NORMALIZED ITERATIONS AND
NON-LINEAR EIGENVALUE PROBLEMS
OF VARIATIONAL TYPE

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1. **Introduction.** Let A and B be self-adjoint, positive definite operators from a real reflexive Banach space X to its dual X^*. Assume that A has a continuous inverse and that B is compact. Then the eigenvalue problem

$$(1.1) \qquad Ax = \mu Bx$$

has an unbounded sequence of positive eigenvalues $\mu_1 \leq \mu_2 \leq \mu_3 \leq \cdots$, with

$$(1.2) \qquad \lim_{n \to \infty} \mu_n = \infty,$$

and corresponding normalized eigenvectors $x_1, x_2, \ldots$, which together with the vectors $\eta_1, \eta_2, \ldots$, where $\eta_n = Bx_n$, form a biorthogonal system; the system $\{x_n\}$ is a basis for X. Finally, Poincare's principle is valid,

$$(1.3) \qquad \mu_n = \inf\{ \sup_{x \in M \setminus \{0\}} \frac{(x, Bx)}{(x, Ax)}, \ M \in \mathbb{m}_n \}$$

where $\mathbb{m}_n$ denotes the class of all subspaces of X of dimension $\geq n$.

Except for the completeness assertion concerning the eigenvectors, this result has a natural nonlinear generalization. The establishment of this generalization is the purpose of this paper.

We consider a problem of the form (1.1), in which A and B are assumed to be odd. The generalization of the self-adjointness requirement is the requirement that A and B be gradients of real functionals $a(x)$ and $b(x)$ on X. B is assumed to be compact and continuous. The generalization of positive definiteness is convexity of $a(x)$ and $b(x)$. The continuous invertibility of the linear operator A, which is equivalent to $(x, Ax) \geq \gamma \|x\|^2, \gamma > 0$, is generalized in the condition of strong monotonicity (see [19]):

there exists a function $d_0(r)$, defined and continuous on $[0,\infty)$ and positive on $(0,\infty)$ with

$$(1.4) \qquad \lim_{r\to\infty} d_0(r) = \infty,$$

and

$$(1.5) \qquad (x-y, Ax-Ay) \geq \|x-y\| d_0(\|x-y\|) \qquad x,y \in X.$$

The continuity of the linear operator A is replaced by the assumption that A is bounded, i.e. there exists a continuous strictly monotone function $d_1(r)$ on $[0,\infty)$ such that

$$(1.6) \qquad \|Ax\| \leq d_1(\|x\|), \qquad x \in X,$$

and that A is hemi-continuous, i.e. continuous from L to $(X^*, X), (X^*$ with the weak topology) for each line L in X. (Notice that (1.6) and (1.5) imply $d_0(r) \leq d_1(r)$ and hence

$$(1.7) \qquad \lim_{r\to\infty} d_1(r) = \infty.$$

Finally, in the non-linear case the eigenvalue problem is reformulated, for obvious reasons, to include a side condition. Thus we consider the problem

$$(1.8) \qquad Ax = \mu Bx, \qquad a(x) = c,$$

where $c > a(0)$.

The basic topological ideas used here date back to Lyusternik [13], see also [14], and have been used by a number of other authors in proving existence theorems for eigenfunctions of nonlinear operators. See for example [5],[12],[20]. Recently these techniques have been extended and exploited by Browder in the study of eigenvalue problems for nonlinear elliptic partial differential equations. Thus the Lyusternik-Schnirelman theory, the topological technique

to which we refer, is, in its standard form, fairly well known.

