THE N-COMPACTIFICATION OF PRABIR ROY'S SPACE A

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ABSTRACT

An N-compact space is one which can be embedded as a closed subspace in a product of countable discrete spaces. Associated with each space X is a unique N-compact space vX (called the N-compactification of X) which plays the same role that px does for compact Hausdorff spaces: vX is the reflection of X in the category of N-compact spaces. In particular, if ind X = 0 the map from X into vX is an embedding. A conjecture of long standing is that every N-compact space X satisfies dim px = 0. Prabir Roy has described a complete metric space A such that ind A = 0, dim A = dim pA = 1. The author showed in Report 70-40 that A is not N-compact, thus eliminating A as a counterexample to the conjecture. But vA remained a promising candidate for the honor. We now give an explicit construction of VA and prove that dim VA = ind vA = 0, so that dim p(vA) = 0.

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Introduction. Prabir Roy's space A is of great importance in dimension theory, being the first example of a metrizable space for which the small inductive dimension is zero and the Lebesgue covering dimension is positive. (These terms are defined below,,) Indeed it is the first example for which the two dimensions are unequal, whatever they are.

A has shed light on the relationship of N-compact spaces (Spaces which can be embedded as closed subspaces of N^M for some cardinal M, where N denotes the space of positive integers with the discrete topology, are called N-compact; see [N_2].) to other well-known categories of topological spaces. If we let § be the category of strongly zero-dimensional (see definition below) realcompact spaces, h the category of N-compact spaces, and & the category of realcompact spaces of small inductive dimension zero, then

§ c n ⇐ &.

The inclusion on the right was immediately apparent once the concept of N-compact spaces was formulated, and the one on the left has been generally known since 1965 [H_2]. A naturally attracted attention as the key to resolving one of these inequalities:

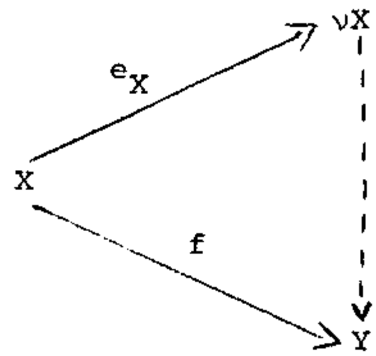
being metrizable of cardinal 2 ° , A is realcompact [GJ, ch.,15] and it is in & but not in S. Had A turned out to be N-compact, the first containment would have been shown to be proper; but, as the author has shown [N_3] A is not N-compact and we have n c &.
≠

But the relevance of A to this problem did not end with this proof, for there remained the chance that the N-compactification (defined below) of A might turn out not to be in S. However, as we shall show in this report, it is in §, and the question remains whether S = n.

Basic Definitions. All spaces are assumed to be Hausdorff. A space X of small inductive dimension zero (ind X = 0) is one having a base of clopen sets. X is of (Lebesgue) covering dimension zero (dim X = 0) if every finite open cover of X can be refined to a partition of X into clopen sets. Equivalently, given a pair of disjoint closed sets ^{F}i>^{F}o} tliere exists a clopen set C such that F_1.cc and C H F_2 = *0.* (To show equivalence, imitate the proof of 16.17 in [GJ].) A space is strongly zero-dimensional if it is completely regular and for every pair of disjoint zero-sets Z_1, Z_2, there exists a clopen set C such that Z_1 c c and C 0 Z_2 = *0.* It is easy to show that a space X is of dim 0 iff it is strongly zero-dimensional and normal (use Urysohn's lemma to get disjoint closed sets contained in disjoint zero-sets). One of the main results of this report is that dim VA = 0.

(Some authors use dim X = 0 to mean that X is strongly zero-dimensional. See [GJ], chapter 16 and Notes, for a discussion of the rival definitions of covering dimension. [A], [D], and [E] give a number of different definitions of dimension and discuss their main properties and relation to each other. In all cases, a definition of an n-dimensional space is given for all n).

Given a space X, there is associated with it an N-compact space νX, called the <u>N-compactification</u> of X, having the same sort of universal property that βX has with respect to compact Hausdorff spaces: νX is N-compact, and we have a map $e_X : X \to \nu X$ such that, for any continuous function f from X into an N-compact space Y, there is a unique continuous $f^\nu : \nu X \to Y$, making the diagram at right commute.



(νX is the reflection of X in the epireflective category of N-compact spaces. We can show the N-compact spaces to be epireflective by noting that every product of N-compact spaces and every closed subspace of an N-compact space is N-compact. See $[H_1]$.)

The image of X under e_ν is dense in νX and e_ν is an embedding iff $\operatorname{ind} X = 0$. $[H_2$, 5.1, (a), (b), (i)].

<u>The Structure of an N-compactification</u>. We indicate how to construct the N-compactification of any space X for which $\operatorname{ind} X = 0$. We let N_1 be the uniformity which has as a base the set of all

equivalence relations partitioning X into countably many clopen sets. (Analogously, for any ordinal a we define N_α to be the uniformity whose base is the set of all equivalence relations partitioning X into $< \aleph_a$ clopen sets.) The uniformity is obviously compatible with the topology on X and plays the same role for vX that C and C* play for uX and px respectively [GJ, ch.15] and that N_0 plays for £X: the completion of X in the uniformity N_1 is the N-compactification of X. This follows from the fact that any continuous function from X into any space of inductive dimension zero is N_1-uniformly continuous (easy proof) and the following theorem.

Theorem. Let ind X = 0. The following are equivalent:

1. X is N-compact, i.e. can be embedded as a closed subspace of a product of countable discrete spaces.

I'. X can be embedded as a closed subspace of a product of discrete spaces of nonmeasurable cardinal.

2. X is N_1- complete.

2'. X is N_α-complete for some $X_\alpha > w_0$' where H is either nonmeasurable or ^{a}is the first measurable cardinal (if any exist).

3. Every clopen ultrafilter with the countable intersection property is fixed on X.

Proof. We can get 2. from 1. and 2'. from I', by noting that a discrete space of cardinality $<N_\alpha$ is clearly N_α-complete and using general results concerning products and closed subspaces of complete uniform spaces. The proof that 1. and 3. are equivalent is in [B_2, Beispiel 6]. Since 1. implies 1'. and 2. implies 2'.

we are done as soon as we show that 2'. implies 3.

A filter Vf is N -Cauchy iff for every partition of X into <N rlopen sets, one (and, of course, only one) member of the partition is in 3. Now let U be a clopen ultrafilter with the countable intersection property; it is enough to show that U contains a member of every partition P of X into <N clopen sets. For each Dcp let 2* = UfU: U€ V) ; then S* is clopen. Define |j,(D*)=l if S*eli, M,(2*)=0 otherwise. UeU for some UeP, as otherwise (a would induce a measure on the discrete space whose points are elements of P, contradiction.

In constructing the N- compactification of a space X of inductive dimension zero, one adds ideal points to X in 1-1 correspondence with the free clopen ultrafilters with the countable intersection property and uses as a base for the topology the closures of the clopen sets in X, with an ideal point in the closure of A if, and only if, A is a member of the clopen ultrafilter associated with that point. There are numerous ways of justifying this procedure and I will mention two.

One can show that countable intersections of clopen sets form a strong delta normal base H [see [AS] for a definition], show that clopen ultrafilters with the countable intersection property (write c.i.p.) are in 1-1 correspondence with S-ultrafilters with the c.i.p., so that &-realcompactness is equivalent to N-compactness. The rest follows from the discussion in [AS] .

Or one can define a proximity relation on X thus: for any two subsets A, B of X we say A5B if every clopen set containing A meets B. Then one can show that clopen ultrafilters with the c.i.p. are in 1-1 correspondence with round N_1-cauchy filters and follow the standard construction of a completion [cf. T, p.206].

<u>The N-compactification of A</u>. In [N_3] we constructed $2^{\aleph_0}$ free clopen ultrafilters with the c.i.p. on A. These were associated with the basic clopen subsets of A given in [R_1] and [R_2], regions of the form R_x and $R_{(p,n)}$. Intuitively, what happens is that the clopen subsets of A which "take a big bite out of" R_x [resp. $R_{(p,n)}$] form a free clopen ultrafilter with the M-intersection property for every cardinal $M < 2^{\aleph_0}$, labeled U_x [resp. $U_{(p,n)}$]. We will now show that there are no other free clopen ultrafilters on A with the c.i.p. In doing this we make use of the equivalence of 3. and 2'. above: <u>a clopen ultrafilter with the c.i.p. must contain a clopen set from any partition of A</u>.

To understand what follows, we need to recall some of the definitions and results of [R_2]. (All notation is taken from there unless otherwise noted). In Section 1, we were given the space A in terms of its points and basic open sets called "regions", divided into families $\mathcal{T}_1$ and $\mathcal{T}_2$. For purposes of this present paper we also include A itself in $\mathcal{T}_1$, identifying it as R where x is the finite sequence consisting of the single term 0, (This notation is fully compatible with that used in $[R_2]$, and all the lemmas that are proven remain true when Γ_1 is taken to include this extra member).

