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VECTOR VALUED ABSOLUTELY CONTINUOUS

FUNCTIONS ON IDEMPOTENT SEMIGROUPS

by

Richard A. Alò, Andre de Korvin,
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ABSTRACT

In this paper the concept of vector valued, absolutely continuous functions on an idempotent semigroup is studied. For F a function of bounded variation on the semigroup S of semicharacters with values of F in the Banach space X, let $A = AC(S,X,F)$ be all those functions of bounded variation which are absolutely continuous with respect to F. A representation theorem is obtained for linear transformations from the space A to a Banach space which are continuous in the BV-norm. A characterization is also obtained for the collection of functions of A which are Lipschitz with respect to F. With regards to the new integral being utilized it is shown that all absolutely continuous functions are integrable.

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Introduction. Absolutely continuous functions have been extensively studied in the literature. For example in [5] the dual space of the space of absolutely continuous functions is characterized. In [7], T. Hildebrandt gives a representation theorem for the linear functionals on $BV[0,1]$ which are continuous in the weak topology. In [6] a representation theorem for linear functionals continuous in the variation norm on $BV[0,1]$ is given. This representation is in terms of a so called v-integral. The techniques of that paper, however, make strong use of the order on $[0,1]$. In [8] absolutely continuous functions and functions of bounded variation on idempotent semigroups are defined and these functions are identified with a certain class of finitely additive set functions.

In [1], the identification in [8] is used to obtain a representation theorem. A characterization of the so called Lipschitz functions in the setting of [8] is also obtained by the authors. The techniques of [1] depend on a result of Darst [2] which states that if u and v are two finitely additive real valued set functions with $v << u$, then v is the limit in the variation norm of finitely additve set functions u_n defined by $u_n(A) = \int_A s_n du$, where s_n is a simple function. This result does not in general hold true when u and v are vector valued.

In this paper we study the concept of vector valued, absolutely continuous functions on an idempotent semigroup. We obtain a representation theorem for linear transformations from the space $AC(S,X,F)$ to Y which are continuous in the BV-norm. A characterization is also obtained for the collection of functions of $AC(S,X,F)$ which are Lipschitz with respect to F. It is also shown that this new integral being utilized has a "wide enough" class of integrable functions. In fact all polygonal functions are integrable (see Lemma 6) and even more so all absolutely continuous functions (Theorem 2) are integrable.

1. Notations and Definitions

Let A be an abelian idempotent semigroup, and let S be a semigroup of semicharacters on A containing the identity. Recall that a semi-character on A is a non-zero bounded, complex valued function on A which is a semigroup homomorphism. Since A is idempotent it is clear that every f in S can be viewed as a characteristic function on A. The notations used here will be consistent with the ones used in [1] and [8]. We recall some of these notations.

For f in S, $A_f = \{a \in A : f(a) = 1\}$ and $J_f = \{a \in A : f(a) = 0\}$. Let T_n be the set of all n-tuples consisting of 0 and 1. Let Q_n be a finite subset of S, that is $Q_n = \{f_1, f_2, \ldots, f_n\}$, and let $\sigma \in T_n$. If $\sigma(i)$ denotes the $i^{\underline{th}}$ component of σ, for $Q = Q_n$ let

$$B(Q,\sigma) = \left(\bigcap_{\sigma(i)=1} A_{f_i} \cap \bigcap_{\sigma(i)=0} J_{f_i} \right).$$

Any set of this form will be called a set of B-type. Let F be any function from S to the reals. Define

$$L(Q,\sigma)F = \sum_{\tau \in T_n} m(\sigma,\tau) F\left(\prod_{i=1}^{n} f_i^{\tau(i)} \right)$$

where m denotes the Mobius function for T_n (see [9]). The function F is said to be of bounded variation if

$\sup \sum_{\sigma \in T_n} |L(Q,\sigma)F| < \infty$, where the supremum is taken over all partitions of A into sets $B(Q,\sigma)$ as σ ranges over T_n. The collection of all real valued functions of bounded variation on S will be denoted by BV(S). Consider $F \in BV(S)$. Then by $AC(S,F)$, we mean all functions $G \in BV(S)$ such that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that for every finite set $Q = Q_n$ of S and any subset H of T_n,

$$\sum_{\sigma \in H} |L(Q,\sigma)G| < \varepsilon \quad \text{if} \quad \sum_{\sigma \in H} |L(Q,\sigma)F| < \delta.$$

From now on F will be assumed to be positive definite, i.e. $L(Q,\sigma)F \geq 0$ for all such Q and σ.