Here are used two variations from the standard theory. First, in place of the Lyusternik-Schnirelman category we use the "genus". The later is obviously better suited to the study of critical points of even functionals in a Banach space, and by its use the need to consider deformations or homotopies, which arises in the standard treatment, is eliminated. Second, exploiting convexity of the functionals and the inequalities arising therefrom we define the "normalized iteration", a mapping which leaves the side condition in (1.8) invariant and increases the functional $b(x)$. To obtain mappings with similar properties, in the standard treatment, what one must do, roughly speaking, is solve

$$\frac{dx}{dt} = T(x)$$

where T is the component tangential to the set $\{x \in X : a(x) = c\}$ of the operator B. This involves a fairly sophisticated theory and necessitates rather strong regularity conditions on $a(x)$ and $b(x)$. See the remarks on pp.41 and 42 of [4]. Thus when convexity is present the use of this normalized iteration seems to have distinct advantage over the standard method. In another approach, described in [3] and [4], Browder has shown that the regularity conditions can be weakened by using Galerkin approximations to obtain the existence theorem. With the additional assumption of convexity, the methods here give an existence theorem in which the other regularity requirements are essentially the same as in [3]. In addition, however, we get the multiplicity result (inequality (2.5)) which, so far as I can tell, is not

easily obtained, if it can be obtained at all, by the techniques of [3].

The use of "iterations" in the variational study of nonlinear problems appears in the investigation by Moore and Nehari, [16], of the boundary value problem

$$y" + p(x)y^{2n+1} = 0, \quad 0 \le x \le 1, \ y(0) = y(1) = 0.$$

The idea was subsequently exploited further by Nehari, [17],[18], and later by the author, [6],[7], and finally, also by the author, used in connection with the Lyusternik-Schnirelman techniques in [8],[9].

The notion of genus (род in Russian) was introduced in [11], see also [12]. The distinct advantages of using the genus in the study of critical points of even functionals seems generally to have been overlooked in spite of the treatment in the well-known monograph [12]. The invariant here referred to as the genus appears also in [10], where it is called the co-index.

2. <u>Statement of results</u>. We first introduce some additional terminology and notation. Let $\mathcal{S}$ denote the class of symmetric subsets of $X\backslash\{0\}$ which are closed in X. For a set $S\in\mathcal{S}$, the <u>genus</u> of S, denoted $\rho(S)$, is zero if S is empty and otherwise $\rho(S)$ is the supremum of the set of integers n such that every odd continuous map $f : S \to R^{n-1}$ has a zero on S. Equivalently one can define the genus of a non-empty set $S\in\mathcal{S}$ to be 1 if no connected component of S contains a pair of antipodal points and to be n if S can be covered by n but not by less than n sets in $\mathcal{S}$ of genus 1.

Let a(x) and b(x) be as indicated in section 1, we shall assume henceforth that

(2.1) $a(0) = b(0) = 0.$

Let the positive number c be fixed. We shall say that $x \in X$ is admissible if $a(x) = c$, and that a set $S \subseteq X$ is admissible if $S \in \mathcal{S}$, S consists of admissible elements, and S is compact. The class of all admissible subsets of X will be denoted by $\mathcal{G}$, and, for $n = 1, 2, \ldots,$

$$\mathcal{G}_n = \{ S \in \mathcal{G} : \rho(S) \geq n \}.$$

The characteristic values of (1.8) are defined to be the numbers $\lambda_n = \lambda_n(c)$ given by

(2.2) $\lambda_n = \sup_{S \in \mathcal{G}_n} \min_{x \in S} b(x).$

Theorem 1. For each n the class $\mathcal{G}_n$ is non-empty. The numbers λ_n defined by (2.2) form a non-increasing sequence of positive numbers with

(2.3) $\lim_{n \to \infty} \lambda_n = 0.$

It follows from Theorem 1 that given the natural number n, there will exist natural numbers k,m such that $k \leq n \leq k + m - 1,$ $\lambda_k = \lambda_{k+m-1} > \lambda_{k+m}$ and either $k = 1$ or $\lambda_{k-1} > \lambda_k$; the natural number m will be called the multiplicity of λ_n.