In Section 2, we were given a sequence $G_1 \supseteq G_2 \supseteq G_3 \supseteq \ldots$ of covers of A by members of Γ_1 and Γ_2. To simplify the notation of this present paper, we also add the cover $G_0 = \Gamma_1 \cup \Gamma_2$.

For the time being, we need only know the definitions of Section 1, the definitions of the covers, and the following facts.

1. The regions form a base for a topology. This is the content of Lemma 2.7; the proof of Lemma 2.7 requires nothing beyond Lemma 2.1.

2. Given $p \in P_1$, the regions R with $x < p$ (we use $x < p$ to stand for: $x(i) = p(i)$ for all $i = 1, \ldots, |x|$) form a local base at p. Given $p \in P_2$, the regions $R_{(p,n)}$ where n runs over infinitely many positive integers form a local base at p. This is the content of Lemma 2.8, whose proof requires nothing beyond Lemma 2.1.

3. Given $R \in \Gamma_1 \cup \Gamma_2$, there exists an N such that $R \notin G_N$. This follows immediately from the definitions.

4. The regions are clopen. This is the content of Lemma 5.1. If the reader has not read all of $[R_2]$ he may either assume this without proof or follow the proof in the Appendix to this present paper.

5. If $H_1 \supset H_2 \supset \ldots$ is an infinite sequence of regions with $H_n \in G_n$ for all n, then $\bigcap_{n=1}^{\infty} H_n \neq \emptyset$. This follows from the above results and 4.1, whose proof can be followed right after 2.2.4.

In addition, we need a result whose proof is put into the Appendix because it requires an exhaustive (and exhausting!) case-by-case analysis which would break the continuity of this report; it is about as long as all the rest of this entire report!

<u>Lemma</u>. Let R be a region in F_1 or F_2, and let C be a clopen subset of R.

a. If $R = R \in F_1$, and $C \neq U$, then there exists a σ-locally finite cover of C consisting of clopen sets each contained in proper subregions of R.

b. If $R = R_{(p,n)} \in F_2$ and $C \subseteq U_{(p,n)}$, then there exists a σ-locally finite cover of C consisting of clopen sets each contained in proper subregions of $R_{(p,n)}$.

We also need the following lemma; its proof also has been deferred to the Appendix.

<u>Lemma</u>. Let $R \in G_N$. Any proper subregion R' of R (i.e. R' is a region which is properly contained in R) is in G_{N+1}.

We now sketch a proof of a rather basic result in general topology.

Lemma. Let $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$ be a σ-locally finite clopen cover of a space Y. Then $\mathcal{V}$ can be refined to a partition of Y into clopen sets.

Proof. For each locally finite $\mathcal{V}_i$ let V_i be the set of all points which are members of sets in $\mathcal{V}_i$, then V_i is clopen and $\mathcal{V}_i$ can be refined to a partition of V_i into clopen sets in two steps. For each $x \in V_i$ let $V_i^x = \bigcap\{V_a : x \in V_a \in \mathcal{V}_i\}$; the V_i^x form a locally finite clopen refinement of $\mathcal{V}_i$ such that if $V_i^x \cap V_i^y \neq \emptyset$ then $V_i^x \subset V_i^y$ and/or $V_i^y \subset V_i^x$. From this it follows that if we let

$$W_i^x = V_i^x \setminus \left(\bigcup\{V_i^y : V_i^y \subsetneq V_i^x\}\right)$$

then $\mathcal{W}_i = \{W_i^x : x \in V_i\}$ is a partition of V_i into clopen sets refining $\mathcal{V}_i$.

Now $\{V_i\}_{i=1}^{\infty}$ is a clopen cover of Y which can be refined to the partition $\{V_n'\}_{n=1}^{\infty} = \{V_n \setminus \bigcup_{i=1}^{n-1} V_i\}_{n=1}^{\infty}$. Then $\{V_i' \cap W_i^x : x \in V_i'\}$ is a partition of each V_i' into clopen sets and the union of all these is a partition of Y refining $\mathcal{V}$.

This lemma will be used both to characterize vA and to show that dim $vA = 0$.

Property VII. The clopen ultrafilters on A with the countable intersection property are precisely the fixed clopen ultrafilters and the free clopen ultrafilters of the form U_x and $U_{(p,n)}$.

Proof. Let U be a free clopen ultrafilter and suppose U belongs to none of these classes. If for each N there exists $R \in G_N$ such that $R \in U$, then by 3,5, and the first Lemma we can select a descending sequence of regions $R_n \in U$ whose intersection is nonempty. Careful scrutiny of the proof of 4.1 in [R_] shows that the intersection is a single point of A and the R_n form a local base at that point. Hence it converges to p and is therefore fixed. Now suppose there exists G_N such that U contains a region $R \in G_N$, but no regions in G_{N+1}. If $R = R_x$ and $U \neq U_x$ then there must exist a clopen set $C \subset R_x$ such that $C \in U$, $C \neq U$. Then c may be partitioned into clopen sets each of which is contained in a region in G_{N+1} thanks to the three lemmas above. None of these clopen sets can be in U. Therefore U does not have the countable intersection property. A similar argument works for R = R, > .

So now vA can be identified as the space whose underlying set is $A \cup X \cup F$, where

$T = \{ [p,n] : p \in P_2^*, n \geq 1$, where $[p,n] = [q,m]$ iff $m = n$ and $q \in R_{(p,n)} \}$

and where the base of (clopen) sets is the set of all $\bar{C}$ where C is clopen in A and $\bar{C} = C \cup \{x : C \in U_x\} \cup \{[p,n] : C \in U_{(p,n)}\}$

(To show $\bar{C}$ is clopen simply consider the unique extension of f : A→{0, 1} to vA, where f(C) = 0 and f(A\C) = 1.)

To shorten the proof of the other main result we write A δ B in place of "every clopen set containing A meets B". (The notation is quite apt, for δ defines a proximity relation.) So "dim vA = 0" is equivalent to "If A and B are closed and A ∩ B = ∅, then A $\not\delta$ B".

Property VIII. dim (vA) = 0.

Proof. Let A and B be two closed sets such that A δ B, Let $\bar{R}$ be a set of the form $\bar{R}_x$ or $\bar{R}_{[p,n]}$, satisfying

$(A \cap \bar{R})\,\delta\,(B \cap \bar{R})$. Suppose $\bar{R} = \bar{R}_x$ and suppose $(A \cap \bar{C})\,\delta\,(B \cap \bar{C})$ for all $C \in U_x$ (i.e. all $\bar{C}$ containing x). Then x is in A ∩ B because the $\bar{C}$ containing x form a local base at x. If this is not true for all $C \in U_x$, then $(A \cap (\bar{R}\setminus\bar{C}))\,\delta\,(B \cap (\bar{R}\setminus\bar{C}))$ for some $C \in U_x$. Partition $R_x \setminus C$ into A-clopen sets $\{K_\alpha\}_{\alpha\in G}$ each contained in a proper subregion of R_x. $\{\bar{K}_\alpha\}_{\alpha\in G}$ is a clopen partition of $\bar{R}\setminus\bar{C}$ (take the discrete space with underlying set G, add an extra isolated point *, map K_α to the point α and A\C to *, use the unique extension property) and if $(A \cap \bar{K}_\alpha)\,\delta\,(B \cap \bar{K}_\alpha)$ for any α, then for each $\bar{K}_\alpha$ there is a clopen set $\bar{D}_\alpha \subset \bar{K}_\alpha$ such that $(A \cap \bar{K}_\alpha) \subset \bar{D}_\alpha$, $B \cap \bar{D}_\alpha = \emptyset$. Then $\bigcup_{\alpha\in G} \bar{D}_\alpha$ is a clopen set containing

AD $(\overline{R_x} \setminus \overline{C})$ and missing B (1 $(\overline{R}_x \setminus \overline{C})$, contradiction. So for some a we must have $(A \cap \overline{K}_{\alpha}) \delta (B \cap \overline{K}_{\alpha})$. Now pick $\overline{R}'$ with $\overline{K}_{\alpha} \subset \overline{R}' \subset \overline{R}$, whence $(A \cap \overline{R}')\delta(B \cap \overline{R}')$. A similar argument works for $\overline{R}$ of the form $\overline{R_{(p,H)}}$. Beginning with $R_0 = A$ where it certainly holds, pick a sequence of regions $\overline{R}_0 \supsetneq \overline{R}_1 \supsetneq \ldots$ such that $(A \cap \overline{R}_n) \delta (B \cap \overline{R}_n)$ for all n. If the sequence terminates after a finite number of steps, we get a point in vA\A in the intersection of A and B, as shown. Otherwise the R_n close down on a single point p in A [R_2>4.1] and since the $\overline{R}_n$ form a local base at p in VA, $p \in A \cap B$. So for two disjoint closed sets A,B we must have $A \not\delta B$.