Let X be a Banach space. Then by the space BV(S,X) we mean all functions from S to X which are of bounded variation in the above sense where absolute value is replaced by the norm in X. For $G \in BV(S,X)$, $\|G\|_{BV}$ will denote $\sup \sum_{\sigma \in T_n} \|L(Q,\sigma)G\|_X$.

Definition. Let Σ be any field of subsets of some set and let u and v be finitely additive set functions defined on Σ where u is scalar valued and v is X valued. We say that v is absolutely continuous with respect to u and write $v << u$ if v is the limit in the variation norm, of X valued set functions of the form $\sum_{i=1}^{n} v_i \cdot x_i$, where $x_i \in X$, and each v_i is a scalar valued, finitely additive, set function on Σ, which is $\varepsilon - \delta$ absolutely continuous with respect to u.

Remark. In the case that X is the reals the above definition reduces to the usual one.

Definition. Consider $G \in BV(S,X)$ and F as above. The function G is called absolutely continuous with respect to F if G is the limit in $\|\cdot\|_{BV}$ of X valued functions of the form $\sum_{i=1}^{n} G_i \cdot x_i$ where $x_i \in X$ and each G_i is a scalar valued function defined on S which is absolutely continuous with respect to F as in [8]. We denote this space by $AC(S,F,X)$.

2. Results. Let u denote a scalar valued finitely additive set function defined on Σ and let m be an X-valued finitely additive set function defined on Σ.

Lemma 1. $m << u$ if and only if m is the limit in the variation norm of finitely additive X-valued set functions defined on Σ whose range is finite dimensional, and which are $\varepsilon - \delta$ absolutely continuous with respect to u.

Proof: Suppose that m is the limit in the variation norm of finitely additive set functions m_i (where the ranges are finite dimensional) which are $\varepsilon - \delta$ absolutely continuous with respect to u. It follows that each m_i can be written as

$$m_i = \sum_{j=1}^{n_i} m_{ij} \cdot x_{ij}$$

where each m_{ij} is a finitely additive, real valued, set function defined on Σ each of which are $\varepsilon - \delta$ absolutely continuous with respect to u, and where the x_{ij} are linearly independent. Hence $m << u$. The converse is clear.

Lemma 2. $m << u$ if and only if m is the limit in the variation norm of X-valued set functions which are represented by integrals of X-valued simple functions with respect to u.

Proof: From lemma 1, $m << u$ if and only if m is the limit in the variation norm of set functions of the form

$$\Sigma\, m_i \cdot x_i$$

where each m_i is real valued and $\varepsilon - \delta$ absolutely continuous with respect to u. From a result due to Darst [1], each m_i is the limit in the variation norm of set functions of the form

$$\int s_{i,k}\, du$$

where each $s_{i,k}$ is a real valued simple function.

It is clear then that m will be approximated in the variation norm by

$$\int \Sigma s_{i,k} \cdot x_i du.$$

From now on $\underline{\Sigma}$ <u>will denote the field generated by all</u> $\underline{J_f}$ <u>as</u> f <u>ranges over</u> $\underline{S}$.

Let m be a finitely additve X-valued set function defined on Σ. To m we associate an X-valued function defined on S, denoted by $\hat{m}$, which is defined by

$$\hat{m}(f) = \int fdm = m(A_f).$$

Let $BV(\Sigma,X)$ denote the collection of all finitely additive X-valued set functions of bounded variation. Then $BV(\Sigma,X)$ is a Banach space under the variation norm [5].

<u>Theorem 1</u>. <u>The map</u> $m \longrightarrow \hat{m}$ <u>is a linear isometry from</u> $BV(\Sigma,X)$ <u>onto</u> $BV(S,X)$. <u>Moreover</u> $m << u$ <u>if and only if</u> $\hat{m} << \hat{u}$ <u>and for each</u> x <u>in</u> X, $\widehat{u\cdot x} = \hat{u}\cdot x$.