Theorem 2. Let a(x) and b(x) be real-valued, even, convex, continuous functionals on the real reflexive Banach space X, and with $a(0) = b(0) = 0$. Let a(x) have a bounded, strongly monotone hemi-continuous Gateaux derivative $A : X \to X^{*}$, and let b(x) be

positive for $x \neq 0$, and have a compact continuous Frechet derivative $B : X \rightarrow X^*$. Then for each $n = 1,2,\ldots,$ there exist solutions x of (1.8) satisfying

$$(2.4) \qquad\qquad b(x) = \lambda_n \, ,$$

moreover, if E_n denotes the set of solutions of (1.8) satisfying (2.4), then E_n is admissible and

$$(2.5) \qquad\qquad \rho(E_n) \geq \text{multiplicity of } \lambda_n \, .$$

Remark. In describing the linear result corresponding to Theorem 2 in the introduction, we did not assert that X was separable. To do so would have in fact been redundant since a compact positive definite self-adjoint operator can not exist on a reflexive space X unless X is separable. The reflexivity assumption is also redundant in the linear case, when taken together with the hypotheses on A. Indeed X becomes a Hilbert space when provided with the equivalent norm (x,Ax). The situation remains the same in the non-linear case. Suppose X is reflexive. By a result in [19], B can be uniformly approximated on bounded sets in X by operators with finite dimensional range in X^*. Thus the range of B lies in a separable subspace of X^*, and hence, since $(x,Bx) > 0$ for $x > 0$, X must be separable. If the functional $a(x)$ has the properties indicated in the hypothesis of Theorem 2 then X can be given an equivalent norm for which the unit ball is $\{x : a(x) \leq c\}$, moreover this norm will be uniformly convex so that X must be reflexive.

The proof of Theorems 1 and 2 is based on the following result concerning the existence of a continuous "normalized iteration"

operator associated with (1.8).

Theorem 3. _Assume_ _the_ _hypothesis_ _of_ _Theorem_ 2. _Then_ _the_ _problem_ $Ax = \alpha By$, $a(x) = c$, $\alpha > 0$ _has_ _a_ _unique_ _solution_ (x,α) _for_ _each_ _non-zero_ $y \in X$. _The_ _resulting_ _solution_ _map_ $\sigma : y \rightarrow x$: $X \backslash \{0\} \rightarrow X$ _has_ _the_ _following_ _properties_:

i) σ _is_ _odd_ _and_ _continuous_ _and_ _maps_ $X \backslash \{0\}$ _into_ _the_ _set_ _of_ _admissible_ _elements_ _in_ X; _the_ _fixed_ _points_ _of_ σ _are_ _precisely_ _the_ _solutions_ _of_ (1.8),

ii) _the_ σ-_image_ _of_ $\{x \in X : a(x) = c, b(x) \geq \lambda\}$ _is_ _precompact_ _for_ _any_ $\lambda > 0$,

iii) _if_ x _is_ _admissible_ _then_ $b(\sigma(x)) \geq b(x)$ _and_ _equality_ _holds_ _only_ _if_ x _is_ _an_ _eigenfunction_ _of_ (1.8).

3. Basic inequalities. We shall collect here the inequalities resulting from the convexity of $a(x)$ and $b(x)$ and from the boundedness and strong monotonicity of A.

Lemma 3.1. a) _For_ _arbitrary_ $x, y \in X$, _there_ _hold_ _the_ _inequalities_

$$(3.1) \qquad (x-y, Ax) \geq a(x) - a(y) \geq (x-y, Ay),$$

$$(3.2) \qquad (x-y, Bx) \geq b(x) - b(y) \geq (x-y, By),$$

and _equality_ _can_ _hold_ _in_ (3.1) _only_ _if_ $x = y$.

