REVERSED GEOMETRIC PROGRAMS
TREATED BY HARMONIC MEANS

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Research Report 71-19

March, 1971

/elb/3/19/71

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ABSTRACT

A "posynomial" is a (generalized) polynomial with arbitrary real exponents, but positive coefficients and positive independent variables. Each posynomial program in which a posynomial is to be minimized subject to only inequality posynomial constraints is termed a "reversed geometric program".

The study of each reversed geometric program is reduced to the study of a corresponding family of approximating (prototype) "geometric programs" (namely, posynomial programs in which a posynomial is to be minimized subject to only upper-bound inequality posynomial constraints). This reduction comes from using the classical arithmetic-harmonic mean inequality to "invert" each lower-bound inequality constraint into an equivalent "robust" family of "conservatively approximating" upper-bound inequality constraints. The resulting families of approximating geometric programs are then studied with the aid of the

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Partially supported by Research Grant DA-AROD-31-124-71-G17

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Partially supported by a Summer Fellowship Grant from Northwestern University.

techniques of (prototype) geometric programming.

This approach has important computational features not possessed by other approaches, and it can easily be applied to the even larger class of well-posed "algebraic programs" (namely, programs involving real-valued functions that are generated solely by addition, subtraction, multiplication, division, and the extraction of roots).

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REVERSED GEOMETRIC PROGRAMS TREATED BY HARMONIC MEANS

1. Introduction.

Originally developed by Duffin, Peterson and Zener [13], prototype geometric programming provides a powerful method for studying many problems in optimal engineering design [28-30, 2, 5, 14, 18, 27]. However, many other important optimization problems can be modelled accurately only by using more general types of algebraic functions. Hence the question of extending the applicability of geometric programming to those larger classes of programs has received considerable attention.

In particular, Section III.4 of [13] presents various techniques for transforming a limited class of algebraic programs into equivalent prototype geometric programs, but many of the most important optimization problems are not within that limited class.

Initial attempts at rectifying this situation were made by Passy and Wilde [21], and Blau and Wilde [6]. They generalized some of the prototype concepts and theorems in order to treat "signomial programs" (namely, programs in which a "signomial"--i.e., the difference of two posynomials--is to be minimized subject to signomial constraints). Subsequently, Duffin and Peterson [11, 12] advanced that work in the still more general setting of algebraic programming. In particular, Appendix A of [11] shows how to transform each well-posed algebraic program into a corresponding equivalent <u>finite</u> family of signomial programs; and Section 2 of [11] shows how to further transform each signomial program into a corresponding equivalent reversed geometric program. Those transformations justify the restriction of our attention to the seemingly irreducible class of reversed geometric programs; and in [11, 12] most of the important prototype concepts and theorems are generalized to that class. However, the important prototype inequality between the primal and geometric dual objective functions is simply not valid in that more general setting.

In an independent and completely different approach, Charnes and Cooper [8] have proposed methods for approximating signomial programs with prototype geometric programs. However, the errors involved in their approximations have never been investigated.

More recently, the preceding difficulties have been at least partially overcome by Avriel and Williams [3], who have shown how to reduce the study of each signomial program to the study of a corresponding <u>infinite</u> family of approximating prototype geometric programs. By introducing an additional independent variable and an additional signomial constraint, they transform each signomial program into a corresponding equivalent signomial program in which only a posynomial is to be minimized subject to signomial constraints. Then, they "condense" certain posynomials in the signomial constraints into approximating single-term posynomials so that the signomial constraints are approximated by upper-bound inequality posynomial constraints. The result of their approximation is that the transformed signomial program is approximated by a prototype geometric program--actually, by an infinite family of prototype geometric programs because the condensation is not unique.

Their approximation is conservative in that Cauchy's "arithmetic-geometric mean inequality" shows that each feasible solution to an arbitrary geometric program in the approximating family is also a feasible solution to the transformed signomial program. Thus, the infimum for each of those geometric programs is not less than the infimum for the original signomial program.

Their approximating family is robust in that each feasible solution to the transformed signomial program turns out to be a feasible solution to at least one of the approximating geometric programs. Hence, the infima for those geometric programs come arbitrarily close to the infimum for the original signomial program.

Under suitable conditions, Avriel and Williams have shown how to choose a sequence of their approximating geometric programs so that the corresponding infima sequence converges monotonely to the infimum for the original signomial program. Thus, a signomial program can frequently be solved by solving a sequence of approximating geometric programs, each of which can be solved by the techniques of prototype geometric programming.

It seems that similar algorithms have been proposed independently by Broverman, Federowicz and McWhirter [7], Pascual and Ben-Israel [17], and Passy [19], but for somewhat smaller classes of programs and without convergence proofs. Each group uses the arithmetic-geometric mean inequality to condense certain posynomials into single-term posynomials so that their programs can be conservatively approximated by corresponding infinite robust families of prototype geometric programs, from which converging sequences can be chosen. Actually, that condensation process can be further exploited to reduce the study of each signomial program to the study of a corresponding infinite robust family of conservatively approximating linear programs. In fact, that process combined with the duality theory of linear programming provides an alternative proof [9] of the "refined duality theory" for prototype geometric programming ([10] and Chapter VI of [13]).

In this paper, we do not use the condensation process just described. Instead, we assume that the transformations given in Section 2 of [11] have been used to transform each signomial program into its corresponding equivalent reversed geometric program. Then, we "invert" each of the posynomials appearing in a lower-bound inequality constraint into an approximating reciprocal posynomial so that each of those constraints is approximated by an upper-bound inequality posynomial constraint. As a consequence, the reversed geometric program is approximated by a prototype geometric program--actually, by an infinite family of prototype geometric programs because this inversion is not unique.

Our approximation is conservative, and the corresponding infinite family is robust, by virtue of the classical "arithmetic-harmonic mean inequality". Thus, this paper presents another approach to solving signomial programs (and hence algebraic programs) by solving a sequence of approximating geometric programs, each of which can be solved by the techniques of prototype geometric programming.

This inversion approach has an important feature not possessed by the condensation approach in that its approximating "exponent matrices" depend only on the program being approximated and not on the given approximation. Only the posynomial coefficients change with the approximation, so many matrix computations need not be repeated during the solution of a sequence of approximating programs. This feature also leads to a variety of strategies for determining such program sequences. For example, one can employ the coefficient sensitivity analyses developed in Appendix B of [13] and further elaborated on in [22, 23]. Those sensitivity analyses cannot be used with the condensation approach because of its lack of invariance for the exponent matrix.

However, the condensation approach has potentially useful features not possessed by the present inversion approach. In particular, its approximations are generally not as conservative as those in the inversion approach, so it may require fewer iterations. Furthermore, it tends to reduce the "degree of difficulty" (page 11 of [13]), an invariant in the inversion approach; so its approximating geometric programs may be easier to solve.

Consequently, the relative computational merits of the two approaches may not become apparent until considerable computational experience is obtained.