<u>Proof</u>: Clearly the map is linear, we now show that it is onto. Consider $G \in BV(S,X)$, then G can be extended to the linear span of S by the equation

$$G(\Sigma a_i f_i) = \Sigma a_i G(f_i),$$

since S is a linearly independent set (see lemma 1.4 [7]). Since S is a semigroup, for each $E \in \mathfrak{a}$, it follows that χ_E is an element of the linear span of S. Thus we define a set function u_G by equation

$$u_G(E) = G(\chi_E).$$

It follows that u_G is a finitely additive X-valued set function defined on Σ. Furthermore for each $f \in S$,

$$\hat{u}_G(f) = u_G(A_f) = G(f).$$

We now show that the map is norm preserving. We have

$$\|m\| = \sup \Sigma \|m(B_i)\|$$

where the B_i's are sets of B-type and form a partition of A. Now

$$\|m\| = \sup \Sigma \|m(B_i)\|$$

$$= \sup \Sigma \|L(B_i)\hat{m}\|$$

$$= \|\hat{m}\|_{BV}.$$

Note that we can now obtain the norm of G directly from the equation

$$\|G\|_{BV} = \sup \Sigma \|G(\chi_A)\|.$$

Now suppose that $G << F$. Then G is the limit in the variation norm of

$$G_n = \sum_i h_{n,i} \cdot x_{n,i}$$

where each $h_{n,i}$ is real valued and $h_{n,i} << F$. Also each $h_{n,i} = \hat{u}_{n,i}$ where $u_{n,i} << u_F$. Hence if we let

$$u_n = \sum_i u_{n,i} \cdot x_{n,i}$$

it follows that u_n converges to u_G in the variation norm once we have shown that for each real valued finitely additive set function u and each $x\epsilon X$

$$\widehat{u\cdot x} = \hat{u}\cdot x$$

since then we have that

$$\hat{u}_n = G_n.$$

This follows since

$$\begin{aligned}\widehat{u\cdot x}(f) &= u\cdot x(A_f)\\ &= u(A_f)\cdot x\\ &= u(f)\cdot x.\end{aligned}$$

Lemma 3. The space $AC(S,X,F)$ is a Banach space.

Proof: Since $BV(S,X)$ is a Banach space, from theorem 1 it is sufficient to show that $AC(S,X,F)$ is closed in $BV(S,X)$. Consider $G\epsilon BV(S,X)$ and $G_n\epsilon AC(S,X,F)$ where $\{G_n\}$ converges to G in the BV-norm. Since each G_n is the limit in the BV-norm of functions of the form

$$\sum_i G_{n,i}\cdot x_{n,i}$$

where each $G_{n,i} << F$ it follows that $G\epsilon AC(S,F,X)$.

Definition. Let $\{A_1,A_2,\ldots,A_n\}$ be a partition of A by sets in Σ and let

$$s = \sum_{i=1}^{n} \chi_{A_i}\cdot x_i.$$

Define

$$V_s(E) \int_E s du_F$$

for each E in Σ, then clearly $V_s << u_F$. The function $P_s \epsilon AC(S,F,X)$ which corresponds to V_s from theorem 1 will be called a <u>polygonal function</u>.

<u>Lemma 4</u>. <u>The collection of polygonal functions is dense in</u> AC(S,F,X).

<u>Proof</u>: Consider $G \epsilon AC(S,F,X)$ and $\mathcal{E} > 0$. There exists an $H = \sum_{i=1}^{n} h_i \cdot x_i$, where $h_i \epsilon AC(S,F)$ and

$$\|G - H\|_{BV} < \mathcal{E}.$$

Furthermore each $h_i = \hat{u}_i$, where $u_i << u_F$. From the result of Darst [1], there exist simple functions s_i such that

$$\|u_i - \int s_i du_F\|_V < \mathcal{E}.$$

Let

$$t = \Sigma\, s_i \cdot x_i$$

and

$$V_t(E) = \int_E t du_F,$$

then if P_t is the polygonal function which corresponds to V_t, we have

$$\begin{aligned}\|H - P_t\|_{BV} &= \|\Sigma h_i \cdot x_i - P_t\| \\ &= \|\Sigma u_i \cdot x_i - V_t\| \\ &= \|\Sigma u_i \cdot x_i - \int \Sigma s_i\ x_i du_F\| \\ &\leq \Sigma(\|u_i - \int \Sigma s_i du_F\|) \|x_i\|\end{aligned}$$

which establishes the lemma.