The reader may enjoy rearranging and rewording the proof of property VII to make it follow the lines of this proof. The two can be made to bear an uncanny resemblance.

<u>Concluding Remarks</u>. With VA eliminated as a possible counter-example, we can ask with increased boldness, "Is every N-compact space strongly zero-dimensional?" If the answer is yes, it would follow that the category of N-compact spaces is precisely the category of strongly zero-dimensional realcompact spaces [cf. H_2, Beispiele 5,6]. This would be a very significant result, because the category of N-compact spaces is closed under the taking of arbitrary products and closed subspaces, and it is not even known whether the product of two strongly zero-dimensional realcompact spaces is strongly zero-dimensional. Indeed the only really promising candidate for an N-compact space that is not strongly zero-dimensional is the Sorgenfrey plane. It is the product of two

copies of the real line with intervals of the form [a,b) as a base for the topology (called the Sorgenfrey line). The Sorgenfrey line has a base of clopen sets and is Lindelöf. Hence [cf. GJ, p.115, pp. 245-7] it is both of (Lebesgue) covering dimension zero and realcompact. Hence it is N-compact, and so the Sorgenfrey plane is also. But the Sorgenfrey plane is not normal and is thus of positive covering dimension. But it may be strongly zero-dimensional; we do not know.

For a further discussion of strongly zero-dimensional spaces see [GJ, ch. 16] [E, ch.6, §2], [H_2, Beispiele 5,6] and [N_2, §2]. The first two references also give an analogue of strongly zero-dimensional spaces in higher dimensions and go into some detail on results in this area; they use "dim X" to refer to the dimension of a space under this system. It turns out to be equal to dim pX where this second dim refers to Lebesgue covering dimension. It is also equal to the <u>analytic dimension</u> of C*(X) as defined by Katětov [A],[GJ, ch.16].

The following two problems generalize certain aspects of the unanswered question above. First: given completely regular spaces X,Y, is it true that

dim $ (XXY) ≤1 dim px + dim pY

or, if one adopts the notation of [GJ], simply: is it true that

dim XXY ≤. dim X + dim Y.

The result is known to be true for metric spaces [N_1,pp.20-28] and for compact Hausdorff spaces [GJ,16j] and as far as this author knows, there is no known counterexample in the general case. (Alasl p is not distributive over products!) Second: given a realcompact space X and a closed subspace Y of X, is it true that dim £Y £_ dim βX? Here we must say "realcompact" rather than "completely regular" since there are counterexamples to the latter, for instance one discovered by Smirnov and included as an exercise in [E, p.307]. The result is true for all normal spaces [GJ, 161], realcompact or otherwise, and even for some non-normal, non-realcompact spaces like the Tychonoff plank, which is hereditarily strongly zero-dimensional.

It should be mentioned that the strict equality dim XxY = dim X + dim Y does not even hold for compact Hausdorff spaces. Pontryagin constructed a counterexample by pasting together Möbius strips of ever-decreasing size in a kind of higher-dimensional analogue of the snowflake curve. By modifying the Möbius strips slightly he got another space whose product with this other one was of covering dimension 3, while both surfaces were of covering dimension 2. [A] mentions this result along with an example of a compact metric space X such that $\dim X^2 < 2(\dim X)$.

APPENDIX

In this Appendix we prove the two Lemmas whose proof was deferred, and supply a proof that the regions are clopen. We also give properties of N»A which, while not needed, follow from results which will be proven along the way and which it would be awkward to introduce in a separate paper.

To facilitate comparison with [R_2], we continue the practice begun in [N_3] of giving results numbers which show where they might logically occur were they incorporated into [R_2]. Thus the first result we prove would best come right after 2.2.4 in [R_2] and has been numbered 2.2.5. After the proof that the regions are clopen we begin a new set of results on A; they will all begin with a 6 and they pick up where [N_3] left off.

At first, we will assume nothing beyond 2.2.4 in [R_2] and everything up to and including that.

2.2.5. Lemma. Let R be a region in G_N. Let R' be a proper subregion of R(i.e. a region such that $R' \subset R$). Then

$R' \in G_{N+1}$.

Proof. Since $G_{N+1} = \bigcup_{i=N+1}^{\infty} G_i$, we need only show $R' \in G_{N+k}$, for some $k \geq 1$.

Case I. R and R' are both in Γ_1. Then $R = R_x$ for some x with $|x| = N$ (2.2.3) and by 2.1.1 $R' = R_{x'}$ for some x' with $|x| \le |x'|$. Since $R' \ne R$ we must have $|x'| \ge N + 1$, whence $R' \in G_{N+k}$ for some $k \ge 1$.

Case II. $R \in \Gamma_1$, $R' \in \Gamma_2$. Then $R = R_x$, $R' = R_{(p,n)}$, and since $R_x \cap R^o_{(p,n)} \ne \emptyset$, $R_x \subset R_{p_x}$, whence $|p_x| \ge N$. $R_{(p,n)} \in G_{|p_x|+n}$.

Case III. R and R' are both in Γ_2. Then $R = R_{(p,n)}$ where $|p_x| + n = N$ and $R' = R_{(q,m)}$. If $|p_x| = |q_x|$ then by 2.1.4 $m > n$ and we are done. Since $R_{(p,n)}$ contains $R^o_{(q,m)}$ we must have $|p_x| \le |q_x|$ (1.3.4-7). If $|p_x| < |q_x|$ then (2.1.5) $R^{\pm}_{(p,n)} \supset R_{(q,\pm 1)}$ and by 1.3.7 $|q_x| \ge |p_x| + m + 1$ so that $R_{(q,m)} \in G_{N+r}$

Case IV. $R \in \Gamma_2$, $R' \in \Gamma_1$. Let $R = R_{(p,n)}$, and $R' = R_x$, then by 2.1.3. $R_x \cap R_{\gamma(p,n,\pm)j} \ne \emptyset$ for some j. If $|x| \le |p_x| + n$, R_x contains some P_2-points q with $|q_x| = p_x + n$ which cannot be in $R_{(p,n)}$ (1.3.5-7). Consequently $|x| \ge |p_x| + n + 1$ and we are done.

2.2.6. <u>Corollary.</u> Let R be a region in G_N, R' a proper subregion of R, then $R' \in G_{N+1}$.

<u>Proof</u>. We need only go back to the definitions, 2.2.3 and 2.2.4, and use 2.2.5.

Next we prove that every region of A is closed, adopting the following notation-. if o and a^1 are finite or infinite sequences, we write $a^1 > a$ to mean that o' extends c, i.e. a' is defined for all positive integers for which *o* is defined, and a' (i) = a(i) for all i^>l for which a (i) is defined.

Lemma. Each region of A is closed.

Proof. Let R eV-, and let peP-, . Choose $x^1 < p$ with
1 X X X
|x¹| = |x|. If p^R then R H R ,= 0 by 2.1.1 and so p is not in the closure of R . If peP~ and p^Rv then if
|xl ≤L |Pvl we ^ave R XH R p_x -θ by 2.1.1, while if |x|> |p k|
and R_X 0 R - vp j 1) ^ θ then x(|pX I +1) = ± py(j) for a unique j
(and the sign is determined also) so that R x0 R/ (p,1) c R Y(p,1,±)j.
Now let n = j + 1. R(p,n) H R Y(p,1,±)j = 0 because
$p_y(n + k - 1)$ ^ p_y (j) for any k (1.1.3). Hence $R_{(pjn)}$ n R_x ^ 0
and p is not in the closure of R_x.

Let R.(p9n)2 eT and let qeP,i. Choose x' < q with
I x| =| Pu. I + n + 1. If qiR, > then R ,0 R , > . = 0 for
ic ^-r (p,n) x! y(pjnji)j
all j by 2.1.1, and R_x, 0 R°(p,n) > = 0 by 1.3.3.. Hence R_x''
is a nbhd of q missing R. . . If qePo and |q I = |p |
(PjIi) zl X X
then R'(p,n) 0 R, (q,n) ^ 0 implies the regions are identical by
2.1.4 and so qeR, .. Hence if q^R, > , R, x misses R, . .
(pj-n) ^T (p5n) (q,n) *(p,n)*

If $|q_X| < |p_X|$ and $R_{(p,n)} \cap R_{(q,1)} \neq \emptyset$ then $p(|q_X| + 1) = \pm q_Y(k)$ for some (unique) k by 1.3.5 - 9, so the intersection of the two regions is contained in $R_{Y(q,1,\pm)k}$. Let $m = k + 1$, then reasoning as above, $R_{(p,n)} \cap R_{(q,m)} = \emptyset$; $R_{(q,m)}$ is a neighborhood of q which misses $R_{(p,n)}$.