Other applications of the harmonic mean to optimization have been given by Avriel [1] and Passy [20]. In fact, a few of the results in this paper have been obtained independently by Passy, and included in a recently revised version of [19].

2. <u>Reversed Geometric Programs and their Equilibrium Solutions</u>.

The study of well-posed algebraic programs is reduced in Appendix A and Section 2 of [11] to the study of equivalent posynomial programs having a special form. Posynomial programs having that special form have been termed "reversed geometric programs" [9], because some of their inequality posynomial constraints have a direction $g(t) \geq 1$ that is the reverse of the direction $g(t) \leq 1$ required for the prototype geometric programs treated in [10, 13].

The most general <u>reversed geometric program</u> is now stated for future reference as the following program.

PRIMAL PROGRAM A. <u>Find the infimum</u> M_A <u>of a posynomial</u> $g_0(t)$ <u>subject to the posynomial constraints</u>

$$g_k(t) \le 1, \qquad k = 1,2,\ldots,p, \tag{1}$$

<u>and</u>

$$g_k(t) \ge 1, \qquad k = p+1,\ldots,p+r \triangleq q \tag{2}$$

<u>Here</u>,

$$g_k(t) \triangleq \sum_{i\in[k]} u_i(t), \qquad k = 0,1,\ldots,q, \tag{3}$$

<u>and</u>

$$u_i(t) \triangleq \begin{cases} c_i t_1^{a_{i1}} t_2^{a_{i2}} \ldots t_m^{a_{im}}, & i\in[k], \quad k=0,1,\ldots,p, \quad (4) \\ \\ c_i t_1^{-a_{i1}} t_2^{-a_{i2}} \ldots t_m^{-a_{im}}, & i\in[k], \quad k=p+1,\ldots,q, \quad (5) \end{cases}$$

where

$$[k] \triangleq \{m_k, \; m_k+1, \ldots, n_k\} \qquad k = 0,1,\ldots,q, \tag{6}$$

and

$$1 \triangleq m_0 \leq n_0, \qquad n_0 + 1 \triangleq m_1 \leq n_1, \quad \ldots,$$

$$n_{q-1} + 1 \triangleq m_q \leq n_q \triangleq n\,. \tag{7}$$

The exponents a_{ij} and $-a_{ij}$ are arbitrary real numbers, but the coefficients c_i and the independent variables t_j are assumed to be positive.

We have placed minus signs in the exponents for the reversed constraint terms (5) in order to obtain a notational simplification in the ensuing developments.

To provide other notational simplifications, we introduce the index sets

$$P \triangleq \{1,2,\ldots,p\}\,, \tag{8}$$

$$R \triangleq \{p+1,\ldots,q\}\,, \tag{9}$$

and

$$[K] = \bigcup_{k \in K} [k] \qquad \text{for each } K \subseteq \{0\} \cup P \cup R . \qquad (10)$$

For purposes requiring pronunciation, $[K]$ is called "block K".

In terms of the preceding symbols, primal program A consists of minimizing the "primal objective function" $g_0(t)$ subject to the prototype "primal constraints" $g_k(t) \leq 1$, $k \in P$, and subject to the reversed primal constraints $g_k(t) \geq 1$, $k \in R$, where: the posynomial $g_k(t) \triangleq \sum_{i \in [k]} u_i(t)$ for each $k \in \{0\} \cup P \cup R$; the posynomial term $u_i(t) \triangleq c_i t_1^{a_{i1}} t_2^{a_{i2}} \dots t_m^{a_{im}}$ for each $i \in [0] \cup [P]$; and the posynomial term $u_i(t) \triangleq c_i t_1^{-a_{i1}} t_2^{-a_{i2}} \dots t_m^{-a_{im}}$ for each $i \in [R]$.

As in prototype geometric programming [10, 13] each posynomial term $u_i(t)$ in primal program A gives rise to an independent "dual variable" δ_i, $i \in [0] \cup [P] \cup [R]$, and each posynomial $g_k(t)$ gives rise to a dependent dual variable $\lambda_k(\delta) \triangleq \sum_{i \in [k]} \delta_i$, $k \in \{0\} \cup P \cup R$. To define the "geometric dual" of primal program A, it is convenient to extend the notation of the preceding paragraph by introducing the symbols

$$K(\delta) \triangleq \{k\in K \mid \lambda_k(\delta) \neq 0\} \qquad \text{for each} \quad K \subseteq \{0\}\cup P\cup R\,, \tag{11}$$

and

$$[K](\delta) \triangleq \{i\in [K] \mid \delta_i \neq 0\} \qquad \text{for each} \quad K \subseteq \{0\}\cup P\cup R\,. \tag{12}$$

Then, corresponding to primal program A is the following <u>geometric dual program</u>.

DUAL PROGRAM B. <u>Find the supremum</u> M_B <u>of the "dual objective function"</u>

$$v(\delta) \triangleq \left\{\left[\prod_{[0](\delta)}\left(\frac{c_i}{\delta_i}\right)^{\delta_i}\right]\left[\prod_{[P](\delta)}\left(\frac{c_i}{\delta_i}\right)^{\delta_i}\right]\left[\prod_{[R](\delta)}\left(\frac{c_i}{\delta_i}\right)^{-\delta_i}\right]\right\}$$
$$\times\left\{\left[\prod_{P(\delta)}\lambda_k(\delta)^{\lambda_k(\delta)}\right]\left[\prod_{R(\delta)}\lambda_k(\delta)^{-\lambda_k(\delta)}\right]\right\}, \tag{13}$$

<u>subject to the "dual constraints" that consist of the "positivity conditions"</u>

$$\delta_i \geq 0\,, \qquad i \in \{1,2,\ldots,n\} \triangleq [0] \cup [P] \cup [R]\,, \tag{14}$$

<u>the "normality condition"</u>

$$\lambda_0(\delta) = 1\,, \tag{15}$$

<u>and the "orthogonality conditions"</u>

$$\sum_{i=1}^{n} a_{ij}\,\delta_i = 0\,, \qquad j = 1,2,\ldots,m\,. \tag{16}$$

<u>Here</u>

$$\lambda_k(\delta) \triangleq \sum_{i\in[k]} \delta_i\,, \qquad k \in \{0,1,\ldots,q\} \triangleq \{0\} \cup P \cup R\,, \tag{17}$$

<u>and the numbers</u> a_{ij} <u>and</u> c_i <u>are as given in primal program</u> A .

The dual constraints are identical to their analogues in prototype geometric programming; and they are linear, so the dual feasible solution set is either empty or polyhedral and convex. The dual objective function differs from its analogue only by the presence of minus signs in the exponents of the factors corresponding to the reversed primal constraints; but those minus signs result in very large theoretical and computational differences between reversed and prototype geometric programming. In particular, Theorem 3A of [11] shows that, unlike prototype geometric programming, reversed geometric programming is not essentially a branch of convex programming.