b) _There_ _exists_ _a_ _non-decreasing_ _function_ $d_2(r)$, _continuous_ _on_ $[0, \infty)$ _and_ _positive_ _on_ $(0, \infty)$ _with_

$$(3.3) \qquad \lim_{r \to \infty} d_2(r) = \infty$$

such _that_

$$(3.4) \qquad a(x) \geq \|x\| d_2(\|x\|), \qquad x \in X.$$

c) B __is__ __bounded__ __and__

(3.5) $\qquad (x,Bx) > 0 \qquad$ __for__ $\ x \neq 0.$

__Proof.__ Let x and y be given, then the convexity and differentiability of a(x) and b(x) and the definition of the Gateaux derivative [19],[21], imply that, as functions of t, a(y+t(x-y)) and b(y+t(x-y)) are convex and differentiable on [0,1] with derivatives (x-y,A(y + t(x-y)) and (x-y,B(y+t(x-y)) respectively. For a convex function $\varphi(t)$ on [0,1], differentiable at 0 and 1, there holds

$$\varphi'(1) \geq \varphi(1) - \varphi(0) \geq \varphi'(0)$$

and either equality holds only if both do and φ is linear on [0,1]. Applied to the two convex functions mentioned just above, this inequality yields (3.1) and (3.2). We also conclude that if either equality holds in (3.1) then (x-y,Ax-Ay) = 0, and, because of the strong monotonicity of A, this implies that x = y. This concludes the proof of part a.

From the differentiability of a(x) and the hemi-continuity of A, since a(0) = 0,

$$a(x) = \int_0^1 (x,A(tx))\,dt.$$

$$\geq \|x\| \int_0^1 d_0(t\|x\|)\,dt,$$

where the latter inequality follows from (1.5). The function $d_2(r) = \int_0^1 d_0(tr)\,dt$ clearly satisfies the conditions of part b, thus the proof of that assertion is completed.

The boundedness of B when B is compact is a standard result, and since b(x) is assumed to be positive for $x \neq 0$, (3.5) follows immediately from (2.1) and (3.2) with y = 0.

Lemma 3.2. The operator A has a continuous inverse $A^{-1} : X^* \to X$ and there exists $r_o > 0$ and a monotone non-decreasing continuous function $d_3(r)$ on $[0,\infty)$ which is positive on (r_o,∞) with

(3.6)
$$\lim_{r \to \infty} d_3(r) = \infty$$

such that

(3.7)
$$\|A^{-1}\eta\| \geq d_3(\|\eta\|), \quad \eta \in X.$$

Proof. The invertibility of A follows from a result in Browder, [1], Minty [15], or see Theorem 3.3 p. 101, [19]. Since d_1 was assumed to be strictly increasing and continuous we can take $r_o = d_1(0), d_3(r) = 0$ for $0 \leq r < r_o$ and $d_3(r) = d_1^{-1}(r)$ for $r \geq r_o$, and (3.7) will then follow immediately from (1.6) with $x = A^{-1}\eta$.

Lemma 3.3. a) The set $N = N_c$ defined by

(3.8)
$$N = \{x \in X : a(x) = c\}$$

is closed, bounded, and bounded away from zero and intersects every ray through 0 in a single point. The mapping

$$\pi : x \to \{tx : t > 0\} \cap N$$

is continuous on $X \backslash \{0\}$.

b) The functional $b(x)$ is positive and bounded on N, and $\|Bx\|$ is bounded away from zero on $N(\lambda) = \{x \in N : b(x) \geq \lambda\}$ for any $\lambda > 0$.

Proof. That N is closed follows from the continuity of $a(x)$, and from the closedness, since $c > a(0) = 0$, it follows that N is bounded away from zero. The boundedness of N follows from (3.3)

and (3.4). Similarly, one sees easily that a(tx) varies continuously from 0 to ∞ as t varies from 0 to ∞, and from (3.1),

$$a(tx) - a(x) > (t-1)(x,Ax), \qquad t > 0,$$

from which it follows that $\{tx : t > 0\} \cap N$ consists of precisely one point. This latter fact, together with the closedness of N, readily yields the continuity of π. Since $0 \notin N$, the functional b(x) is clearly positive on N. The boundedness of b(x) on N follows from the boundedness of N, the boundedness of B, and (3.2) with y = 0. Finally, for $x \in N(\lambda)$,

$$\lambda \leq b(x) \leq (x,Bx) \leq \|x\|\|Bx\|,$$

and since N is bounded, it follows that $\|Bx\|$ has a positive lower bound on $N(\lambda)$, $\lambda > 0$.