Finally, suppose $|p_X| < |q_X|$ and $R_{(p,n)} \cap R_{(q,1)} \neq \emptyset$. Since $R^{\circ}_{(p,n)} \cap R_{(q,1)} = \emptyset$ (1.3.5-7). It follows that the intersection of the two regions is contained in $R^{+}_{(p,n)}$. If p' is a point in the intersection,[1] then $p'(|q_X| + 1)$ and $p'(|q_X| + 2)$ are of opposite sign; on the other hand, $p'(|p_X| + i)$ are of the same sign for $i = 2,\ldots,n + 1$(1.3.9-11). Since $|p_X| + 2 \leq |q_X| + 1$ it follows that $|q_X| \geq |p_X| + n$. If $|q_X| \geq |p_X| + n + 1$ then $R_{q_X} \subset R_{Y(p,n,\pm)j}$ for some j by 2.1.1 and so $q \in R_{(p,n)}$. So if $q \notin R_{(p,n)}$ we must have $|q_X| = |p_X| + n$. If $n = 1$, $p'(|q_X| + 1) = q_X(|q_X| + 1) = \pm p_Y(j)$ for a (unique) j, so that $p'(|q_X| + 1) = \pm F^{-1}_{p_Y(j)}(p_Y) = \pm q_Y(k)$ for a unique k--in other words, the intersection of $R_{(p,n)}$ and $R_{(q,1)}$ is contained in $R_{Y(q,1,\pm)k}$ for a k which is determined by p,n, and q alone. If $n > 1$, $p'(|q_X| + 1) = \pm p_Z(n - 1) = \pm_Y q_Y(k)$ for a unique k. So in both cases, we can let $m = k + 1$ and, reasoning as above, $R_{(p,n)} \cap R_{(q,m)} = \emptyset$.

Since the regions are open, this shows A has a base of clopen sets.

[1] If $p' \in P_X$, read as is; if $p \in P_Y$, imagine a subscript X for p'.

Now we come to the last and greatest hurdle, the problem of covering a clopen set that does not "take up a big chunk" of a region R with a a-locally finite collection of clopen subsets, each contained in a proper subregion of R.

In what follows we will assume every numbered item in this report and in [R~2] up to 2.2.6, as well as 2.7, 2.8, and 4.1 (which, as pointed out earlier, can be proved right after 2.1), and also some definitions and results from [^3!

1. The fact that U_x and $U_{(p,n)}$ are free clopen ultrafilters, and we have precisely one associated with each R eT, (including the case where R = A) and each R, >er$_o$:U has R for a member but no proper subregion of R_x is in U_x, and similarly for $\mathcal{U}_{(p,n)}$s. This is all included in 5.15.1 and 5.15.2.

2. Let A be a clopen set and let x be fixed. If, for uncountably many x^1 with, $x^1 > x$, |x' | = |x | + 1, we have AeU ,, then AeU • This follows immediately from 5.4.1 and the definition of U (strangely enough, we do not need to know the definition of U or U, at any point in this appendix, once we accept the results listed herel) and in referring to this result we will use the number 5.4.1.

3. Let A be a clopen set and $p \in P_2$, n fixed. If, for uncountably many $r \in R^+$ there exists $q^r \in R?_{(p,n)}$ with $q^r_Z(n) = r$ such that $A \in U_{(q^r,n+l)}$, then $A \in U/_{(p,n)}$. The number for this result is 5.4.2.

4. Let A be a clopen set such that $A \in U/_{(p,n)}$. Then $A \in U /$ » . for all but finitely many j, and both + and -. The number for this result is 5.7'.

5. Let R be a region and let A be a clopen set contained in R. If $A \in U_X$ for some x, then $R_X \subset R$, and if $A \in U_{(p,n)}$ for some p, n, $R_{(p,n)} \subset R$. This follows from 5.3,5 and 5.3.6 in [Ng] and the definitions of the U_x and the $\backslash x,_{(p,n)}$.

We will make use of the following simple topological lemma without comment: a collection of disjoint clopen sets in a space X is locally finite if, and only if, the union has no boundary points (equivalently, the union is clopen). This follows immediately from the definition of "locally finite". An even more elementary fact is that if \s is a a-locally finite cover of clopen set A by sets clopen in X, then V' = (V 0 A:Ve\r} is likewise a-locally finite, a cover of A, and composed of sets clopen in X. So all we need aim for is a a-locally finite cover of A (if $A \subset R$ and A is not in the ultrafilter associated with R) by clopen subsets of A, each a subset of a proper subregion of the region R under consideration.

To keep notation down to a minimum, we will refer to the regions in $G^{\wedge}$ as "the regions of level n". Note that every region is on some level (namely, if the region is R_{x} the level is $|x|$, if it is $R_{(p,n)}$, it is $|p_x| + n$) and no region can belong to more than one level.

Basically, our plan of attack will be to use these levels to obtain a cover of A, as follows: for each point p_α in A we take the "largest subregion R' of R "(or: "the subregion R^1 of R on the earliest level") containing p_a such that A is in the clopen ultrafilter associated with R^1 (there may be more than one). We break up each level N into sublevels: the sublevel of F_I-regions and the N sublevels of r_o-regions $R_{(p,n)}$ with a fixed $|p_x|$. We try to show that the regions R_α on each sublevel which we have thus associated with each $p_\alpha \epsilon A$ form a locally finite collection by showing that their union has no boundary points, use 2.1.1 and 2.1.4 to show they are disjoint, and use the observation above. The tricky part is showing their union contains no boundary points, and a major part of the effort in the following pages is expended in this direction. The basic idea is to try to obtain a contradiction by showing that a boundary point will have a region R^1 associated with it so that A is in the clopen ultrafilter associated with R'--here is where those results 1,2,3,4 above will come in handy--and so that R^1 contains the regions R_α whose unions we are forming as proper subregions, thereby contradicting the way the R_α were defined. Then we put all these locally finite collections together (there are countably many of them).

As it turns out, we have to be a bit more subtle than this to get a a-locally finite clopen cover. What we have just outlined is a good "first approximation", though. We will modify it as we go along.

The reader may find it helpful to prepare a rough sketch for himself of the way the various regions of A intersect, using as a guide the description of A given in [R..]. (Note: The reference there to Hilbert space is somewhat misleading; the reader can prepare a crude but quite serviceable series of illustrations on a two-dimensional manifold, such as the surface of a sheet of paper or a blackboard.

6.1.1 Lemma. Let $[R_{x})_{\alpha \in a}$ be a family of disjoint regions in T->, with $|x_n| = N$ for all n. Then, $\bigcup_{a \in a} R_{x_a}$ is not closed <=•> there exists P^P?' P^R_{x_a} for an^ a* a pos:ı-t:ı-ve integer k such that, for infinitely many a, there is a j_α for which

$R_{x_\alpha} \supset R_{\gamma(p,k,\pm) j_\alpha}$.

Remarks prior to proof. If such a p exists, the k described must satisfy IPyl + k + 1 ^ N, i.e. k ^ N - |Pvl ~ !• (2.1.1.)

Furthermore, let $k^1 \geq k$, then R , , ,> . 3 R . . VfP^^)] Y(pk f +)j -(k'k) '

Proof, ^rr ^^ : If there is such a p and k, then $j_\alpha = j_\beta \Longleftrightarrow \alpha = \beta$, so that there are infinitely many distinct j_α. Then for each $k^1 \geq k$

there exists α such that $j_\alpha > (k'-k)$, so that $j_\alpha - (k'-k)$ is a positive integer. Therefore, for each $R_{\gamma(p,k')}$ there exists j_α so that $R_{x_\alpha} \supset R_{\gamma(p,k', \pm) D_\alpha - (k'-k)}$ and hence $R_{x_\alpha} \cap R_{\gamma(p,k')} \neq \emptyset$ so p is an accumulation point of $\bigcup R_{x_\alpha}$.

$\Rightarrow$: If $\bigcup R_{x_\alpha}$ is not closed then any point on the boundary is in P_0, for if $p \in P$, $p \notin \bigcup R_{x_\alpha}$, then R where $x < p$, $|x| = N$, is disjoint from all the R_{x_α} and hence p is outside the closure of $\bigcup R_{x_\alpha}$.

So let $p \in P_?$ be on the boundary of $\bigcup R_{x_\alpha}$, we will show that it is of this form. Suppose on the contrary that there exists M such that no R_{x_α} contains $R_{\gamma(p,k,\pm)j}$ for $j > M$, and any k.

Then $R_{x_\alpha} \cap R_{\gamma(p,k,\pm)j} = \emptyset$ for all α and all $j > M$ when $k \geq N - |p_-|$, which means that, if p is in the closure of $\bigcup R_{x_\alpha}$ it must be in the closure of the union of finitely many of them, which implies that it is already in one of them, contradiction.