The convex nature of prototype geometric programming is reflected in its "main lemma" (Lemma 1 on page 114 of [13]), which asserts that the primal objective function evaluated at each primal feasible solution is greater than or equal to the dual objective function evaluated at each dual feasible solution; with equality holding if, and only if, the primal and dual feasible solutions satisfy certain "extremality conditions" (a term that is used in [22-26] although not in [10, 13]).

With suitable but very weak hypotheses, one of the main duality theorems of prototype geometric programming asserts the existence of primal and dual feasible solutions that satisfy the extremality conditions; in which event the primal infimum equals the dual supremum, and the primal and dual optimal solutions (namely, "minimizing points" for the primal program and "maximizing points" for the dual program) are characterized as those primal and dual feasible solutions that satisfy the extremality conditions.

The preceding facts and the linearity of the dual constraints lead to algorithms for finding primal and dual optimal solutions to prototype geometric programs; and it is our ultimate goal to devise such algorithms for reversed geometric programming. However, the lack of total convexity in reversed geometric programming will force us to be content with devising algorithms for finding "equilibrium solutions" that need not always be optimal.

Thus, the preceding remarks and the extremality conditions for prototype geometric programming help to motivate the following definition.

DEFINITION 1. A feasible solution t^* to primal program A is termed a primal equilibrium solution if there is a feasible solution δ^* to dual program B such that

$$\delta^*_i \, g_0(t^*) = u_i(t^*) , \qquad i \in [0] , \tag{18a}$$

and

$$\delta^*_i = \lambda_k(\delta^*)\, u_i(t^*) , \qquad i \in [k] , \quad k \in P \cup R ; \qquad (18b)$$

in which case δ^* is termed a dual equilibrium solution. Given corresponding primal and dual equilibrium solutions t^* and δ^*, the numbers $E_A \triangleq g_0(t^*)$ and $E_B \triangleq v(\delta^*)$ are said to be corresponding primal and dual equilibrium values.

The practical significance and many important mathematical properties of equilibrium solutions are studied in [11]. In this paper we concentrate on other important mathematical properties that lead to the use of the harmonic mean and prototype geometric programming for obtaining equilibrium solutions.

3. Harmonized Geometric Programs and their Optimal Solutions.

The study of each reversed geometric program will now be reduced to the study of either of two different corresponding families of approximating prototype geometric programs. This reduction is based on the classical inequalities relating the arithmetic, geometric, and harmonic means [4,15,16].

For our purposes, it is convenient to state those inequalities in somewhat disguised form as the following lemma.

Lemma 3a. *If* $u_1,\ldots,u_N$ *are positive quantities, and if* $\alpha_1,\ldots,\alpha_N$ *are positive numbers such that*

$$\sum_{i=1}^{N} \alpha_i = 1 ,$$

then

$$\left(\sum_{i=1}^{N} u_i\right)^{-1} \leq \prod_{i=1}^{N} \left(\frac{\alpha_i}{u_i}\right)^{\alpha_i} \leq \sum_{i=1}^{N} \left(\frac{\alpha_i^{\,2}}{u_i}\right) .$$

Moreover, these inequalities are strict unless

$$u_i = \alpha_i\left(\sum_{j=1}^{N} u_j\right), \qquad i = 1,\ldots,N,$$

in which case they are equalities.

Proof. Given positive quantities $T_1,\ldots,T_N$ and the positive "weights" $\alpha_1,\ldots,\alpha_N$, Cauchy's arithmetic-geometric mean inequality [4, 13, 15, 16] asserts that

$$\left(\sum_{i=1}^{N} \alpha_i T_i\right) \geq \prod_{i=1}^{N} (T_i)^{\alpha_i} , \tag{1}$$

with equality holding if, and only if,

$$T_i = \sum_{j=1}^{N} \alpha_j \, T_j \,, \qquad i = 1,\ldots,N \,. \tag{2}$$

Replacing the positive quantities T_i with their positive reciprocals T_i^{-1} gives the classical geometric-harmonic mean inequality

$$\prod_{i=1}^{N} (T_i)^{\alpha_i} \geq \left(\sum_{i=1}^{N} \alpha_i \, T_i^{-1} \right)^{-1} , \tag{3}$$

with equality holding if, and only if,

$$T_i^{-1} = \sum_{j=1}^{N} \alpha_j \, T_j^{-1} \,, \qquad i = 1,\ldots,N \,. \tag{4}$$

Moreover, it is easily seen that the normalization $\sum_{i=1}^{N} \alpha_i = 1$ implies the equivalence of the equality conditions (2) and (4). Now, choose $T_i = u_i/\alpha_i$ for $i = 1,\ldots,N$, and invert each of the inequalities resulting from (1) and (3) to complete our proof of Lemma 3a .

Given a posynomial

$$g(t) \triangleq \sum_{i=1}^{N} u_i(t)$$

and positive weights $\alpha_1, \ldots, \alpha_N$, the corresponding <u>geometric</u> <u>inverse</u> $g'(\cdot\,;\alpha)$ of g, and the corresponding <u>harmonic</u> <u>inverse</u> $g''(\cdot\,;\alpha)$ of g, are posynomials defined by the following formulas:

$$g'(t;\alpha) \triangleq \prod_{i=1}^{N} \left(\frac{\alpha_i}{u_i(t)} \right)^{\alpha_i} ,$$

and

$$g''(t;\alpha) \triangleq \sum_{i=1}^{N} \frac{\alpha_i^{\,2}}{u_i(t)} .$$

Then, Lemma 3a shows that

$$1/g(t) \leq g'(t;\alpha) \leq g''(t;\alpha)$$

for each $t > 0$, so we have the following implications:

$$g'(t;\alpha) \leq 1 \Rightarrow g(t) \geq 1 , \qquad (5)$$

and

$$g''(t;\alpha) \leq 1 \Longrightarrow g'(t;\alpha) \leq 1 \Longrightarrow g(t) \geq 1 . \tag{6}$$

Given a reversed geometric program A, the implication (5) suggests the introduction of a <u>condensed program</u> $A'(\alpha)$ in which the reversed inequality constraints $g(t) \geq 1$ are replaced by the corresponding prototype inequality constraints $g'(t;\alpha) \leq 1$. Then, the resulting condensed program $A'(\alpha)$ is a prototype geometric program, and the implication (5) shows that the infima M_A and $M_{A'(\alpha)}$ for programs A and $A'(\alpha)$ respectively satisfy the inequality

$$M_{A'(\alpha)} \geq M_A .$$

A detailed analysis of the family of all condensed programs $A'(\alpha)$ has essentially been given by Avriel and Williams [3].

In this paper we introduce a <u>harmonized program</u> $A''(\alpha)$ in which the reversed inequality constraints $g(t) \geq 1$ are replaced by the corresponding prototype inequality constraints $g''(t;\alpha) \leq 1$. Then, the resulting harmonized program $A''(\alpha)$ is a prototype geometric program, and the implication (6) shows that

$$M_{A''(\alpha)} \geq M_{A'(\alpha)} \geq M_A \ ,$$

where $M_{A''(\alpha)}$ is the infimum for program $A''(\alpha)$.