4. <u>Proof of Theorems 1 and 2.</u> In this section we shall assume Theorem 3 and derive from it Theorems 1 and 2. We require to begin with the following basic properties of the genus; the verification of these is straightforward and is indicated in [8]. Below the letter S, with or without subscript, will denote a set in the class $\mathbf{S}$.

G.1. If there exists an odd continuous map $f : S_1 \to S_2$, in particular if $S_1 \subseteq S_2$, then $\rho(S_1) \leq \rho(S_2)$.

G.2. $\rho(S_1 \cup S_2) \leq \rho(S_1) + \rho(S_2)$.

G.3. If S is compact then $\rho(S) < \infty$ and S has a neighborhood U with $\overline{U} \in \mathbf{S}$ and $\rho(\overline{U}) = \rho(S)$.

G.4. If $\{S_n\}$ is a decreasing sequence of compact sets from $\mathbf{S}$, then $S = \bigcap_{n=1}^{\infty} S_n \in \mathbf{S}$ and

$$\rho(S) = \lim_{n \to \infty} \rho(S_n).$$

G.5. If there exists an odd homeomorphism of the n-sphere onto S then $\rho(S) = n + 1$.

Proof of Theorem 1. Let $\Sigma = \{x \in X : \|x\| = 1\}$, the unit sphere in X, then $\pi|\Sigma$, being a radial projection, is one-to-one and, by part a of Lemma 3.3, $\pi|\Sigma$ is continuous and onto N. The inverse of $\pi|\Sigma$, $x \to x/\|x\| : N \to \Sigma$ is continuous so $\pi|\Sigma$ is a homeomorphism. If M is an n-dimensional subspace of X, then clearly $\pi(\Sigma \cap M)$ is admissible, and since $\pi|\Sigma$ is a homeomorphism, it follows from G.5 that $\rho(\pi(\Sigma \cap M)) = n$, and therefore $\mathcal{a}_n$ is non-empty.

From part b of Lemma 3.3, and the fact that each $\mathcal{a}_n$ is non-empty, it follows that (2.2) defines a sequence of positive numbers, and this sequence is clearly non-increasing.

For the completion of the proof of Theorem 1 and for the proof of Theorem 2 we require the following.

Lemma 4.1. Let S be an admissible set, then

(4.1)
$$\min_{x \in S} b(x) \geq \lambda$$

if and only if $S \subseteq N(\lambda)$. Thus (4.1) implies

(4.2)
$$\rho(S) \leq \rho(N(\lambda)).$$

Lemma 4.2. Let σ be the mapping in Theorem 3. Then $\overline{\sigma(N(\lambda))}$ is admissible and

(4.3)
$$\rho(\overline{\sigma(N((\lambda))}) = \rho(N(\lambda)) < \infty, \quad \lambda > 0,$$

and

(4.4)
$$\rho(N(\lambda)) = n, \quad \lambda_n \geq \lambda > \lambda_{n+1}.$$

Proof of Lemma 4.1. The first assertion follows directly from

the definition of an admissible set and the definition of $N(\lambda)$;
(4.2) then follows from (4.1) by G.1.

Proof of Lemma 4.2. From properties i) and iii) of σ it follows
that $\sigma(N(\lambda)) \subseteq N(\lambda)$ and then from the continuity of $a(x)$ and $b(x)$,

$$(4.5) \qquad \overline{\sigma(N(\lambda))} \subseteq N(\lambda);$$

it is clear that $\overline{\sigma(N(\lambda))} \in \mathcal{S}$.
The inclusion (4.4) implies directly, by G.1, that $\rho(\overline{\sigma(N(\lambda))}) \leq \rho(N(\lambda))$;
the opposite inequality follows from applying G.1 with $f = \sigma$.
Property ii) of σ implies that $\overline{\sigma(N(\lambda))}$ is compact, hence
admissible, for $\lambda > 0$ and by G.3 this completes the proof of (4.3).