Note that there are essentially two distinct cases: one where $|p_x| = N - 1$, in which case R_{x_α} contains $R_{\gamma(p,1,\pm)j_\alpha}$ properly, or $|p_x| < N - 1$, in which case we can take $k = N - |p_x| - 1$ and then $R_{x_\alpha} = R_{\gamma(p,k,\pm)j_\alpha}$ for infinitely many distinct α. Note that

$|p_X| \geq N$ is not allowed because then $R_{X_\alpha} \cap R_{(p,k)} \neq \emptyset$ for any k, a would imply $R_{X_a} \supset R_{p_X}$ by 2.1.1.

6.1.2. Lemma. Let $\{R_{(p^a,n)}\}_{a \in a}$ be a set of disjoint regions in T_2 with $|p^a| = M$ for all a and with n fixed. Then $\bigcup_{a \in a} R_{(p^a,n)}$ is not closed $\Longleftrightarrow$ there exists $q \in P_2^0$ such that q is an accumulation point of the union of the $R^{\pm}_{(P^a,n)}$ and $|q_X| \leq M + n$, $|q_X| \neq M$.

Proof. $\Rightarrow$: $R_{(p,n)}$ is dense in $R_{(p,n)}$ for any $R_{(p,n)} \in r_2$, because it contains all the P_1-points of $R_{(p,n)}$, and every nbhd of every point in $R_{(p,n)}$ contains P_1-points. Putting the $R^+_{(p^\alpha,n)}$ together gives us a collection of disjoint regions $R_{(p,n,\pm)j}$, all on level M + n + 1. By the previous lemma any extra accumulation point of these regions is a $q \in P_2$, and if $|q_X| \geq M + n + 1$ then already $R_{(q,\ldots)} \subset R_{(p^a,n,\ldots)j}$ for some (unique) a, j. If $|q_X| = M$ then either $R_{(q,n)} \cap R_{(p^a,n)} = \emptyset$ for all a or else $R_{(q,n)} = R_{(p^a,n)}$ for some a by 2.1.4.

$\Leftarrow$: q cannot be in any of the $R_{(p_\alpha,n)}$ under these conditions,

Now let us look at what happens when we take accumulation points of regions of the second kind.

Let $\{R_{(p^a,m)}\}_{a \in a}$ be such that $|p_x^a| = N$ for all a, m fixed. Any point on the boundary of $\bigcup_{a \in a} R_{(p^a,m)}$ is on the boundary of $\bigcup_{a \in a} R^{\pm}_{(p^a,m)}$, the union of a collection of (disjoint) regions R_x with $|x| = N + m + 1$. Hence it must be a point $q \in P_2$, with $|q_x| < N + m$. Without loss of generality we may assume q is a boundary point of $\bigcup_{n=1}^{\infty} R_{x_n}$ where $x_n = y(p^n, m, +)j$ for some infinite collection of p^n. We have the following profile for x^n: $(0,\ldots,p^n(N),+,-,\ldots,-)$

Since $|q_x| \leq N + m$, $R_{x_n} \cap R_{(q,m)} = R_{x_n} \cap R^{+}_{(q,m)}$ for all k,n.

Case I. $|q_x| < N$. Let $k \geq N + 2$ and let $x = y(q,k,+)j$. Then in the $(N + 1)$th place we have a negative number, so $R_x \cap R_{x_n} = \emptyset$ for all n. If $x = y(q,k,-)j$, then in the $(N + 2)$th place we have a positive number, and again $R_x \cap R_{x_n} = \emptyset$ for all n. So q is not in the closure by 6.1.1.

Case II. $|q_x| = N$. Then, if q is an accumulation point, we have in particular $R_{(q,m)} \cap R_{x_n} \neq \emptyset$ for some n and so by 2.1.4,

$R_{(q,m)} = R_{(p^n,m)}$.

Case III. $|q_\lambda| > N$. Then, if $|q_\lambda| < N + m$, the $(|q_\lambda| + 1)$th term of any $y(q,'k,\pm)j$ is of opposite sign from the $(|q_\lambda| + 2)$th term, while the corresponding terms of x_n have the same sign. So $|q_\lambda| = N + m$, and $q_\lambda > p^n_\lambda$ for all λ, so that p^n_λ is the same for infinitely many n.

IIIa. $m > 1$. Then we can choose the p^n so that each R_n is contained in R_{q_λ} (otherwise, since $|x_n| > |q_\lambda|$, we have $R \cap R = \emptyset$) and contains a ball of the form $R_{,\ ,\ *}$. (note the minus sign) properly (see the comment following 6.1.1.). So we have $x_n = (0,\ldots,q_\lambda(N+m),-q_\lambda(j))$ for some j; the j's will of course be different for different n because the x_n are all different and can only vary in the (N+m)th term. In particular, $R_{q_\lambda} = R_{\lambda(p^n,m-1,+)j_n}$ for all n.

The p^n are all distinct: $-q_\lambda(j) = -p^n_\lambda(m-1)$ and the terms are different for different n. But $R_{(p^n,m-1)}$ is the same for all n.

IIIb. $m = 1$. Then $q_\lambda = (0,\ldots,p^n_\lambda(N), p^n_\lambda(j_n))$. There are two possibilities: either p^n_λ is the same for all p^n (in which case the j^n are all different) or p^n_λ is not the same for all p^n_λ, when j^n could be the same integer for different n(in fact if the p^n are all different it could even be the same integer for all n).

There is not much else to be said here except that we can choose p^n so that R_{qX} contains $R_{Y(P^n, l,+)j_n}$ properly while $R_{p^n_X}$ contains R_{qX} and also $R_{(p^n, l)}$ for all n--and that the p's themselves are all distinct.

We are now in a position to establish our result for regions of the second kind.

6.2. Lemma. Let A be a clopen subset of $R_{(p,n)}$ such that $A \nmid li_{(p,n)}\backslash$. Then there exists a a-locally finite cover of A by proper subregions of $R_{(p,n)}$.

Proof. For each $q \in R^o_{(p,n)} \cap A$ there exists n_q such that $R_{(q,n_q)} \subset A$ and therefore $A \in U_{(q,n_q)}$. For each q let n_q be the least such integer. (In what follows, always assume $q \in R^*_{(p,n)} \cap A$.)

Consider the collection $\{R_{(q,n+l)}: n_q = n+1\}$. By 5.4.2. this is a collection of at most countably many regions.

Next consider the collection $\{R_{(q,n+2)}: n_q = n+2\}$. Any boundary point of the union comes under the heading Ilia and so must be the boundary point of the union of subregions of the same $R_{(q,n+1)}$. Each such $R_{(q,n+1)}$ can contain only countably many such subregions (otherwise $n_q \leq n+1$, contradicting the assumption on the subregions). Furthermore if we take a collection of regions $R_{(q,n+2)}$ with $n_q = n + 2$, such that the regions $R_{(q,n+1)}$ are all disjoint, then their union must be clopen. Hence by the axiom of choice we can split $\{R_{(q,n+2)} : n_q = n+2\}$ into countably many locally finite families.

Using essentially the same argument we obtain for each $N \geq n + 1$ a splitting up of $\{R_{(q,n_q)} : n_q = N\}$ into countably many locally finite families; now let $\mathcal{V} = \{R_{(q,n_q)}\}_{n=1}^{\infty} \cup \{R_{(p,n,\pm)}\}_{j=1}^{\infty}$; $\mathcal{V}$ is a σ-locally finite clopen cover of A.

We note in passing the fact that any two distinct regions $R_{(p,n)}$ and $R_{(q,m)}$ in Γ_2 of level M are disjoint. This follows immediately from 2.1.4 if $|p_x| = |q_x|$ (and, of course, $n = M - |p_x| = M - |q_x| = m$). Otherwise assume without loss of generality that $|p_x| < |q_x|$; then Case III above shows that if they were not disjoint we'd have $|q_x| + m > |p_x| + n$ even for $i = 1$; contradiction. Of course by 2.1.1, any two distinct regions in Γ_1 of level M are disjoint.

Next we look at accumulation points of Γ_1 regions in greater detail. Let $\{R_{x_\alpha}\}_{\alpha \in S}$ with $|x_\alpha| = N$ for all α be such that p is a boundary point of the union, where $|p_x| < N$.

Case I. $|p_x| < N - 1$.