We now proceed more formally to investigate the properties of harmonized programs. Thus, corresponding to the reversed primal program A is the following family of harmonized geometric programs $A''(\alpha)$.

PRIMAL PROGRAM $A''(\alpha)$. <u>Find the infimum</u> $M_{A''(\alpha)}$ <u>of the posynomial</u> $g_0(t)$ <u>subject to the posynomial constraints</u>

$$g_k(t) \leq 1 \,, \qquad k = 1,2,\ldots,p \,, \tag{7}$$

<u>and</u>

$$g_k''(t;\alpha) \leq 1 \,, \qquad k = p+1,\ldots,p+r \triangleq q \ . \tag{8}$$

<u>Here</u>

$$g_k(t) \triangleq \sum_{i\in[k]} c_i \, t_1^{a_{i1}} t_2^{a_{i2}} \cdots t_m^{a_{im}} \,, \qquad k=0,1,\ldots,p, \tag{9}$$

and

$$g_k''(t;\alpha) \triangleq \sum_{i\in[k]} (\alpha_i^{\,2}\, c_i^{-1})\, t_1^{a_{i1}} t_2^{a_{i2}} \cdots t_m^{a_{im}} ,$$

$$k = p+1,\ldots,q, \qquad (10)$$

where α *satisfies the weight conditions*

$$\alpha_i > 0 \quad \textit{for} \quad i\in[k] \quad \textit{and} \sum_{i\in[k]} \alpha_i = 1 , \qquad k = p+1,\ldots,q . \qquad (11)$$

The index sets $[k] \triangleq \{m_k,\ldots,n_k\}$, *the exponents* a_{ij} , *and the coefficients* c_i *and* c_i^{-1} *are as given in primal program* A.

From a computational point of view it may be worth noting that the linear equations $\sum_{i\in[k]} \alpha_i = 1$ can be relaxed by replacing them with the linear inequalities $\sum_{i\in[k]} \alpha_i \geq 1$, because the defining formula for the harmonic inverse $g_k''(t;\alpha)$ implies that the resulting enlargement of the family of all harmonized geometric programs does not lower the infimum of all its program infima. Moreover, it is easy to show that the infimum for this larger family is identical to the infimum for the single program obtained by letting α vary simultaneously with t. This single program is clearly a reversed geometric program of a very special type with reversed constraints $\sum_{i\in[k]} \alpha_i \geq 1$. However, there seems to be no theoretical advantage in pursuing these observations, so we shall not discuss them any further.

Instead, we shall investigate the family of harmonized geometric programs $A''(\alpha)$. This family of approximations is conservative and robust in the following sense.

Theorem 3A. _If_ t _is a feasible solution to a harmonized primal program_ $A''(\alpha)$, _then_ t _is a feasible solution to primal program_ A, _and hence_ $M_{A''(\alpha)} \geq M_A$. _On the other hand, if_ t _is a feasible solution to primal program_ A, _then_ t _is a feasible solution to the harmonized primal program_ $A''(\alpha)$ _with weights_

$$\alpha_i \triangleq u_i(t)/g_k(t) \quad \underline{\text{for}} \quad i\in[k] \quad \underline{\text{and}} \quad k = p+1,\ldots,q\ ;$$

and hence the non-negative infima difference $M_{A''(\alpha)} - M_A$ _can be made arbitrarily small._

<u>Proof</u>. By virtue of Lemma 3a, each solution t to the constraint inequality $g_k''(t;\alpha) \leq 1$ is also a solution to the corresponding reversed constraint inequality $g_k(t) \geq 1$. This observation and the fact that programs A and $A''(\alpha)$ have the same objective function $g_0(t)$ clearly imply the validity of the first part of Theorem 3A.

Now, given $t > 0$ and the corresponding weights $\alpha_i \triangleq u_i(t)/g_k(t)$, we infer from Lemma 3a that

$$g_k''(t;\alpha) = [g_k(t)]^{-1} \quad \text{for each} \quad k \in R .$$

From this, we see that each feasible solution t to program A produces a harmonized program $A''(\alpha)$ to which t is also a feasible solution. Hence, the fact that programs A and $A''(\alpha)$ have the same objective function clearly implies that their infima difference can be made arbitrarily small. This completes our proof of Theorem 3A.

Each harmonized primal program $A''(\alpha)$ is a prototype geometric program, so its geometric dual program is also a prototype geometric dual program and is stated here for future reference as the following program.

DUAL PROGRAM $B''(\alpha)$. <u>Find the supremum</u> $M_{B''(\alpha)}$ <u>of the product function</u>

$$v''(\delta;\alpha) \triangleq \left\{\left[\prod_{[0](\delta)} \left(\frac{c_i}{\delta_i}\right)^{\delta_i}\right]\left[\prod_{[P](\delta)} \left(\frac{c_i}{\delta_i}\right)^{\delta_i}\right]\left[\prod_{[R](\delta)} \left(\frac{\alpha_i^{2} c_i^{-1}}{\delta_i}\right)^{\delta_i}\right]\right\} \times \left\{\left[\prod_{P(\delta)} \lambda_k(\delta)^{\lambda_k(\delta)}\right]\left[\prod_{R(\delta)} \lambda_k(\delta)^{\lambda_k(\delta)}\right]\right\}, \tag{12}$$

<u>subject to the constraints</u>

$$\delta_i \geq 0, \qquad i \in \{1,2,\ldots,n\} \triangleq [0] \cup [P] \cup [R], \tag{13}$$

$$\lambda_0(\delta) = 1, \tag{14}$$

<u>and</u>

$$\sum_{i=1}^{n} a_{ij}\, \delta_i = 0, \qquad j = 1,2,\ldots,m. \tag{15}$$

<u>Here</u>,

$$\lambda_k(\delta) \triangleq \sum_{i\in[k]} \delta_i, \qquad k \in \{0,1,\ldots,q\} \triangleq \{0\} \cup P \cup R, \tag{16}$$

and the weights α_i *along with the numbers* a_{ij} *and* c_i *are as given in primal program* $A''(\alpha)$.

Notice that the constraints for the harmonized dual program $B''(\alpha)$ are identical to those for dual program B. This observation establishes the following useful analogue of Theorem 3A.

Theorem 3B. *A vector* δ *is a feasible solution to dual program* B *if, and only if,* δ *is a feasible solution to at least one harmonized dual program* $B''(\alpha)$; *in which case* δ *is a feasible solution to each harmonized dual program* $B''(\alpha)$.

We can now obtain important information about the reversed primal program A and its geometric dual program B by applying prototype geometric programming theory to the prototype primal programs $A''(\alpha)$ and their geometric dual programs $B''(\alpha)$.

Theorem 3C. *If primal program* A *is consistent and has a positive infimum* M_A, *then its dual program* B *is consistent.*

Proof. From Theorem 3A we infer the existence of weights α such that the harmonized program $A''(\alpha)$ is consistent and has an infimum $M_{A''(\alpha)} \geq M_A$. It follows that $M_{A''(\alpha)} > 0$, and hence the duality theory for prototype geometric programming (Theorem 1 on page 166 of [13]) asserts that dual program $B''(\alpha)$ is consistent. Thus, by virtue of Theorem 3B, our proof of Theorem 3C is now seen to be complete.