Suppose now that $\lambda_n > \lambda_{n+1}$ (otherwise the assertion (4.4) is
vacuous) and let $\lambda_n > \lambda > \lambda_{n+1}$. Then by (2.2), (we are also
implicitly using that portion of Theorem 1 which is already proved)
there exists an admissible set S with $\min\limits_{x \in S} b(x) \geq \lambda$ and
$\rho(S) \geq n$. It then follows by Lemma 4.1 that

$$(4.6) \qquad \rho(N(\lambda)) \geq n, \qquad \lambda > \lambda_n,$$

From (4.5),

$$\min\{b(x) : x \in \overline{\sigma(N(\lambda))}\} \geq \lambda,$$

so by (2.2), since $\sigma(N(\lambda))$ is admissible and $\lambda > \lambda_{n+1}$, this implies
$\rho(\overline{\sigma(N(\lambda))}) < n + 1$. Together with (4.3) and (4.6) this gives the
equality in (4.4) except for $\lambda = \lambda_n$. To obtain this inequality
when $\lambda = \lambda_n$ we observe that

$$\bigcap_{0 < \lambda < \lambda_n} \overline{\sigma(N(\lambda))} \subseteq N(\lambda_n)$$

and use G.4 and G.1. This completes the proof of Lemma 4.2.

<u>Completion</u> <u>of</u> <u>the</u> <u>proof</u> <u>of</u> <u>Theorem 1</u>. Let $\lambda_o = \lim\limits_{n\to\infty} \lambda_n$, so

that $\lambda_o \geq 0$. By G.1 and the definition of $N(\lambda), \rho(N(\lambda))$ is

monotone nonincreasing, so it follows from (4.4) that $\rho(N(\lambda_o)) = \infty$,

and hence, by (4.3), $\lambda_o = 0$. This proves (2.3).

<u>Proof</u> <u>of</u> <u>Theorem 2</u>. Let $n \geq 1$ with $\lambda_n > \lambda_{n+1}$. Let E_n

denote the set of solutions x of (1.8) satisfying

$$(4.7) \qquad\qquad b(x) = \lambda_n;$$

equivalently E_n can be characterized as the set of fixed points

of σ satisfying (4.7). From the properties of σ, and the latter

characterization it is clear that E_n is an admissible set.

By G.3 we choose a neighborhood U of E_n with $\overline{U} \in \mathcal{S}$,

$\rho(\overline{U}) = \rho(E_n)$ (U is empty if E_n is). Consider the compact set

$S = \overline{\sigma(N(\lambda_n))}\backslash U \subseteq N(\lambda_n)$, the σ-image, $S_1 = \sigma(S)$, of this set satisfies

$$(4.8) \qquad\qquad \min_{x \in S_1} b(x) > \lambda_n$$

To see this assume the contrary, then

$$\min_{x \in S_1} b(x) = \lambda_n$$

and there is an x_o in the compact set S_1 with $b(x_o) = \lambda_n$.

However $x_o = \sigma(x_1)$, $x_1 \in S$, and thus by property iii) of σ, since

$S \subseteq N(\lambda_n)$,

$$\lambda_n = b(x_o) = b(\sigma(x_1)) \geq b(x_1) \geq \lambda_n,$$

and x_1 must be a fixed point of σ. But by the construction of S

this is impossible, hence $S_1 \subseteq N(\lambda)$ for some $\lambda > \lambda_n$.

It follows that

$$\rho(S) \leq \rho(S_1) \leq \rho(N(\lambda)) \leq n - \text{(multiplicity of } \lambda_n).$$

Since $\sigma(N(\lambda_n)) \subset S \cup \overline{U}$, it follows from G.2 and G.1 that

$$n \leq \rho(S) + \rho(\overline{U})$$

and hence $\rho(E_n) = \rho(\overline{U}) \geq$ multiplicity of λ_n. This completes the proof of Theorem 2.

