ARITHMETICAL REDUCIBILITIES, II

by

Alan L. Selman

Report 71-15

February 1971

/ps

ARITHMETICAL REDUCIBILITIES, II

by

Alan L. Selman

Abstract

Certain reducibilities which generalize many-one reducibility are studied. Let $\leq_{rm}$ be the result of eliminating the bounded quantifier in the definition of $\mathcal{S}_1$. It is shown that $\mathcal{S}_1$ differs from the reducibility $\leq_{rm}$ on sets of the same Kleene-Post degree. Also, a characterization of "Σ_n in" is given, which for $n = 1$ enables us to make more precise the difference between "$A \in \Sigma_1^B$" and "$A \ \mathcal{S}_1 \ B$".

Arithmetical Reducibilities, II

by

Alan L. Selman

Introduction.

Concepts and notation present in this paper refer to our paper [3]. For brevity, Theorem x.y of [3] will be cited here as Theorem I.x.y. For the convenience of the reader we repeat here the following two definitions.

Definition 1. If $\mathfrak{R}$ and $\mathfrak{T}$ are binary relations defined on the set of all subsets of ω, then $\mathfrak{R}$ is an $\mathfrak{T}$-*reducibility relation*, if $\mathfrak{R}$ is reflexive, $\mathfrak{R}$ is transitive, and for all sets A and B, if $A\mathfrak{R}B$, then $A\mathfrak{T}B$.

Definition 2. $A \mathfrak{S}_n B \leftrightarrow \forall X[B\in\Sigma_1^X \rightarrow A\in\Sigma_n^X]$, $n \geq 1$. $A \mathfrak{P}_n B \leftrightarrow \forall X[B\in\Pi_n^X \rightarrow A\in\Pi_n^X]$, $n \geq 1$. $A \mathfrak{I}_1 B \leftrightarrow$ there exist recursive functions f and g so that $\forall x(x\in A \leftrightarrow \exists y\forall z_{z<f(y)} g(x,y,z)\in B)$.

The Σ_n-reducibilities $\mathfrak{S}_n$, $n \geq 1$, (and to a lesser extent the Π_n-reducibilities $\mathfrak{P}_n$, $n \geq 1$), were studied in Chapter $\underset{\sim}{2}$ of [3]. Also, citing Theorem I.2.8, $A \mathfrak{S}_1 B \leftrightarrow A \mathfrak{I}_1 B$ for all sets A and B so that $B \neq \emptyset$ and $B \neq \omega$. It was shown that none of the reducibilities $\mathfrak{S}_n$ generalize relative recursion, but it is an immediate consequence of Theorem I.2.8 and the hierarchy theorem, Theorem I.2.3, that each $\mathfrak{S}_n$ does generalize many-one reducibility.

One aim of the present paper is to make clearer the difference between $\mathcal{S}_1$ and "Σ_1 in". The first two sections are largely devoted to this end. Central to this discussion is the concept of a positive reducibility to be introduced in section 1. Also, this concept will enable us to elaborate on the principal open questions raised in [3].

Another aim of this paper is to study certain other reducibilities which also generalize many-one reducibility. In this direction, our attention is restricted to certain Σ_1-reducibilities which arise naturally from our considerations of the sequence $\mathcal{S}_n$, $n \geq 1$. This study will be taken up in sections 3 and 4. In section 3 we study a reducibility, $\leq_{rm}$, which is the result of eliminating the bounded quantifier in the definition of $\mathcal{J}_1$. It is proved in this section that $\mathcal{S}_1$ differs from $\leq_{rm}$ on sets of the same Kleene-Post degree. In section 4 we study the reducibility $\mathcal{S}_1 \cap \mathcal{P}_1$. As is easily seen (Theorem 6), $\mathcal{S}_1 \cap \mathcal{P}_1$ is a proper subrecursive reducibility.

1. Positive Reducibilities.

Definition 3. Let A and B be any two sets. If $A \in \Sigma_n^B$, then $A \in \Sigma_n^B$ in a positive sense if there is a predicate $\exists y S(x,y)$ which satisfies the following two properties:

(i) $\forall x(x \in A \leftrightarrow \exists y S(x,y))$; and

(ii) S is constructed using the propositional connectives $\wedge$ and $\vee$, together with bounded quantifiers, from predicates $P_1, \ldots, P_k, P_i \in \Sigma_n$, $i = 1, \ldots, k$, $k \geq 1$, and from predicates $f(x,y) \in B$ and $f(x,y,x_1,\ldots,x_n) \in B$, f recursive, $x_1,\ldots,x_n$ not free in S, $n \geq 1$.

Definition 4. A Σ_n-reducibility relation $\mathfrak{R}$ is positive if for each set A and B so that $A\mathfrak{R}B$, $A\epsilon\Sigma_n^B$ in a positive sense.

Theorem 1. If $\mathfrak{R}$ is a positive Σ_n-reducibility, then $\mathfrak{R} \subseteq \mathfrak{S}_n$.

Proof. The proof consists of an easy induction argument.