The conclusion to Theorem 3C remains valid under the weaker hypothesis that primal program A is "subconsistent" and has a positive "subinfimum" (as defined in [10] and Chapter VI of [13]), but we shall not need that fact in this paper.

Theorem 3C shows that dual program B (and hence each harmonized dual program $B''(\alpha)$) is usually consistent when primal program A is consistent. In the next section we shall investigate the possibility of solving such reversed programs A by solving appropriate harmonized programs.

Due to the following fundamental theorem, the algorithm to be studied in the next section is inherently limited in its ability to obtain (globally) optimal solutions to programs A and B.

Theorem 3D. *Given a vector* $t^* > 0$ *and the associated weights*

$$\alpha^*_i \triangleq u_i(t^*)/g_k(t^*) \quad \textit{for} \quad i\in[k] \quad \textit{and} \quad k = p+1,\ldots,q\,,$$

consider the harmonized primal program $A''(\alpha^*)$ *and its harmonized dual program* $B''(\alpha^*)$. *Then, the vector* t^* *and another vector* δ^* *are optimal solutions to programs* $A''(\alpha^*)$ *and* $B''(\alpha^*)$ *respectively if, and only if,* t^* *and* δ^* *are corresponding primal and dual equilibrium solutions to programs* A *and* B *respectively; in which case*

$$E_A = M_{A''(\alpha^*)} = M_{B''(\alpha^*)} = E_B\,.$$

Proof. According to the "main lemma" of prototype geometric programming (given on page 114 of [13]), vectors t^* and δ^* are optimal solutions to programs $A''(\alpha^*)$ and $B''(\alpha^*)$ respectively if, and only if, t^* and δ^* are feasible solutions to $A''(\alpha^*)$ and $B''(\alpha^*)$ respectively, and satisfy the conditions

$$\delta^*_i = \begin{cases} u_i(t^*)/g_0(t^*)\,, & i\in[0]\,, \\ \lambda_k(\delta^*)u_i(t^*)\,, & i\in[k]\,, \quad k\in P\,, \\ \lambda_k(\delta^*)[\alpha^*_i]^2\, u_i^{-1}(t^*)\,, & i\in[k]\,, \quad k\in R\,. \end{cases} \qquad (17)$$

If t^* and δ^* are feasible solutions to programs $A''(\alpha^*)$ and $B''(\alpha^*)$ respectively, then Theorems 3A and 3B assert that t^* and δ^* are also feasible solutions to programs A and B respectively. If, in addition, t^* and δ^* satisfy the conditions (17), then the defining equations for α^*_i show that

$$\begin{cases} \delta^*_i g_0(t^*) = u_i(t^*), & i\epsilon[0], \\ \delta^*_i = \lambda_k(\delta^*)u_i(t^*), & i\epsilon[k],\quad k\epsilon P, \\ \delta^*_i[g_k(t^*)]^2 = \lambda_k(\delta^*)u_i(t^*), & i\epsilon[k],\quad k\epsilon R. \end{cases} \tag{18}$$

Given an arbitrary index $k\epsilon R$, either $\lambda_k(\delta^*) = 0$ or $\lambda_k(\delta^*) > 0$. If $\lambda_k(\delta^*) = 0$, the corresponding conditions (18) show that $\delta^*_i = 0$ for each $i\epsilon[k]$, so $\delta^*_i = \lambda_k(\delta^*)u_i(t^*)$ for each $i\epsilon[k]$. If $\lambda_k(\delta^*) > 0$, a summation of the corresponding conditions (18) shows that $\lambda_k(\delta^*)[g_k(t^*)]^2 = \lambda_k(\delta^*)g_k(t^*)$; so $g_k(t^*) = 1$, and hence $\delta^*_i = \lambda_k(\delta^*)u_i(t^*)$ for each $i\epsilon[k]$. Thus, we have shown that arbitrary optimal solutions t^* and δ^* to programs $A''(\alpha^*)$ and $B''(\alpha^*)$ respectively are always feasible solutions to programs A and B respectively, and also satisfy the conditions

$$\begin{cases} \delta^*_i g_0(t^*) = u_i(t^*) & i \in [0] \\ \delta^*_i = \lambda_k(\delta^*) u_i(t^*), & i \in [k] \quad k \in P \cup R; \end{cases} \tag{19}$$

that is, t^* and δ^* are corresponding primal and dual equilibrium solutions to programs A and B respectively.

Conversely, if t^* and δ^* are corresponding primal and dual equilibrium solutions to programs A and B respectively, then t^* and δ^* are feasible solutions to programs A and B respectively, and also satisfy the conditions (19). If $k \in R$ and $\lambda_k(\delta^*) = 0$, the corresponding conditions (19) show that $\delta^*_i = 0$ for each $i \in [k]$; so $\delta^*_i = \lambda_k(\delta^*)[\alpha^*_i]^2 u_i^{-1}(t^*)$ for each $i \in [k]$. If $k \in R$ and $\lambda_k(\delta^*) > 0$, a summation of the corresponding conditions (19) shows that $\lambda_k(\delta^*) = \lambda_k(\delta^*) g_k(t^*)$; so $g_k(t^*) = 1$, and hence $\alpha^*_i = u_i(t^*)$ for each $i \in [k]$, which implies that $\delta^*_i = \lambda_k(\delta^*)[\alpha^*_i]^2 u_i^{-1}(t^*)$ for each $i \in [k]$. Thus, we have shown that corresponding primal and dual equilibrium solutions t^* and δ^* to programs A and B respectively always satisfy the conditions (17).

Furthermore, such solutions t^* and δ^* are feasible solutions to programs $A''(\alpha^*)$ and $B''(\alpha^*)$ respectively by virtue of the definition of α^* and Theorems 3A and 3B. Hence, it follows from the first paragraph of this proof that corresponding primal and dual equilibrium solutions t^* and δ^* to programs A and B respectively are always optimal solutions to programs $A''(\alpha^*)$ and $B''(\alpha^*)$ respectively.

Now, the relation $E_A = M_{A''(\alpha^*)}$ results from the fact that programs A and $A''(\alpha^*)$ have the same objective function $g_0(t)$. The relation $M_{A''(\alpha^*)} = M_{B''(\alpha^*)}$. is one of the fundamental relations in prototype geometric programming (given on page 80 of [13]). Finally, the relation $E_A = E_B$ is a consequence of Corollary 1 to Theorem 4A in [11]. This completes our proof of Theorem 3D.