Remark. It is interesting to note that if $\rho(E_n) = \mu$, then one can choose elements $x_1, x_2, \ldots, x_\mu \in E_n$ so that the finite sequences $\{x_1, \ldots, x_\mu\}$ and $\{Bx_1, \ldots, Bx_\mu\}$ are biorthogonal. This is easily established by induction. If $1 \leq k < \mu$ and $\{x_1, \ldots, x_k\}$ and $\{Bx_1, \ldots, Bx_k\}$ are biorthogonal, then an argument similar to that above shows that

$$\rho(\{x \in E_n : (x, Bx_j) = 0, \ j=1, \ldots, k\}) \geq \mu - k.$$

5. Proof of Theorem 3. Let $\eta \in X^*$, $\eta \neq 0$. By Lemma 3.2, A^{-1} is continuous and from (3.7), for $\alpha > 0$,

$$\|A^{-1}(\alpha\eta)\| \geq d_3(\alpha\|\eta\|),$$

so that by (3.6),

$$\lim_{\alpha \to \infty} \|A^{-1}(\alpha\eta)\| = \infty.$$

Since $a(0) = 0$, from the continuity of $a(x)$ and A^{-1} and from (3.3) and (3.4) of Lemma 3 it follows that $a(A^{-1}(\alpha\eta))$ varies continuously from 0 to ∞ as α varies from 0 to ∞. The strong monotonicity of A (or merely the strict convexity of $a(x)$) implies, for $\alpha > \beta > 0$,

$$(5.1) \qquad (\alpha-\beta)(A^{-1}(\alpha\eta) - A^{-1}(\beta\eta), \eta) > 0,$$

and from (3.1)

$$a(A^{-1}(\alpha\eta)) - a(A^{-1}(\beta\eta)) > (A^{-1}(\alpha\eta) - A^{-1}(\beta\eta),\eta).$$

It follows therefore, because of (5.1), that $a(A^{-1}(\alpha\eta))$ is a strictly increasing function of α, for any $\eta \neq 0$, and hence, for each $\eta \neq 0$, there is a unique $\alpha > 0$ such that

$$(5.2) \qquad\qquad a(A^{-1}(\alpha\eta)) = c.$$

If Ω denotes the set $\{\eta\epsilon X^*: a(A^{-1}\eta) = 0\}$, then by what we have just shown, Ω intersects every ray through the origin in precisely one point. From the continuity of $a(x)$ and A^{-1} it follows that Ω is closed, and since $c > a(0)$, $0\notin\Omega$. From (3.4) and (3.7)

$$a(A^{-1}\eta) \geq d_3(\|\eta\|)d_2(d_3(\|\eta\|)), \quad \eta\epsilon X^*,$$

so that from (3.3) and (3.6) it follows that Ω is bounded. From these facts it follows, as for the mapping π in section 3, that the "Ω-normalization" $\omega : \eta \rightarrow \alpha\eta$, where α is determined by (5.2), is continuous from $X^*\backslash\{0\}$ to Ω.

From part c) of Lemma 3.1, $Bx \neq 0$ for $x \neq 0$, and thus the composite mapping

$$(5.3) \qquad\qquad \sigma = A^{-1}\circ \omega \circ B$$

is defined and continuous on $X\backslash\{0\}$. Since A^{-1},ω,B are odd and continuous, and from the way ω was defined, the first part of assertion i) of Theorem 3 follows immediately. It is also clear that $x\epsilon X$ is a fixed point of σ if and only if it satisfies (1.8).

To prove part ii) of Theorem 3, let $\lambda > 0$ be given, and consider $BN(\lambda)$. By part a of Lemma 3.3, $N(\lambda)$ is bounded and hence

since B is compact, $BN(\lambda)$ is precompact. By part b) of Lemma 3.3, $0 \notin \overline{BN(\lambda)}$ so from the continuity of A^{-1} and ω it follows that

$$\overline{\sigma(N(\lambda))} = A^{-1}(\overline{\omega(\overline{BN(\lambda)})}),$$

and the assertion is proved.