Now to each G in $AC(S,X,F)$ we associate a special polygonal function which, in the case that S is the set of characteristic functions on half open intervals, coincides with the usual idea of polygonal function, see [3]. Let $G \in AC(S,X,F)$, and let Y be a finite subset of S. Let

$$W_{Y,G} = \sum_{\sigma \in T_n} \frac{u_G(B(Y,\sigma))}{u_F(B(Y,\sigma))} \cdot \chi_{B(Y,\sigma)}.$$

Since $u_F(B(Y,\sigma)) = 0$ implies $u_G(B(Y,\sigma)) = 0$, we define the ratio to be zero in this case. Let

$$V_{Y,G} = \int W_{Y,G} du_F,$$

then since $V_{Y,G} << u_F$, we denote the corresponding polygonal function by pG_Y.

<u>Lemma 5</u>. <u>The collection of all</u> pG_Y <u>is dense in</u> $AC(S,X,F)$ <u>in the</u> BV-<u>norm</u>. <u>In fact for</u> $\varepsilon > 0$, <u>there exists a finite subset</u> Y_0 <u>of</u> S <u>such that if</u> $Y \supset Y_0$, <u>then</u>

$$\|G - pG_Y\| < \varepsilon.$$

<u>Proof</u>: Let $\varepsilon > 0$, then there exists an X-valued simple function s such that

$$\|u_G - V_s\| < \frac{\varepsilon}{2},$$

since $u_G << u_F$. If

$$s = \sum_{\sigma} \chi_{B(Z,\sigma)}$$

then for each $B(Z,\sigma)$

$$\begin{aligned} v_s(B(Z,\sigma)) &= \int_{B(Z,\sigma)} s\,du_F \\ &= u_F(B(Z,\sigma))\cdot x_\sigma. \end{aligned}$$

Thus

$$x_\sigma = \frac{1}{u_F(B(Z,\sigma))}\cdot v_S(B(Z,\sigma)).$$

Similarily, if $Z' \supset Z$, we have

$$s = \sum_\sigma \frac{1}{u_F(B(Z',\sigma))}\cdot \chi_{B(Z',\sigma)}.$$

which we shall write as

$$\sum_\sigma \frac{v_s(B(Z',\sigma))}{u_F(B(Z',\sigma))}\cdot \chi_{B(Z',\sigma)}.$$

Now

$$\begin{aligned} \|v_s - u_{pG_{Z'}}\| &\le \int \|s - w_{Z',G}\|\, du_F \\ &\le \sum_\sigma \int_{B(Z',\sigma)} \left\| \frac{v_s(B(Z',\sigma)) - u_G(B(Z',\sigma))}{u_F(B(Z',\sigma))} \right\| du_F \\ &= \sum_\sigma \|v_s(B(Z',\sigma)) - u_G(B(Z',\sigma))\| \\ &\le \|v_s - u_G\| \\ &< \frac{\varepsilon}{2}. \end{aligned}$$

Hence the result follows from the triangular inequality and theorem 1.

We will denote the space of all bounded linear maps from a Banach space X to a Banach space Y by $L(X,Y)$.

<u>Definition</u>. Let K be a function defined on all sets of B-type with

values in $L(X,Y)$. We say that K is <u>convex relative to F</u>, if whenever $\{B(Z',\tau)\}$, $\tau \in T_m$ is a partition of $B(Z,\sigma)$, then

$$K(B(Z,\sigma)) = \sum_{\tau} \lambda_{\tau} K(B(Z',\tau))$$

where

$$\lambda_{\tau} = \frac{u_F(B(Z',\tau))}{u_F(B(Z,\sigma))} .$$

The set function K will be called <u>bounded</u> if K is bounded in the $L(X,Y)$ norm over all sets of B-type. By $\|K\|$, we will mean the least upper bound of the bounds for K.

<u>Definition</u>. For $G: S \to X$ and K convex, by the <u>v-integral of G with respect to K</u>, we mean the limit, if it exists, of

$$\sum_{\sigma} K(B(Z,\sigma)) L(Z,\sigma) G,$$

where the limit is taken over the net of all finite subsets of S. We denote the integral when it exists by

$$v \int G dK.$$

<u>Lemma 6</u>. <u>All polygonal functions are v-integral with respect to every convex and bounded K. In fact</u>

$$v \int p_S dK = \sum_{\sigma} K(B(Z,\sigma)) L(Z,\sigma) p_s$$

for all $Z \supset Z_0$, Z_0 some finite subset of S.

