

(b): *there exists $h \in C(K)$ with $h(K) = [0,1]$.*

Proof. If T is a locally compact Hausdorff space, let $T+T$ be the topological sum of T and T; the points of $T+T$ are, say, (t,j), $t \in T$, $j = \pm 1$. $T+T$ is a locally compact Hausdorff space; let $K = (T+T) \cup \{\infty\}$ be its one-point compactification (if T is itself compact, ∞ is isolated and will do no harm). The mapping $\sigma : K \to K$ defined by $\sigma(t,j) = (t,-j)$, $t \in T$, $j = \pm 1$, and $\sigma\infty = \infty$ is an involutory automorphism of K, with $K^{\sigma} = T+T$. It is easily verified that the mapping $f \mapsto f'$: $C_0(T) \to C_{\sigma}(K)$ is a congruence, where f' is defined by $f'((t,j)) = jf(t)$, $t \in T$, $j = \pm 1$, and $f'(\infty) = 0$. By Theorem 5 applied to K, σ as constructed, statements (F1)-(F4) for $X = C_0(T)$ are indeed equivalent, and equivalent to "$T+T$ is not scattered" and "there exists $h \in C_0(T)$ with $h'(T+T) = h(T) \cup -h(T) = [-1,1]$". The first of these is equivalent to statement (a). The second implies that statement (b) is satisfied with $|h| \in C_0(T)$ instead of h; and if h satisfies (b) then indeed $h(T) \cup - h(T) = [-1,1]$.

The result for compact K follows from this, since $C(K) = C_0(K)$.

Remark 1. For compact K and $X = C(K)$, the equivalence of statements (a), (b), (F3) appears in [10; Main Theorem, (0), (3), (12)].

Remark 2. A result closely analogous to Theorem 5 can be formulated for $C_{0\sigma}(T)$, where T is a locally compact Hausdorff space and σ an involutory automorphism of T, since σ has an obvious unique extension to an involutory automorphism of the one-point compactification of T.

Remark 3. Theorem 6 implies that $C_0(A)$ (sometimes called $\ell_0^\infty(A)$) is not flat for any set A. This is in contrast to $\ell^\infty(A)$ (or $m(A)$), which is flat for every infinite set A (Corollary 8 or Theorem 10).

7. Theorem. *If T is a completely regular Hausdorff space, then* (F1), (F2), (F3), (F4) *are equivalent for* $X = C(T)$, *and hold if and only if either* T *is not pseudocompact or there exists a continuous mapping of* T *onto* $[0,1]$.

Proof. Let T be embedded (as a dense set) in its Stone-Čech compactification βT. The mapping $f \mapsto f'$: $C(T) \to C(\beta T)$ is a congruence, where f' is the unique continuous extension of f to βT. We may therefore apply Theorem 6 to $K = \beta T$ and conclude that statements (F1)-(F4) are equivalent for $X = C(T)$ and hold if and only if

(*) *there exists* $h \in C(T)$ *such that* $h(T)$ *is a dense subset of* $[0,1]$.

If T is pseudocompact, every continuous image of T in R is pseudocompact, hence compact; in this case, (*) is equivalent to the existence of $h \in C(T)$ with $h(T) = [0,1]$. If, on the other hand, T is not pseudocompact, we use an argument adapted from [4]. Let $f : T \to R$ be an unbounded continuous function; then there exists a countably infinite set $S \subset f(T)$ that is closed and discrete in R. Since R is normal, there exists, by the Tietze Extension Theorem, a continuous $\varphi : R \to [0,1]$ such that $\varphi(S)$ is dense in $[0,1]$. Then $h = \varphi \circ f$ satisfies (*).

Remark. An analogous theorem can be formulated for $C_\sigma(T)$, where T is a completely regular Hausdorff space and σ is an involutory automorphism of T, since σ has a unique extension to an involutory automorphism of βT.

8. Corollary. If T is a metrizable space, (F1), (F2), (F3), (F4) are equivalent for $X = C(T)$, and hold unless T is compact and scattered.

Proof. From Theorems 6 and 7, since a metrizable pseudocompact space is compact.

For non-compact pseudocompact spaces, Theorem 7 remains unsatisfactory: pseudocompactness itself has a simple intrinsic characterization for completely regular Hausdorff spaces [3; p. 232], but we lack such a characterization of those pseudocompact spaces that can be mapped continually onto $[0,1]$, or, equivalently, have a Stone-Čech compactification that is not scattered. We point out that such a pseudocompact space may well be scattered itself: it is easy to construct a suitable instance of the scattered pseudocompact space Ψ described in [5; 5I] so that it has a continuous mapping onto [0,1]; the construction is suggested by [5;6Q].

A topological space is _basically disconnected_ if the closure of every co-zero set is open. Extremally disconnected spaces are basically disconnected.

9. _Theorem_. _If_ T _is a completely regular Hausdorff space that is basically disconnected, then_ (F1), (F2), (F3), (F4) _are equivalent for_ $X = C(T)$ _and hold unless_ T _is finite_.

Proof. If T is infinite, βT contains a subset homeomorphic to βN [5; 9H]; but βN is not scattered, hence βT is not scattered. Since $C(T)$ and $C(\beta T)$ are congruent, the conclusion follows from Theorem 6.

10. _Theorem_. _If_ $(S, \underset{\sim}{S}, \mu)$ _is any measure space_, (F1), (F2), (F3), (F4) _are equivalent for_ $X = L^{\infty}(\mu)$ _and hold unless this space is finite-dimensional_.

Proof. $L^{\infty}(\mu)$ is congruent to $C(K)$, where K is the Stone space of the σ-complete Boolean measure algebra of μ [14; pp. 206-207]. K is compact and basically disconnected; a proof might use [14; pp. 85-86] and [5; Theorem 16.17]. The conclusion follows from Theorem 9.

11. Corollary. *If Y is an infinite-dimensional abstract L-space, then* $X = Y^*$ *satisfies* (F1), (F2), (F3), (F4).

Proof. By Kakutani's Representation Theorem [2;pp. 107-108], Y is congruent to $L^1(\mu)$ for a measure space $(S, \underset{\sim}{S}, \mu)$ that is *localizable* [13]; then Y^* is congruent to $L^\infty(\mu)$ [13; p. 301]. Y^* is infinite-dimensional, since Y is. By Theorem 10, $X = Y^*$ satisfies (F1)-(F4).

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