Then, for infinitely many a, (Label them $a_1, \ldots, a_n, \ldots$) there exists J_n such that $R_{x_{a_n}} = R_{Y(p,N-1-|p_x|,\pm)J_n}$ C«P»iity

holds because the regions are on the same level). Without loss of generality assume that the sign at the end of $y(P>^N - 1 - |p_x|>)j_n$ is +. Now for each n, x_{a_n} starts off with $(0,\ldots,p_v(|p_v|),$ $p_x(N - 2 - |p_x| + j_n),\ F^{-1}_{x_{a_n}(|p_x|+1)}(P_v)>\ldots)$ and continues with

negative reals as long as it lasts. The variance between the x_{a_n} takes place at the two terms following $p_v(|p_v|)$, and the second of these depends upon the first and upon p_y. It is clear that any $q \in R^{*}_{(p,N-1-|p_x|)}$ is a boundary point of the $R_{x_{a_n}}$. What else?

la. $|q_x| < |p_x|$. If $R^{+}_{(q,m)} \cap R_{x_{a_n}} \neq \emptyset$ then we must have the term $|p_v| + 1$ positive, so that $p_Y(N - 2 - |p_X| + j_n)$ is equal either to $+F^{-1}_{x_{a_n}(|q_x|+1)}(q_Y)$ or else $+q_z(i)$ for some i, and the signs remain positive. But then, for $m \geq N$, $R_{(q,m)} \cap R_{x_{a_n}} = \emptyset$. So q is not an accumulation point.

Ib. $|q_x| = |p_x|$. if $R_{(q,m)} \cap R_{(p,m)} \neq \emptyset$, where $m = N - 1 - |p_x|$, then the two are equal and $q \in R^{\circ}_{(p,m)}$.

Ic. $|p_v| < |q_v| < N$. (The second inequality follows from our not wanting q to be already in one of the R_{x_a}.) Since all the terms of these x after $|p| + 1$ are negative, and since there must occur a switching of signs between $|q_A| + 1$ and $|q_A| + 2$, the only case to consider is $|q_A| = N - 1$. If $R_{(q, i)} \neq 0$ $R_{x_{\alpha_n}} \neq 0$, then $q_X(|p_X| + 1) = p_Y(m + j_n)$ and this can be true for at most one n, so that q is not an accumulation point of the R_{α_n}.

Case II. $|p_A| = N - 1$. We then have infinitely many α_n such that (without loss of generality) there exists j_n such that

$$x_{\alpha_n} = (0, \ldots, p_X(|p_X|), P_Y(J_n))$$

What other boundary points are there?

IIa. $|q_X| < N - 1$. Then the Nth term of "$i(q., m, \pm)$ j is either fixed for all m and all j or else (in the case $|q_A| = N - 2$) depends upon the previous term, which is the same for all the x_{α_n}. Hence $R_{x_{\alpha_n}}$ OR $_{(q, m, \pm)} \neq 0$ for at most one n and so $R_{(q, m, \pm)}$ is not an accumulation point.

IIb. $|q_A| = N - 1$. if $q_A = p_A$, and q_Y has infinitely many of the j_n as terms, then q is indeed an accumulation point; otherwise it is not.

Let us try to apply these results to the task of finding an analogue of Lemma 6.2 for r_f-regions. Take a clopen set A and take the set of regions R_x with the property that $A \in U_{(\ldots)}, A \subseteq U_x$, [resp. $A \subseteq U_{(p, n)}$]

for any R_x, [resp. $R_{(p,U.)}$] containing R_x properly.

Suppose we take the family of all regions R eT, of level N which are chosen in this way. Any boundary point of the union of this collection must be a pePp with |Pvl < ^. We can now show that IPyl is not less than N - 1 by using Ib and the following lemma.

6.3. Lemma. Let $p \in P_2$ and let A be a clopen set such that AeU$_{\gamma(p,n,\pm)j}$• for infinitely many j (for some fixed n). Then AeU$_{(p,n)}$•

Proof: Suppose AeU$_{(p,n)}$, then A e\x$_{(p,n)}$•> and so from 5.7' there exists a j such that A^ceU$_{\gamma(p,n;\pm)k}$ for a^ ^ ^. J• This implies that AeU$_{\gamma(p,n;\pm)k}$ only for k < j.

So if that boundary point p had $|p_X|$ < N - 1, we would have AeU$_{(p,n)}$ for n = N " IP_X' " 1} and $R_{(pn)} \supset R_X$ for any R_X R on level N such that R,.. . Pi R ^ j6, which would contradict the way that the R.. were chosen.

The case $|p_x|$ = N - 1 is more difficult. We need a strengthening of 5.4.1.

6.4. Lemma. Let A be a clopen set and let xeX. If for infinitely many x^1 with |x'| = |x| + 1, x < x^1, x'(|x| +1) > 0 [resp. x' (|x| + 1) < 0] AeU ,, then AeU .

Proof. We give the proof for x' (|x| +1) > 0.

By hypothesis there is a sequence of x^1 of the form $(0,\ldots, x(|x|), r_n)$ where $r_n \in R^+$, for which $A \in U_{x}$.

We split the remainder of R^+ up into $2^{\aleph_0}$ disjoint sets of distinct positive real numbers each. Take any one of these sets *H* and now let S be the set of all sequences which contain the numbers r_n in the 2n th place and the numbers of the set chosen, exactly once and no others (so that they will all be permutations of each other). It is a simple exercise in set theory to show that S has cardinality $2^{\aleph_0}$.

For each $a \in S$ pick $q^\sigma \in P_0$ so that $q^\wedge = x$, $q^\wedge = a$, and now for all q^σ, $y(q^\sigma,l,+)_{2j}(|x|) = r_j$ for all j. And, for all but countably many $a \in S$, $y(q^\circ,1,+)_0.(|x|+1)$ is such that $A \in U / o, >_0.$, because of 5.4.1 and the fact that $A \in U_{x'}$.

This is true for any fixed j, and so, for all but countably many $a \in S$,

$$A \in U_{Y(q^a,l,+)2j} \text{ for all } j.$$

Then, for these same a, (let this set be called S_1) $A \in U_{(q^0,l)}$; for either A or A^C must be in, and if A^C were in $U_{(q^a,D}$ for any a we would get a contradiction from 5.7', 5.15.1 and 5.15.2: A^C would have to be in $U_{Y(q^\sigma,l,+)2j}$ for all but finitely many j.

Indeed by the same argument $A \in U_{y(q^a,l,+)2j+l}$ for all but finitely many j given any $\sigma \in S_1$. Order the reals in « in a sequence

$f s_n\}$. Since S_i is uncountable there exists m such that, for uncountably many $c \in S_1$, $A \in U_{\gamma(c|\ell+)2j_m}$ where j_m is the term of a which is equal to s. Since the $U_{\gamma(q,1,+)2j_m}$ are all in R_m, where $x_m = (0, \ldots, x(|x|), s_m)$ and are all disjoint because of the way F_{s_m} is defined, it follows from 5.4.1 that $A \in U_{y}$.

We can now repeat the argument for all the other sets into which R^+ was split, thus obtaining $2^{\aleph_0}$ distinct x' with $x' > x_1$, $x'(|x| + 1) > 0$, $|x'| = |x| + 1$, such that $A \in U_{x'}$. Another application of 5.4.1 gives us the fact that $A \in U_x$.

Now consider the collection of all regions in r_i of level N with the property that each is maximal with respect to $A \in U_x$. Any boundary point p of this collection must be of the form indicated by 6.1.1., and could only have $|p_x| = N - 1$ by the reasoning following 6.3. But now any such p can only be the accumulation points of those regions contained in R_{p_x}. $|p_x| = N - 1$ means that the regions involved satisfy the conditions of 6.4. Because they are closed, there must be infinitely many of them, whence $A \in U_{p_x}$, contradicting the way these regions were chosen. So the collection is locally finite.

Next we show that we can indeed speak of "a largest region containing p" satisfying the conditions given in the discussion preceding 6.1.1.

6.5. Lemma. Let A be a clopen set. The collections

$\{R_x : A \in U_x$ but $A \notin U_x$, [resp. $A \in U_{(p,m)}$] for any R_x, [resp. $R_{(p,n)}$] containing R_x properly$\}$

and

$\{R_{(p,n)} : A \in U_{(p,n)}$ but $A \notin U_x$ [resp. $A \notin U_{(q,m)}$] for any R_x [resp. $R_{(q,m)}$] containing $R_{(p,n)}$ properly.$\}$

together form a cover of A.

<u>Proof</u>. Let $p \in A$, then by 2.8 there is a region R containing p and contained in A. If $p \in P_1$, say $p \in R \subset A$, then $A \in U$. Similarly for $p \in P_2$. If R is not in either collection, there is a region R_1 such that A is in the clopen ultrafilter associated with R_1 and $R \subset R_1$. Repeat the argument, obtaining a sequence $R_1 \subset R_2 \subset \ldots$ if necessary; but if R is on level N, then R_1 is on an earlier level, etc. (2.2.5.) The sequence must stop somewhere and the last term is in one of the above collections.

Note that these regions are not necessarily disjoint, so that a point may be in more than one "largest" region. As it turns out, this does not prevent us from reaching our goal, it merely makes our work much more complicated.

We will apply this lemma to $A^c R$ for a fixed region R by noting that if $A \in U_x$, then $R_x \subset R$, and if $A \in U_{(p,n)}$, then $R_{(p,n)} \subset R$.

