MORE DISTANT THAN THE ANTIPODES

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Abstract

In a normed space X, 6 denotes the inner metric on the surface d£ of the unit ball. We consider M(X) = sup{6(-p,p): p e d£} , D(X) = sup{6(p,q): p,q e 5E) . Exploding the conjecture that M(X) = D(X) for all X (previously verified in several cases), it is shown that M(X) = 2 , D(X) = 3 for $X = C_0((O,1]))$.

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Introduction.

Let X be a real normed linear space, and let **Σ(X)** be its unit ball, with the boundary dE(X) . If dim X J≥ *2* , *6y* denotes the inner metric of d£(X) induced by the norm (cf. [1; Section 3]). If no confusion is likely, we write *T, hT,* 6 • In [1J we introduced and discussed parameters of X based on the metric structure of d£ ; among them are D(X) = sup{6(p,q): p,q € 3L) *,* the inner diameter of 3£, and M(X) = sup{6(-p,p): p € *bZ) ,* half the perimeter of £. Obviously, M(X) ^ D (X) , and it was conjectured [1; Conjecture 9.1] that M(X) = D (X) in every case, i.e., that ftno pair of points of 3E is more distant in dE than the most distant antipodes". This equality was shown to hold if dim X = 2 or dim X = 3 [1; Theorems 5.4, 5.8], if D(X) = 4 [3], if X is an L-space [4].

In this paper we explode this conjecture by showing that M(X) = 2 , D(X) =3 for X = C_Q((O,1]) , the space of continuous real-valued functions on (0,1] that tend to 0 at 0, with the supremum norm. We observe that this failure of the conjecture is "as strong as possible" since $2D(X) \le M(X) + 4$ for every normed space X [3; Theorem 1]. The present result is a simple specific instance of the evaluation of $M(X)$, $D(X)$ for many spaces of continuous functions, which will be carried out in a forthcoming paper. It has appeared useful, however, to give a separate account of this very simple example. In addition, Lemma 1 is required for the general theory. The conjecture remains unresolved, and interesting, for spaces of finite dimension greater than three.

We shall use the terminology, notations, and elementary results of Sections 1-3 of [1]. In particular, a *subspace* of X is a linear manifold in X, not necessarily closed, provided with the norm of X. If Y is a subspace of X, we obviously have

$$\text{(i)} \qquad \delta_Y(p,q) \le \delta(p,q), \qquad p,q \in \partial\Sigma(Y).$$

Instead of dealing with the space $C_0((0,1])$, we prefer, for technical reasons, to consider the space $C_\pi([-1,1])$ of odd continuous real-valued functions on $[-1,1]$ with the supremum norm. The two spaces are obviously congruent, and therefore any metrical property of one implies the same metrical property of the other. *In the rest of this paper*, X shall always stand for $C_{\pi}([-1,1])$.

2. The Perimeter

We consider the special function u e X defined by $u(t) = t$, t €[-1,1] .

Lemma 1. 6(-u,u) = 2 .

Proof. For each given integer $n > 1$, let $R_n = I^{\infty}(\{1,...,n\})$ be the Banach space of sequences of length n of real numbers, with the maximum norm. The proof will depend on the computation of the length of certain polygonal curves in d£(R_n), carried out in [2].

Let Y_n be the closed subspace of X consisting of the piecewise linear odd continuous real-valued functions on [-1,1] with "corners" at most at ;± $(2k-l)(2n-l)^{-1}$, k = l^..#<,n . Define the linear mapping θ_n : • Y_n -* R_n by ($*_n$f) (j) = f((2n-4j +3) (2n-l)"^{1}) , j = 1, ...,n . Since the mapping j H* 2n - 4j + 3 : {1, . . . ,n) —* {±;(2k-l) : k = lj...,n} is injective and the image, contajLns exactly one of each pair of opposites,, $\$_n$ is bijective; since a piecewise linear function attains its extrema at "corners", $<>_n$ is isometric. Hence $*_n$ is a congruence.