According to Corollary 1 to Theorem 3D in [11], equilibrium solutions t^* to primal program A are "tangentially optimal in a certain weakly global sense". That nature of primal equilibrium solutions t^* is also indicated by the preceding Theorem 3D and the easily verified fact that each posynomial $(\sum_{i\in[k]} u_i)$ has the same value and the same partial derivatives as its harmonic approximant $(\sum_{i\in[k]} [\alpha^*_i]^2 u_i^{-1})^{-1}$ when $u_i = u^*_i$ and $\alpha^*_i \triangleq u^*_i/(\sum_{j\in[k]} u^*_j)$ for $i\in[k]$. The preceding Theorem 3D also shows that equilibrium solutions δ^* to dual program B are tangentially optimal in a strongly global sense, because programs B and $B''(\alpha^*)$ have identical feasible solution sets by virtue of Theorem 3B.

4. A Direct Method for Obtaining Equilibrium Solutions.

Before proceeding we need to classify reversed geometric programs in essentially the same way that prototype geometric programs are classified in [10] and Section VI.2 of [13].

DEFINITION 2. *Primal program A and its geometric dual program B are said to be canonical if there is a feasible solution δ^+ to program B with strictly positive components. Programs A and B that are not* canonical are said to be degenerate.

It seems that properly formulated primal programs A arising from physical and economic considerations are canonical programs. In any event, in [12] the study of degenerate programs is reduced by linear algebra to the study of equivalent canonical programs. Hence, there is no loss of generality in restricting our attention to canonical programs. This restriction implies that each harmonized primal program $A''(\alpha)$ and its geometric dual program $B''(\alpha)$ are also canonical, because the feasible solution δ^{+} to dual program B is also a feasible solution to each harmonized dual program $B''(\alpha)$, by virtue of Theorem 3B. It is then a consequence of one of the main duality theorems of prototype geometric programming (Theorem 1 on page 169 of [13]) that each consistent harmonized primal program $A''(\alpha)$ has at least one optimal solution.

The following algorithm depends on Theorem 3A and the existence of such optimal solutions.

ALGORITHM. *Given a feasible solution* t^{0} *to primal program* A, *use prototype geometric programming to find an optimal solution* t^{1} *to the harmonized primal program* $A''(\alpha^{1})$ *where*

$$\alpha^1{}_i \triangleq u_i(t^0)/g_k(t^0) \quad \underline{\text{for}} \quad i \in [k] \quad \underline{\text{and}} \quad k = p+1, \ldots, q \, .$$

<u>Then, use prototype geometric programming again to find an optimal solution</u> t^2 <u>to the harmonized primal program</u> $A''(\alpha^2)$ <u>where</u>

$$\alpha^2{}_i \triangleq u_i(t^1)/g_k(t^1) \quad \underline{\text{for}} \quad i \in [k] \quad \underline{\text{and}} \quad k = p+1, \ldots, q \, .$$

<u>Continuing in this fashion, generate a sequence</u> $\{A''(\alpha^\nu)\}_1^\infty$ <u>of harmonized primal programs and corresponding sequences</u> $\{M_{A''(\alpha^\nu)}\}_1^\infty$ <u>and</u> $\{t^\nu\}_1^\infty$ <u>of harmonized minima and optimal solutions respectively</u>.

It is a consequence of Theorem 3A that, for each positive integer ν, the optimal solution t^ν to the harmonized primal program $A''(\alpha^\nu)$ is also a feasible solution to the succeeding harmonized primal program $A''(\alpha^{\nu+1})$, and hence t^ν can serve as a starting point for finding the optimal solution $t^{\nu+1}$ to program $A''(\alpha^{\nu+1})$. This fact also implies that

$$M_{A''(\alpha^1)} \geq M_{A''(\alpha^2)} \geq M_{A''(\alpha^3)} \geq \cdots \geq M_A \, , \tag{1}$$

because each harmonized primal program has the same objective function g_0. Now, either at least one of these inequalities is an equality, or each of these inequalities is strict.

In the former case the harmonized optimal solutions t^{ν} are clearly identical for sufficiently large ν, say $\nu \geq \nu^*$; so Theorem 3D shows that this common harmonized optimal solution t^* and any harmonized optimal solution δ^* to the geometric dual program $B''(\alpha^{\nu^*})$ are corresponding primal and dual equilibrium solutions to programs A and B respectively. Such harmonized optimal solutions δ^* almost always exist, because Theorem 1 on page 80 of [13] guarantees their existence when the harmonized primal program $A''(\alpha^{\nu^*})$ is "superconsistent" (i.e. satisfies Slater's condition that all inequality constraints be strict inequalities for at least one feasible solution). Of course, it is an immediate consequence of Theorem 3D that all the harmonized optimal solutions t^{ν} are, in fact, identical to the initial feasible solution t^0 when t^0 happens to be a primal equilibrium solution. Hence, our algorithm can not always be expected to produce a globally optimal solution to primal program A. It may produce only a (tangentially optimal) primal equilibrium solution, and it might not do even that.

In the (more likely) latter case, in which each of the inequalities (1) is strict, the harmonized optimal solutions t^{ν} are clearly distinct, so we need to further investigate the convergence properties of $\{t^{\nu}\}_1^{\infty}$. There are canonical primal programs A that give rise to sequences $\{t^{\nu}\}_1^{\infty}$ that have no limit points in the domain of the posynomials $g_k(t)$, namely, the "positive orthant" $\{t \in E_m \mid t_j > 0, \quad j = 1,2,\ldots,m\}$ of E_m. However, it seems that properly formulated primal programs A arising from physical and economic considerations are canonical programs with the additional property that for each positive number b the set of all those primal feasible solutions t for which the primal objective function

$$g_0(t) \leq b$$

is a compact subset of the positive orthant of E_m. Actually, each canonical prototype primal program (with an exponent matrix (a_{ij}) of rank m) possesses this compactness property, as can be seen by examining the proof given on page 120 of [13]. For an arbitrary primal program A with this compactness property it is clear from Theorem 3A that each sequence $\{t^{\nu}\}_1^{\infty}$ generated by our algorithm is in the nonempty compact subset obtained by choosing $b = g_0(t^0)$; and hence each such sequence $\{t^{\nu}\}_1^{\infty}$ has at least one limit point in the positive orthant of E_m. Limit points produced in this fashion are termed _sequential solutions_ to primal program A.

We have already observed from Theorem 3D that all equilibrium solutions t^* to primal program A are sequential solutions to primal program A (obtained by choosing $t^0 = t^*$ in our algorithm). We now show that almost all sequential solutions t^* to primal program A are, in fact, equilibrium solutions to primal program A.