<u>Proof</u>: Suppose

$$s = \sum_{\sigma} B(Z_0,\sigma) \cdot x_{\sigma}$$

then

$$V_s(B(Z,\sigma)) = u_F(B(Z,\sigma))\cdot x_\sigma$$

for all $Z \supset Z_0$. So

$$L(Z,\sigma)p_s = u_F(B(Z,\sigma))\cdot x_\sigma.$$

Consider $Z_0 \subset Z \subset Z'$, then

$$L(Z',\tau)p_s = u_F(B(Z',\tau))\cdot x_\tau$$

where $x_\tau = x_\sigma$ if $B(Z',\tau) \subset B(Z,\sigma)$. By convexity

$$K(B(Z_0,\sigma)) = \Sigma\lambda_\sigma B(Z',\sigma)$$

where

$$\lambda_\sigma = \frac{u_F(B(Z',\sigma))}{u_F(B(Z_0,\sigma))} .$$

Thus

$$\sum_\sigma K(B(Z_0,\sigma))L(Z_0,\sigma)p_s = \sum_\tau K(B(Z',\tau))L(Z',\tau)p_s.$$

Theorem 2. Let T be a linear operator from the space $AC(S,X,F)$ into Y which is continuous in the BV-norm. Then there exists a unique convex and bounded set function K, with values in $L(X,Y)$, such that every G in $AC(S,X,F)$ is K-integrable, and moreover

$$T(G) = v\int GdK.$$

Furthermore $\|T\| = \|K\|$.

Conversely if K is any convex and bounded, $L(X,Y)$ valued, set function, then each $G \in AC(S,X,F)$ is K-integrable and $v\int GdK$ defines a continuous linear operator from $AC(S,X,F)$ into Y.

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Proof: Let Z be any finite subset of S and let

$$V_{Z,\sigma}(E) = \frac{u_F[B(Z,\sigma) \cap E]}{u_F\ B(Z,\sigma))}$$

then $V_{Z,\sigma}$ is finitely additive and $V_{Z,\sigma} << u_F$. Let $\psi_{Z,\sigma}$ be the corresponding function in $AC(S,F)$. Define the function K by the equation

$$K(B(Z,\sigma))\cdot x = T(\psi_{Z,\sigma}\cdot x),$$

then

$$\begin{aligned}\|K(B(Z,\sigma))\cdot x\|_Y &= \|T(\psi_{Z,\sigma}\cdot x)\|_Y \\ &\le \|T\|\ \|\psi_{Z,\sigma}\cdot x\|_{BV}.\end{aligned}$$

Since

$$\begin{aligned}\|\psi_{Z,\sigma}\cdot x\|_{BV} &= \|V_{Z,\sigma}\cdot x\| \\ &= \|V_{Z,\sigma}\|\cdot\|x\| \\ &\le \|x\|,\end{aligned}$$

we have that

$$\|K\| \le \|T\|.$$

Now,

$$\begin{aligned}V_{Z,G}(E) &= \int_E \sum_\sigma \frac{u_G(B(Z,\sigma))}{u_F(B(Z,\sigma))}\cdot \chi_B(Z,\sigma)\, du_F \\ &= \sum_\sigma u_G(B(Z,\sigma))\, V_{Z,\sigma}(E).\end{aligned}$$

Thus by theorem 1,

$$pG_Z = \sum_\sigma L(Z,\sigma)\, G\psi_{Z,\sigma}.$$

From Lemma 5, we have

$$T(G) = \lim_{Z} T(pG_Z)$$

$$= \lim_{Z} T(\sum_{\sigma} L(Z,\sigma) G \psi_{Z,\sigma})$$

$$= \lim_{Z} \sum_{\sigma} K(B(Z,\sigma)) L(Z,\sigma) G$$

$$= v \int G dK.$$

Also

$$\|T\| \geq \sup_{\|x\| \leq 1} \|T(\psi_{Z,\sigma} \cdot x)\|_Y$$

$$= \sup_{\|x\| \leq 1} \|K(B(Z,\sigma)) \cdot x\|_Y$$

$$= \|K(B(Z,\sigma))\|_{L(X,Y)} .$$

Hence

$$\|T\| = \|K\|.$$

Conversely suppose that K is a bounded, convex, $L(X,Y)$ valued set function. Then

$$\|v \int pG_{Z_1} dK - v \int pG_{Z_2} dK\| \leq \|pG_{Z_1} - pG_{Z_2}\| \; \|K\|.$$

Since Y is complete this shows that G is K-integrable and moreover that

$$v \int G dK = \lim_{z} v \int pG_Z dK.$$

We now define the concept of a Lipschitz function and characterize the space of all such functions in terms of convex and bounded set functions.