In spite of all these preliminaries the case of a region $R \in T_1$ remains much more difficult than that of a region in T_2, because the $p \in P_0$ with $|p_v|$ minimal do not all belong to one R_{σ}, as they do for a $V_?$ region, and because there is no analogue here of the $R_{Y(p,n,\pm)j}$'s which provided such a convenient countable cover of the rest of the points in $R_{(p,n)}$. Even after we are done with the $p \in P_9$ of $|p_Y|$ we still have to go down within each $R_{x'}$, with $x' > x$ and take care of the points in there.

Even the cover we have obtained for a clopen set $A \subset R^x$ by 6.5 is not as well-behaved. True, the R_x, of a given level N form a locally finite collection as explained above, and we can take care of the $R_{(p,n)}$ with $n > 1$ by a technique like that employed in 6.2, but we have not yet accounted for possible boundary points of regions $R_{(p,1)}^*$ (case lib preceding 6.2).

So now, suppose we have a clopen set $A \subset R_x$ $A \wedge U_x$. Take those p^α such that $p^\alpha_v = x$, $A \in U_{1N}$. If q is a boundary point of $\bigcup_a R_{(p,1)}$ then $q^A \subset R^{x'}$, $|q^x| = |x| + 1$, and R_{q_x} contains infinitely many balls of the form $R_{Y(p^\alpha,1,\pm)j_n}$. If $A \in U_{Y(p^a,1,\pm)j}$ for infinitely many of these balls in R_{q_x}, then $A \in U_{q_x}$ (6.4) and there can be only finitely many such distinct R_{q_x} (6.4). But there is apparently no restriction on the number of R_{q_x} for which $A \in U_{Y(p^a,1,\pm)j_\alpha}$ for only finitely many a. No matter: we can get around this by noting that $A \notin U_{Y(p^\alpha,1,\pm)j}$ for only finitely many j, given a fixed a (5.7[1]), and producing a refinement of the cover presented in 6.5. (We keep the assumption that $A \subset R_x$, $A \notin U_x$).

Take all the $R_{(P^a,1)}$ in that cover and replace each one by $R_{(p^a,l)} \setminus \bigcup R_{Y(p^a,l,\pm)J}$ where the union is taken over exactly those subregions for which $A \notin U_{(P^a,l,\pm)j}$ and those which are subregions of R_{q} where $A \in U_{q}$ and q is a boundary point of $\bigcup R_{(p^a,l)}$.

We have just seen that there are only finitely many $Y(p^a,l,\pm)j$ in each category for a fixed a. Call the resulting set R_{p^a}.

Now we go down within each $R_{x'}$ with $|x'| = |x| + 1$, $x' > x$, and $A \notin U_{x'}$. Take $A \cap (R_{x'} \setminus \bigcup R_{p^a})$ (where the union involved is effectively over a finite number of regions, so that the resulting set is still clopen and of course is not in $U_{x'}$) and repeat the process for this set relative to $R_{x'}$: construct a cover like that given in 6.5 and then truncate the $R_{(p,1)}$ regions in the cover for which $p_x = x'$. as above. Then go by induction to the r_1 subregions of each $R_{x'}$ on the next level, the level following that, and so on as long as necessary. At each stage, the $R_{(p,n)}$ regions with $n > 1$ and the $R_{x'}$-regions with $A \in U_{x'}$ remain as in the original cover.

It is a routine but important exercise to verify that the regions (or truncations of regions) in the refinement thus constructed are incomparable, i.e. if R is a member of the refinement and R' is another member so that $R \subset R'$, then $R = R'$. This was trivially true for the original cover and remains true for the refinement.

What results is indeed a cover: let $p \in A \cap P_i$, then we obtain a member of the original cover containing p as in 6.5. Should it be a region of the form $R_{(q,i)}$ (the only nontrivial case) then if $A \in U_{x'}$, where $|x'| = |x| + 1$, $x' < p$, we are done. If $A \notin U_{x'}$, we then obtain a cover of $A \cap (R_x \setminus \bigcup R_{x_p})$ as in 6.5, and if $p \notin \bigcup R_{x_p}$ then one of the members of this cover contains p...eventually the process must end either with a region $R^{(q,n)}$ with $n > 1$, or with an $R_{x'}$, or with an $R_{(q,1)}$ such that $A \in U_{x'}$, (where x' is defined as above). It cannot go on indefinitely because there exists an R_{x_p} with $p \in R_{x_p} \subset A$, so that A contains every subregion of R_{x_p} and is a member of the free clopen ultrafilter associated with each such subregion; then, since $|q_x|$ increases by at least 1 at each step of the process just outlined, we eventually must get to a point where $|q_x| > |x_p|$ and then $R_{(q,1)} \subset R_{x_p}$ if $p \in R_{(q,1)}$.

If $p \in A \cap P_p$, the only nontrivial case is where every region of the original cover containing p is of the form $R_{(q,1)}$ for some $|q_A| < |p_A|$, then we use an argument like that above, and since $R^{(p,n)} \subset A$ for some n, the process must stop at or somewhere before $|p_A| + n$.

At last we are ready to prove the main result.

6.6. Lemma. Let A be a clopen subset of R . If A<£u , then there exists a a-locally finite cover of A by clopen sets each of which is contained in a proper subregion of R .

Proof. Define a clopen cover for A as in 6.5 and take the refinement described above. We wish to show that this refinement is a-locally finite. Let |x| = M.

For each fixed N > M take the set of all f_i-regions of level N which are members of the refinement. This is a collection of disjoint regions (2.1.1). Suppose its union is not closed; let p be a boundary point, so that p is as described in 6.1.1. Then if $|p_v| < N-2$, the reasoning following 6.3 applies to give a contradiction, while if $|p_v| = N-1$, the reasoning following 6.4 gives a contradiction. Of course we must have $|p_v| \leq N-1$. Finally in the case where $|p_x| = N-2$ (so that the T_2-region involved in 6.3 is $R_{(p;j,\gamma)}$) the truncations only eliminate finitely many $R_{\gamma(p;1,\pm)}$ and there are still infinitely many R_x with $R_x \subset R_{(p,1)} \cup R_{\gamma(p;1,\pm)}$, contradiction. So for each N we have a locally finite collection.

Next we look at the truncated regions $R_\wedge - R_{(p,1)} \cup R_{\gamma(p;1,\pm)}$. The union is finite and so the resulting set is still clopen for all p. For each fixed $N \geq M$ take the family of all such truncated regions in the refined cover with $|p_v| = N$. Any boundary points of these truncated regions must come under Case III of the discussion following 6.1.2., specifically IIIb. But the truncations have eliminated all these accumulation points and were in fact designed with this end in mind. So we have a locally finite collection for each N.

Finally, for each pair (N_1, N_2) with $N_1 \geq |x|$, $N_2 > 1$, we take the family of all $R_{(p,n)}^*$ in the refined cover so that $|p_{|x|}| = N_1$, $n = N_2$. Each such region is contained in $R_{(p,n-1)}$ along with at most countably many other regions $R_{(p,n)}$ of the refined cover (5.4.2). And since we are in Case IIIa the only possible boundary points are points $q \in P_0$ with $|q| = N_1 + N_0$, $q \in R_{(p,n)}$ and on the boundary of the union of those countably many subregions of $R_{(p,n-1)}$. So we can, as in the proof of 6.2, apply the axiom of choice to get a countable collection of locally finite families for each pair (N_1, N_2).

And so, putting everything together, we have a countable collection of locally finite families, each of them composed of disjoint clopen subsets of regions each properly contained in R_x.

<u>The Topology on the N-compactification</u>. The N-compactification $\nu\Delta$ of Roy's space is obtained by adjoining ideal points in 1-1 correspondence with the free clopen ultrafilters with the countable intersection property, so a convenient way of labeling them would be x (in correspondence with U_x) and [p,n] (in correspondence with $U_{(p,n)}$) where it is understood that [p,n] = [q,m] iff m = n and $q \in R^{o}_{(p,n)}$ (i.e. iff $R_{(p,n)} = R_{(q,m)}$ --see 2.1.4). The topology has as a base the clopen sets $\bar{A}$ where A is a Δ-clopen set and

$$\bar{A} = A \cup \{x : A \in \mathcal{U}_x\} \cup \{[p,n] : A \in \mathcal{U}_{(p,n)}\}.$$

Since the ultrafilters U_x and $\mathcal{U}_{(p,n)}$ have the M-intersection property for all $M < 2^{\aleph_0}$ t^J > it follows that the corresponding points in VA are not G_δ's; indeed, every intersection of M neighborhoods of either kind of point has nontrivial intersection with Δ. Nevertheless, it is possible to get every point of vA as a limit of a sequence of distinct points.