Theorem 4A. _Suppose that primal program_ A _is canonical and that our algorithm generates a sequence_ $\{t^{\nu}\}_1^{\infty}$ _of harmonized primal optimal solutions_ t^{ν}, _and let_ $\{\alpha^{\nu}\}_1^{\infty}$ _be the resulting sequence of weights_ α^{ν} _with components_

$$\alpha^{\nu}{}_i \triangleq u_i(t^{\nu-1})/g_k(t^{\nu-1}) \quad \underline{\text{for}} \quad i \in [k] \quad \underline{\text{and}} \quad k = p+1,\ldots,q\,.$$

If the sequence $\{t^{\nu}\}_1^{\infty}$ _produces a sequential solution_ t^* _to primal program_ A, _and if the resulting weights_

$$\alpha^*{}_i \triangleq u_i(t^*)/g_k(t^*) \quad \underline{\text{for}} \quad i \in [k] \quad \underline{\text{and}} \quad k = p+1,\ldots,q$$

give rise to a harmonized primal program $A''(\alpha^*)$ _that_ is superconsistent, then

(i) there is an optimal solution δ^{ν} to the harmonized dual program $B''(\alpha^{\nu})$ for sufficiently large ν, say $\nu \geq \nu_0$,

(ii) each such sequence $\{\delta^{\nu}\}_{\nu_0}^{\infty}$ of harmonized dual optimal solutions δ^{ν} is bounded and hence has at least one limit point δ^*,

(iii) each such limit point δ^* is an optimal solution to the harmonized dual program $B''(\alpha^*)$,

(iv) the vector t^* is an optimal solution to the harmonized primal program $A''(\alpha^*)$,

(v) the vector t^* and each optimal solution δ^* to the harmonized dual program $B''(\alpha^*)$ are corresponding primal and dual equilibrium solutions to programs A and B respectively.

Proof. Consider the prototype primal program $A''(\alpha;\epsilon)$ that is obtained from the harmonized primal program $A''(\alpha)$ by multiplying its coefficients with $[1+\epsilon]$. The assumed superconsistency of program $A''(\alpha^*)$ clearly implies the existence of a vector t' and a sufficiently small $\epsilon > 0$ such that t' satisfies the constraints of program $A''(\alpha^*;\epsilon)$ as strict inequalities.

Now, without loss of generality, we suppose that t^{ν} converges to t^*, which in turn implies that α^{ν} converges to α^* by virtue of the definitions of α^{ν} and α^* along with the continuity of the posynomials $u_i(t)$ and $g_k(t)$.

Thus, each coefficient $([1+\epsilon][\alpha^{\nu}{}_i]^2 c_i{}^{-1})$ in program $A''(\alpha^{\nu};\epsilon)$ converges to the corresponding coefficient $([1+\epsilon][\alpha^*{}_i]^2 c_i{}^{-1})$ in program $A''(\alpha^*;\epsilon)$. Hence, the vector t' satisfies the constraints of program $A''(\alpha^{\nu};\epsilon)$ as strict inequalities for sufficiently large ν, say $\nu \geq \nu_0$. It is then clear that program $A''(\alpha^{\nu};0)$, namely program $A''(\alpha^{\nu})$, is superconsistent for $\nu \geq \nu_0$. Consequently, Theorem 1 on page 80 of [13] asserts, for $\nu \geq \nu_0$, the existence of an optimal solution δ^{ν} to program $B''(\alpha^{\nu})$ such that $g_0(t^{\nu}) = v''(\delta^{\nu};\alpha^{\nu})$, because of our assumption that t^{ν} is an optimal solution to program $A''(\alpha^{\nu})$ for $\nu \geq 1$.

Now, our choice of ν_0 and the fact that $g_0(t)$ is the objective function for each harmonized program $A''(\alpha^{\nu})$ imply that the infimum for program $A''(\alpha^{\nu};\epsilon)$ is uniformly bounded from above for $\nu \geq \nu_0$ by the positive constant $[1+\epsilon]g_0(t')$. It is then a consequence of the main lemma of prototype geometric programming (on page 114 of [13]) that feasible values for the objective function corresponding to the geometric dual $B''(\alpha^{\nu};\epsilon)$ of program $A''(\alpha^{\nu};\epsilon)$ are also uniformly bounded from above for $\nu \geq \nu_0$ by $[1+\epsilon]g_0(t')$. The formula for the objective function of a prototype geometric dual program and Theorem 3B show that these feasible functional values have the form $(1+\epsilon)^{\sum_1^n \delta_i} v''(\delta;\alpha^{\nu})$ where δ is an arbitrary feasible solution to dual program B. Hence, we have actually shown that

$$(1+\epsilon)^{\sum_1^n \delta^{\nu}_i} g_0(t^{\nu}) \leq [1+\epsilon]g_0(t')$$

for $\nu \geq \nu_0$, by virtue of the previously established equation $g_0(t^{\nu}) = v''(\delta^{\nu};\alpha^{\nu})$. Since $g_0(t^{\nu})$ converges to $g_0(t^*) > 0$ by virtue of the continuity of g_0, and since each $\delta^{\nu}_i \geq 0$ by virtue of the dual feasibility of δ^{ν}, we infer from the condition $\epsilon > 0$ and the preceding displayed inequality that the sequence $\{\delta^{\nu}\}_{\nu_0}^{\infty}$ is bounded and hence has at least one limit point δ^*.

Without loss of generality, we suppose that δ^{ν} converges to such a limit point δ^*. Using the previously established equation $g_0(t^{\nu}) = v''(\delta^{\nu};\alpha^{\nu})$ for $\nu \geq \nu_0$, we infer from the continuity of g_0 and v'' that

$g_0(t^*) = v''(\delta^*;\alpha^*)$. Moreover, we deduce from both the feasibility of t^ν and δ^ν and the continuity of the constraint functions in programs $A''(\alpha^\nu)$ and $B''(\alpha^\nu)$

that t^* and δ^* are feasible solutions to programs $A''(\alpha^*)$ and $B''(\alpha^*)$ respectively. It is then a consequence of the main lemma of prototype geometric programming that t^* and δ^* are optimal solutions to programs $A''(\alpha^*)$ and $B''(\alpha^*)$ respectively.

The optimal solution t^* to program $A''(\alpha^*)$ and each optimal solution δ^* to program $B''(\alpha^*)$ are corresponding primal and dual equilibrium solutions to programs A and B respectively, by virtue of Theorem 3D. This completes our proof of Theorem 4A.

The most restrictive hypothesis in Theorem 4A is the superconsistency of the harmonized primal program $A''(\alpha^*)$. The following theorem gives a simple method for testing the superconsistency of an arbitrary harmonized primal program $A''(\alpha^0)$ in terms of the constraint functions in primal program A.

Theorem 4B. <u>Given a feasible solution</u> t^0 <u>to primal program</u> A <u>and the associated weights</u>

$$\alpha^o_i \triangleq u_i(t^o)/g_k(t^o) \quad \underline{\text{for}} \quad i \in [k] \quad \underline{\text{and}} \quad k = p+1,\ldots,q ,$$

<u>the resulting harmonized primal program</u> $A''(\alpha^o)$ <u>is superconsistent if, and only if, there is a vector</u> d <u>in</u> E_m <u>such that</u>

(I) $<\nabla g_k(t^o), d> < 0$ <u>for each</u> $k \in P$ <u>for which</u> $g_k(t^o) = 1$,

<u>and</u>

(II) $<\nabla g_k(t^o), d> > 0$ <u>for each</u> $k \in R$ <u>for which</u> $g_k(t^o) = 1$.