Now we prove part iii) of Theorem 3. Let $x \in N$, then $\sigma(x)$ satisfies

(5.4) $A\sigma(x) = \alpha Bx$

for some $\alpha > 0$. From (5.4) and (3.2) and (3.1),

$$b(\sigma(x)) - b(x) \geq (\sigma(x) - x, Bx)$$

$$\geq \alpha^{-1}(\sigma(x) - x, A\sigma(x))$$

$$\geq \alpha^{-1}(a(\sigma(x)) - a(x))$$

$$\geq 0,$$

and by Lemma 3.1, part a),

$$(\sigma(x) - x, A\sigma(x)) > a(\sigma(x)) - a(x)$$

unless $\sigma(x) = x$. This proves part iii) of Theorem 3.

6. <u>Variants of Theorem 3</u>. The proof of Theorem 2 depends only on Theorem 3, hence the former remains valid in the presence of any variation of the hypothesis under which the latter remains valid. In particular, Theorem 3 will remain valid, the other assumptions being the same, if the conditions on A are replaced by the following: $A : X \to X^*$ <u>is the Gateaux derivative of the continuous strictly convex even functional</u> $a(x)$ <u>on</u> X, <u>and</u> A <u>is coercive</u> (i.e. satisfies (1.5) with $y = 0$ for all $x \in X$ and

where (1.4) holds), <u>bounded</u>, <u>and</u> <u>has</u> <u>a</u> <u>continuous</u> <u>inverse</u> $A^{-1} : X^* \quad X$.

Indeed in section 3, except in proving Lemma 3.2, we use only the coercivity of A and the strict convexity of $a(x)$; Lemma 3.2 clearly is valid under the assumption above. Once the results of section 3 are seen to remain valid under the above assumption, then the proof of Theorem 3 is based, as before, on the results of that section.

A particular condition under which A is invertible, and which is useful for applications to partial differential equations, is the following.

Lemma 6.1. <u>Let</u> $a(x)$ <u>be</u> <u>a</u> <u>strictly</u> <u>convex</u> <u>functional</u> <u>on</u> <u>a</u> <u>separable</u> <u>reflexive</u> <u>Banach</u> <u>space</u> X <u>and</u> <u>let</u> $a(x)$ <u>have</u> <u>a</u> <u>bounded</u> <u>continuous</u> <u>coercive</u> <u>Frechet</u> <u>derivative</u> $A : X \rightarrow X^*$ <u>which</u> <u>satisfies</u> <u>the</u> <u>condition</u>: (S) <u>if</u> $\{x_n\}$ <u>is</u> <u>a</u> <u>sequence</u> <u>in</u> X <u>weakly</u> <u>convergent</u> <u>to</u> x <u>and</u> <u>if</u>

$$\lim_{n \to \infty} (x_n - x, Ax_n - A_x) = 0$$

<u>then</u> $\{x_n\}$ <u>converges</u> <u>strongly</u> <u>to</u> x. <u>Then</u> A <u>has</u> <u>a</u> <u>continuous</u> <u>inverse</u> $A^{-1} : X^* \rightarrow X$.

<u>Proof</u>. It follows immediately from a result of Browder [4,Theorem 5], that A is surjective. The strict convexity of $a(x)$ implies, as in the proof of Lemma 3.1, that for $x, y \in X$

$$(x-y, Ax-Ay) > 0$$

unless $x = y$, thus A is injective. To see that A^{-1} is continuous let $\{x_n\}$ be a sequence in X such that $\{Ax_n\}$ is Cauchy. By the coercive property of A, $\{x_n\}$ is bounded. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ weakly convergent to an element x_0 in X.

We have then

$$\lim_{k \to \infty} (x_{n_k} - x_o, Ax_{n_k} - Ax_o) = 0$$

so $\{x_{n_k}\}$ converges strongly to x_o. If $\tilde{x}_o$ is any other weak limit point of $\{x_n\}$ we see by the same argument that $A\tilde{x}_o$ is a limit point of Ax_n, hence $A\tilde{x}_o = Ax_o$, since $\{Ax_n\}$ is Cauchy, and $x_o = \tilde{x}_o$. It follows that A^{-1} is continuous.

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