<u>Definition</u>. Let g be a real valued function defined on S. Then g is called <u>Lipschitz with respect to</u> F if there exists a constant P such that

$$|L(z,\sigma) g| < PL(z,\sigma) F$$

for all sets $B(x,\sigma)$. We denote this space of functions by <u>Lip(F)</u>.

Definition. By the space Lip(X,F) we mean all functions $G \in BV(S,X,F)$ which are approximable in the BV-norm by functions of the form $\sum_{i=1}^{n} g_i \cdot x_i$, where $x_i \in X$ and $g_i \in LIP(F)$ for $i = 1,2,\ldots,n$.

We now want to give a characterization of the space Lip(X,F) in terms of convex and bounded set functions. For this purpose we introduce a special class of convex and bounded set functions which we denote by $M_C(X,F)$.

Definition. Let K be a convex and bounded X-valued set function. We say that $K \in M_C(X,F)$ if and only if for each $\varepsilon > 0$, there are finite collections $\{K_1,K_2,\ldots,K_n\}$ and $\{x_1,x_2,\ldots,x_n\}$, where each K_i is scalar, convex, and bounded and each $x_i \in X$, and such that

$$\Sigma u_F(B_j) \; \|K(B_j) - \sum_{i=1}^{n} K_i(B_j) \cdot x_i\|_X < \varepsilon$$

for all partitions $\{B_j\}$ of A into sets of B-type. Clearly $M_C(X,F)$ is a linear space.

Theorem 3. The spaces $M_C(X,F)$ and Lip(X,F) are linearly isomorphic.

Proof: Consider $H \in Lip(X,F)$ and $\varepsilon > 0$, then there exists a finite set $\{h_1,h_2,\ldots,h_n\}$ where each $h_i \in Lip(F)$ and a finite set $\{x_1,x_2,\ldots,x_n\}$, $x_i \in X$ such that

$$\|H - \sum_{i=1}^{n} h_i \cdot x_i\|_{BV} < \varepsilon .$$

Let u_H correspond to H and define K_H by the equation

$$K_H(B) = \frac{u_H(B)}{u_F(B)} ,$$

and if m_i corresponds to h_i define k_i by the equation

$$k_i(B) = \frac{u_i(B)}{u_F(B)} ,$$

for each i, for all sets B of B-type. It follows that the set functions $K_H, K_1, \ldots, K_n$ are all convex and bounded. We now show that $K_H \in M_C(X,F)$. Let $\{B_j\}$ be a partition of A into sets of the B-type, then

$$\sum_j u_F(B_j) \ \|K_H(B_j) - \sum_{i=1}^{n} k_i(B_j) \cdot x_i\| =$$

$$= \sum_j \|u_H(B_j) - \sum_{i=1}^{n} m_i(B_j) \cdot x_i\| < \varepsilon .$$

Conversely, consider $K \in M_C(X,F)$. Then for $\varepsilon > 0$ there exists $\{K_1, K_2, \ldots, K_n\}$ and $\{x_1, x_2, \ldots, x_n\}$ such that

$$\sum_j u_F(B_j) \ \|K(B_j) - \sum_{i=1}^{n} K_i(B_j) \cdot x_i\| < \varepsilon$$

for all partitions $\{B_j\}$ of A into sets of B-type. If we define

$$u_K(B) = u_F(B) K(B)$$

and

$$m_i(B) = u_F(B) K_i(B)$$

then it is easy to check that u_K and the m_i's are finitely additive and absolutely continuous with respect to F. Let H_K correspond to u_K where $H_K \in AC(S,X,F)$ and h_i correspond to m_i where $h_i \in AC(S,F)$, then

$$\|H_K - \sum_{i=1}^{n} h_i \ x_i\|_{BV} < \varepsilon .$$

Now the maps $K \longrightarrow H_K$ and $H \longrightarrow K_H$ are inverses of one another. Consequently the theorem is shown since linearity is immediate.

Remark. It should be pointed out that the above characterization is rather different from the scalar case as in [1]. While the map $H \longrightarrow K_H$ was straight forward in the scalar case, we have seen that in our vector setting a weighted-type of variation is needed to define the map.

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