<u>Proof</u>. We shall begin by showing that a vector d in E_m satisfies conditions (I) and (II) if, and only if, it satisfies the conditions

(I") $<\nabla g_k(t^o), d> < 0$ for each $k \in P$ for which $g_k(t^o) = 1$,

and

(II") $<\nabla g_k''(t^o;\alpha^o), d> < 0$ for each $k \in R$ for which $g_k''(t^o;\alpha^o) = 1$.

Since conditions (I) and (I") are identical, it is sufficient to show the equivalence of conditions (II) and (II"). From Lemma 3a and our defining equation for α^o, we know that $[g_k(t^o)]^{-1} = g_k''(t^o;\alpha^o)$ for each $k \epsilon R$; so for each $k \epsilon R$ we see that $g_k(t^o) = 1$ if, and only if, $g_k''(t^o;\alpha^o) = 1$. The equivalence of conditions (II) and (II") is then a consequence of the easily established equation $\nabla g_k''(t^o;\alpha^o) = -[1/g_k(t^o,\alpha^o)]^2 \nabla g_k(t^o)$. Thus, we need only prove that program $A''(\alpha^o)$ is superconsistent if, and only if, there is a vector d in E_m such that conditions (I") and (II") are satisfied.

From Theorem 3A and our defining equation for α^o, we infer that t^o is a feasible solution to program $A''(\alpha^o)$, by virtue of our hypothesis that t^o is a feasible solution to program A. It is then an immediate consequence of the differential calculus and the continuity of posynomials that $t^o + \epsilon d$ satisfies the constraints of program $A''(\alpha^o)$ as strict inequalities for sufficiently small $\epsilon > 0$, when d satisfies conditions (I") and (II"). Hence, program $A''(\alpha^o)$ is superconsistent if there is a vector d in E_m such that conditions (I") and (II") are satisfied.

To prove the converse, we need to make the change of independent variables

$$t_j \triangleq e^{z_j}, \qquad j = 1,2,\ldots,m,$$

so that program $A''(\alpha^o)$ is transformed into an equivalent convex program to which we can apply an elementary theorem from convex analysis. This equivalent convex program clearly consists of minimizing the convex function $G_0(z)$ subject to both the convex constraints

$$G_k(z) \leq 1 , \qquad k \in P ,$$

and the convex constraints

$$G_k''(z;\alpha^o) \leq 1 , \qquad k \in R ,$$

where

$$G_k(z) \triangleq \sum_{i \in [k]} c_i \, e^{a_{i1}z_1 + a_{i2}z_2 + \ldots + a_{im}z_m} , \qquad k \in \{0\} \cup P ,$$

and

$$G_k''(z;\alpha) \triangleq \sum_{i \in [k]} (\alpha_i^{\,2} c_i^{-1}) e^{a_{i1}z_1 + a_{i2}z_2 + \ldots + a_{im}z_m} , \qquad k \in R.$$

(Of course, the convexity of these functions $G_k(\cdot)$ and $G_k''(\cdot;\alpha)$ follows easily from the positivity of the coefficients c_i and $\alpha_i^{\,2} c_i^{-1}$.) If program $A''(\alpha^o)$ is superconsistent, then so is the preceding equivalent program;

and hence there is a vector z' in E_m such that

$$G_k(z') < 1, \qquad k \in P,$$

and

$$G_k''(z';\alpha^o) < 1, \qquad k \in R.$$

It follows that the vector z' satisfies the conditions

$$G_k(z') - G_k(z^o) < 0 \quad \text{for each } k \in P \text{ for which } G_k(z^o) = 1,$$

and

$$G_k''(z';\alpha^o) - G_k''(z^o;\alpha^o) < 0 \quad \text{for each } k \in R \text{ for which } G_k''(z^o;\alpha^o) = 1.$$

From the convexity of $G_k(\cdot)$ and $G_k''(\cdot\,;\alpha^o)$ we know that $\langle \nabla G_k(z^o), z' - z^o \rangle \leq G_k(z') - G_k(z^o)$ and that $\langle \nabla G_k''(z^o;\alpha^o), z' - z^o \rangle \leq G_k''(z';\alpha^o) - G_k''(z^o;\alpha^o)$, so the preceding displayed inequalities imply that

$$\langle \nabla G_k(z^o), z' - z^o \rangle < 0 \quad \text{for each } k \in P \text{ for which } G_k(z^o) = 1,$$

and

$$\langle \nabla G_k''(z^o;\alpha^o)\ ,\ z' - z^o \rangle < 0 \quad \text{for each} \quad k \in R \quad \text{for which } G_k''(z^o;\alpha^o) = 1.$$

Using the easily established relation between $\nabla G_k(z^o)$ and $\nabla g_k(t^o)$, and using the easily established relation between $\nabla G_k''(z^o;\alpha^o)$ and $\nabla g_k''(t^o;\alpha^o)$, we see that the preceding displayed inequalities are identical to the conditions (I") and (II") when d is chosen so that $d_j = e^{z^o_j}(z'_j - z^o_j)$, $j = 1,2,\ldots,m$. This completes the proof of Theorem 4B.

5. Concluding Remarks.

It is worth mentioning that the results in this paper can be extended by using "means" other than the harmonic mean to invert the reversed constraints in primal program A. The basic tool used in such extensions is the classical $r - s$ mean inequality

$$\left(\sum_{i=1}^{N} \alpha_i T_i^{\,r}\right)^{1/r} \geq \left(\sum_{i=1}^{N} \alpha_i T_i^{\,s}\right)^{1/s} ,$$

which is known to be valid when the α_i are weights, the T_i are non-negative numbers, and $r \geq s$; with equality holding if, and only if, there is a number T such that $T_i = T$ for $i = 1,2,\ldots,N$. (For a proof of this inequality see [15].) The $r-s$ mean inequality reduces to Cauchy's arithmetic-geometric mean inequality when $r = 1$ and $s = 0$; and it reduces to the classical arithmetic-harmonic mean inequality when $r = 1$ and $s = -1$. The results in this paper can be extended without any difficulty to all cases in which $s < 0$ (with $r = 1$), but the details are left to the interested reader. As mentioned previously, the case in which $s = 0$ has already been treated by Avriel and Williams [3]. Other applications of this spectrum of means to optimization have recently been given by Avriel [1] and Passy [20].

Finally, it is worth noting that the choice of weights α determines only the coefficients in the harmonized primal program $A''(\alpha)$. The resulting invariance of the exponent matrix (a_{ij}) means that the sensitivity analyses (i.e. "perturbation" analyses) developed in [13, 22, 23] can be used to choose sequences $\{\alpha^{\nu}\}_1^{\infty}$ of weights such that

$$M_{A''(\alpha^1)} \geq M_{A''(\alpha^2)} \geq M_{A''(\alpha^3)} \geq \cdots \geq M_A .$$

Each such method for choosing $\{\alpha^{\nu}\}_1^{\infty}$ provides an alternative algorithm to the algorithm studied here.

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