Now $u \in Y_n$; we consider $\Phi_n u \in \partial\Sigma(R_n)$ and compute

(2) $$(\Phi_n u)(j) = (2n-4j+3)(2n-1)^{-1}, \qquad j = 1, \ldots, n .$$

On the other hand, we consider $P_o \in$ 9i;(R_n) given by

(3) $$P_o(j) = (n-2j+1)(n-1)^{-1}, \qquad j = 1, \ldots, n ;$$

we know from [2; Lemma 4] that

(4) $$\delta_{R_n}(-p_O, p_O) = 2n(n-1)^{-1}$$

(« ^ct, equality holds). Now (^u)(1) · p_O(1, m ± ,

so the straight-line segment with endpoints * u , p

lies entirely in ^^ , thereforSj from ^ ° (3),

(5) $$\delta(p_O, \Phi_n u) = \|\Phi_n u - p_O\|$$

$$= 2(2n-1)^{-1}(n-1)^{-1} \max\{j-1 : j=1, \ldots, n\}$$

$$= 2(2n-1)^{-1}$$

Since $\Phi_n : Y_n \to R_n$ is a congruence, (1), *(4),* (5) yield

$$2 = \|u-(-u)\| \le \delta(-u,u) \le \delta_{Y_n}(-u,u) = \delta_{R_n}(-\Phi_n u, \Phi_n u)$$

$$\le \delta_{R_n}(-\Phi_n u, -p_0) + \delta_{R_n}(-p_0, p_0) + \delta_{R_n}(p_0, \Phi_n u)$$

$$\le 4(2n-1)^{-1} + 2n(n-1)^{-1} = 2 + 2(4n-3)(n-1)^{-1}(2n-1)^{-1} .$$

The integer n was arbitrarily great; we conclude that $\delta(u,-u) = 2$.

2. Theorem. *For every* $f \in \partial S$, $\delta(-f,f) = 2$. *Consequently,* $M(X) = 2$.

Proof. Since $[-1,1]$ is connected and f is odd, we have $f([-1,1]) = [-1,1]$. Since the composition of odd functions is odd, we conclude that the linear mapping $g \mapsto g \circ f : X \to X$ is isometric, hence a congruence of X onto a subspace Y of X . Now $(\pm u) \circ f = \pm f \in Y$; by Lemma 1 and (1) we therefore have

$$2 \le \delta(-f,f) \le \delta_Y(-f,f) = \delta_Y(-u \circ f, u \circ f) = \delta(-u,u) = 2 .$$

3. The inner diameter

3. Lemma. Define v,w *e* ∂Σ by

v(t) = -v(-t) = t - - | + | t - - | | 0 £ t £ 1

w(t) = -w(-t) = ~t - $\frac{1}{2}$~ + |t - ^{1}j I   O £ t £ 1 .

Then 6(v,w) ^ 3 .

Proof. Let c be any curve from v to w in ∂Σ, and r a given number, 0 <£ r < 1 . Since jjv - v|| = 0 , ||v- w|| = 2 , there exists a point z on c such that ||z- v|| = r . Since z *e* SE there exists t e [-1,1] such that z(t) = 1 . Now v(t) ^ z(t) - ||z-v|| = 1 - r > 0 . From the definition of v and w we have t > $\frac{1}{j}$, whence w(t) = - 1 . Then

l(o) ^ ||w- z|| + ||z- v|| ^ |w(t) -z(t) | + r = 2 + r .

Since r was arbitrarily close to 1 , we have £(c) ;≥ 3 . Since c was an arbitrary curve from v to w in 9E * we indeed have 6(v,w) ^ 3 .

Remark. It is easy to show directly that 6(v,w) = 3 ; there exists, in fact, a curve from v to w in ∂Σ consisting of two straight line segments end-to-end, of respective lengths 1 and 2 : the intermediate end point is $z \in X$ defined by

[illegible]

The verification is left to the reader.

4. Theorem. $D(X) = 3$.

Proof. By [3; Theorem 1], $2D(X) \le M(X) + 4$; since $M(X) = 2$ by Theorem 2, we conclude, using Lemma 3, that

$$3 \le \delta(v,w) \le D(X) \le \tfrac{1}{2}(2+4) = 3 ,$$

so that equality holds.

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