If $x \in X$, take a sequence of distinct points $\{x_n\}$ with $x_n > x$, $|x_n| = |x| + 1$. Any basic clopen set containing x is of the form $\bar{A}$ where $A \in U_x$. Then $A \in U_{x_n}$ for all but finitely many n (6.4) and so $x_n \in \bar{A}$ for all but finite many n. And if the point is of the form [p,n] let x_j = Y(P*n>+)J- Let $\bar{A}$ *>e any clopen set containing [p,n], so that $A \in U$, N. Then $A \in U$ for all but finitely many j.

Furthermore, we can get any point of A as a limit of a sequence of points in vA\A. If $p \in P_1$, take the (unique) sequence $\{x_n\}$ such that $|x_n| = n$, $x_n < p$. Then $x_n \in \overline{R_{x_n}}$ and $\overline{R_{x_n}}$ is a local base at p (2.8.1). If $p \in P_2$ take *the* sequence {[p,n]}; it converges to p by a similar reasoning.

Nevertheless, VA\A is not a sequential space, nor even a c-space. (A space is <u>sequential</u> [resp. a <u>c-space</u>] if every sequentially closed [resp. c-closed] set is closed, where a set A is sequentially closed if every limit of a sequence of points in A is itself in A, c-closed if every accumulation point of a countable subset of A is itself in A. (See [F_1], [F_2], [B_1, pp. 146-152].) The most obvious demonstration of this is to take a point $x \in X$ and take A to be the set of all [p>l] such that $p_A = x$. x is an accumulation point of A (see the proof of 6.4: any clopen set belonging to \x,. ,v for all p with $p_A = x$ must belong to U..). But no sequence of points in A has an accumulation point: given a sequence [$R_{(p^n,1)}$} we may take all the $R_{x'}$ with $x' > x$, $|x'| = |x| + 1$ such that $R_{\gamma(p^n,1,\pm)j} \subset R_{x'}$ for infinitely many pairs n,j. There are only countably many of these and so order them in a sequence [$R_{x'_m}$}. Truncate each $R_{(p^n,1)}$ by deleting those $R_{\gamma(p^n,1,\pm)j}$ which occur in $R_{x'_1},\ldots,R_{x'_n}$. The resulting set (call it R'_{p^n}) is still clopen in A for each p^n, and each $R_{x'_n}$ contains only finitely many $R_{\gamma(p^n,1,\pm)j}$ still in R'_{p^n}, namely those for which $n < m$, $\gamma(p^n,1,\pm)j > x'$. Thus R will contain at most one, R at most two, etc. We have eliminated all possible boundary points of $\bigcup_{n=1}^{\infty} R'_{p^n} = B$ which is thus clopen in A. $B \not\ni U_x$ because B is a union of countably many clopen sets each contained in some $R_{(p^n,1)}$ and $R_{(p^n,1)} \not\ni U_x$ for any n. So x is not an accumulation point of ([p^n,1]}. A point $p \in \bar{A}\setminus A$ is not a boundary point of A, for we can find a sufficiently small region R containing p so that $VA\setminus\bar{R}$ is a clopen set containing A. Any $x' \in X$ is not an accumulation point of A, for if $x' > x$, $\overline{R_x} \setminus \overline{R_{x'}}$, is a clopen set containing A and missing x'; if $x > x'$, and $x \neq x'$, the same is true of $\overline{R_x}$; we have just taken care of the case $x = x'$, while in any other case $\overline{R_x} \cap \overline{R_{x'}} = \emptyset$. Similar arguments work for $[p^n,k]$ with $k > 1$ and $[q,k]$ with $q \neq p^n$ for some n and all k. So, actually, A has the single accumulation point x and that cannot be gotten as an accumulation point of countably many points of A.

A similar argument shows that T taken by itself is not a c-space. Take for each $[p,k]$ the set A of points $[q,k+1]$ such that $q \in R^{\circ}_{(p,k)}$. By 5.4.2, $[p,k]$ is an accumulation point of A; and there are no others. And $[p,k]$ is not in the closure of a countable subset of A because we can get a clopen set containing any countable subset of A by truncating the associated regions $R_{(q^n,k+1)}$ as we did above for the p^n and taking the union of what is left. This shows that we cannot get an analogue of 6.4 for r_2-regions. 5.4.2 is the best result we can get.

X taken by itself is not a c-space, either. This we can show in the following manner: split the positive reals into 2^{ω} disjoint countably infinite sets (the group $/Q$ comes to mind), arranging each into a sequence. Take one of these sequences (think of Q^+), a, and for each remaining sequence a_α let β_α be the sequence whose (2n) th term is a(n) and whose (2n + 1) th term is $a_\alpha(n)$.

Now pick any $x \in X$, take the set of all $R_{x'} \subset R_x$ on the level just after that of R_x, and within each $R_{x'}$, take $R_{x''}$, such that

$$x'' = (0,\ldots,x(|x|),x'(|x'|), \ldots)$$

where a is the unique index such that $a(n) = x'(|x'|)$ for some n; there are, of course, countably many $R_{x'}$ which contain no such $R_{x''}$, these being the ones such that $a(n) = x'(|x'|)$ for some n. These we ignore for the moment. So now each ball on the $|x| + 1$ level in the upper row of R_x (see the description in [R]) contains at most one ball of the form $R_{x''}$. Any clopen set A such that $A \in U_{x''}$ for all x'' must also be in $U_{(p^a S l)}$ for all a by 6.3. Then for each a there exists n_a such that $A \in U_{Y(p^a,l,i)}$ for all $i > n_a$ (5.7'). Pick $2m$ such that $2m > n_a$ for infinitely many a, then $A \in U_{x'_a}$ where $x' = (0,\ldots,x(|x|),a(n))$ for all $n > m$. (6.4). Therefore $A \in U_x$, again by 6.4, and so x is in the closure of the x''. It is easy to verify that no other point of X is an accumulation point of A. Any countable subset of A is contained in countably many $\overline{R_{(p,i)}}$ and by a truncation method like that used above we can find a clopen subset of each $\overline{R_{(p,i)}}$ that contains each $x' \in \overline{R_{(p,i)}} \cap A$, and such that their union is clopen. It follows that x is not in the closure of any countable subset of A.

Yet X does contain subspaces which are c-spaces. For instance: take any x, and take all $x' > x$ such that $x'(i) > 0$ for all $i > |x|$.

In other words, within any R_x ball we take the top row of r_1-balls in it, the top row of each of these, and so on. What results (call it P(x)) is actually a sequential space, an uncountable analogue of the space S^ [AF] . Each point x' of P(x) is the extra point of the one-point compactification of the set fx'':x''eP(x), x'' > x', |x''| = | x'| + l]--the points "below x on the following level". What makes P(x) a sequential space is the lack of interference from r_2 regions. Suppose we have any number of R..,, "two levels down from x, below it"[1] such that each R , one level down from x contains at most finitely many R..,,; then U R ,, is a clopen set pure and simple. With this in mind the proof that P(x) is sequential is quite easy. Let A be a sequentially closed subset of P(x) not containing x, then we show x is not an accumulation point of A: by 6.4 there are at most finitely many x^1 on the next level down from x in A. U $\overline{R_{..,}}$ = A, for all such x^1 is a clopen set. In each other x' on this level there are at most finitely many x^{1}' in A for the next level below x'; and U $\overline{R_{x,,}}$ = A_2 for all x'' associated with these x' is a clopen set. Proceeding thus we cover A by infinitely many disjoint clopen sets. Their intersections with A form an infinite collection of disjoint clopen sets, and it is easy to show that no point of A is a boundary point of this collection, so the union of all these sets is clopen in A and so is the closure in VA ,

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[1]Strictly speaking,we should say "above it" since if x is on level G_N, the x^f are in G_{N+1}, the x'' are in G_{N+2} *(the* subscripts are increasing) but if one has before him a visual image of A as described in [R.^] , it is much more natural to think of x as being above all the x' extending it.

which is U A_n. We have covered !J A_n by a discrete collection of $\overline{R_A}$ each properly contained in $\overline{R}$ and so x is not an accumulation point of A. To complete the proof that A is closed, take any point $\hat{x}$ in the closure of A, then x^{-1} is in the closure of A 0 P(x) which is likewise sequentially closed. If ££A we adapt the above argument to obtain a contradiction.

There is an interesting parallel of the P(x) in T: for each [p,n] take the set P([p,n]) = { [q,m] , qeR, > , m ^ n]. This is not a sequential space (see above) but its topology is an uncountable parallel of that of P(x). For instance, [p,n] is the extra point of the "one-point Lindelofization" of the set {[q,n+l]:qeR? }. And if we delete all countable limit points from a_1 + 1 and define a "convergent N_1 -sequence" in a space X as a continuous function from this space to X and define H_1-sequential spaces accordingly, we can show P([p,n]) is an $\aleph_1$-sequential space. Details are left to the reader.

It may be that the applications of A to general topology are not yet exhausted!

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