Essentially, if $A\mathfrak{R}B$ and $B\epsilon\Sigma_n^C$, then there is a predicate $S(x,y)$ which satisfies properties (i) and (ii) of Definition 3, and there is a predicate R^C which is recursive in C so that $x\epsilon B \leftrightarrow \exists z_1 \forall z_2 \ldots Q z_n R^C(x,z_1,\ldots,z_n)$. If all occurrences of B in S are replaced by $\exists z_1 \forall z_2 \ldots Q z_n R^C(x,z_1,\ldots,z_n)$, then, because S contains no occurrences of $\sim$ and no occurrences of unbounded quantifiers, the resulting predicate can be put into prenex normal form ΠM, where the prefix Π consists of n-alternating quantifiers, and the matrix M is recursive in C. Thus $A\epsilon\Sigma_n^C$.

Remark. It is clear that Theorem 1 will not hold if material implication and negation are used in the underlying propositional logic of Definition 3 (ii). (Also, see Theorem 3 and the discussion preceding Theorem 3). Moreover, suppose Φ is an arbitrary truth function of two arguments and suppose φ is the binary connective whose truth-table is given by Φ. Direct examination of the sixteen distinct truth-functions of two arguments shows that at least one of the following holds:

(1) φ is defined in the logic generated by $\{\wedge,\vee\}$;

(2) Φ is a constant function;

(3) negation is definable in the propositional logic generated by $\{\varphi,\wedge,\vee\}$;

(4) $\exists x A(x) \varphi B$ is not equivalent to $\exists x[A(n) \varphi B]$, or $B \varphi \exists x A(x)$ is not equivalent to $\exists x[B \varphi A(x)]$.

Therefore, except for the constant truth-functions, $\{\wedge, \vee\}$ generates the largest underlying propositional logic which can be used in Definition 3.

Theorem 2. $\mathcal{S}_1$ is a positive Σ_1-reducibility.

Proof. The theorem is a corollary of Theorem I.2.8 for all but the special cases. For the special cases, $B = \emptyset$ and $B = \omega$, observe that if $A \in \Sigma_1$. then $A \in \Sigma_1^B$ in a positive sense for all B.

Corollary 1. If $A \in \Sigma_1^B$ in a positive sense, $B \neq \emptyset$ and $B \neq \omega$, then there exist recursive functions f and g so that $\forall x(x \in A \leftrightarrow \exists y \forall z_{z < f(y)}\, g(x,y,z) \in B)$.

Corollary 1 is interesting. since Definition 3 allows for predicates $\exists yS$ of arbitrary finite length.

Is $\mathcal{S}_n$, for $n > 1$, a maximal Σ_n-reducibility? Is there something analogous to Theorem I.2.8 for $n > 1$? We conjecture that the converse of Theorem 1 is true. We state this in the following Conjecture 2.

An argument identical to the proof of Theorem 1 proves the following lemma.

Lemma 1. If $A \in \Sigma_n^B$ in a positive sense and $B \in \Sigma_n^C$ in a positive sense, then $A \in \Sigma_n^C$ in a positive sense.

<u>Conjecture</u> 1. $\mathfrak{S}_n$ is a positive Σ_n-reducibility.

<u>Conjecture</u> 2. $A \in \Sigma_n^B$ in a positive sense $\leftrightarrow \forall X[B \in \Sigma_n^X$ in a positive sense $\rightarrow A \in \Sigma_n^X]$.

By Lemma 1, the implication from left to right of Conjecture 2 is true. By Corollary I.2.1, Theorem 1, and Theorem 2, both Conjectures 1 and 2 are true for the case $n = 1$. Conjecture 2 implies both Conjecture 1 and the maximality of $\mathfrak{S}_n$. In fact for $n > 1$, let $\mathfrak{J}_n$ denote the relation defined by $A\ \mathfrak{J}_n\ B \leftrightarrow A \in \Sigma_n^B$ in a positive sense. (By Corollary 1, Theorem I.2.8, and Theorem I.2.2, if $B \neq \emptyset$ and $B \neq \omega$, then $A\ \mathfrak{J}_1\ B \leftrightarrow A \in \Sigma_1^B$ in a positive sense.) Then, suppose $\mathfrak{J}_n \subset \mathfrak{R} \subseteq$ "Σ_n in", and suppose Conjecture 2 is true. There exist sets A and B so that $A \mathfrak{R} B$ and $A \not\leq_{\mathfrak{J}_n} B$. Thus $\exists X[B\ \mathfrak{J}_n\ X\ \&\ A \notin \Sigma_n^X]$. $A \mathfrak{R} B$ and $B \mathfrak{R} X$, but $A \notin \Sigma_n^X$. Therefore, $\mathfrak{R}$ is not transitive. By Lemma 1, $\mathfrak{J}_n$ is transitive. Hence $\mathfrak{J}_n$ is a maximal Σ_n-reducibility relation. By Theorem 1, $\mathfrak{J}_n \subseteq \mathfrak{S}_n$. Hence $\mathfrak{J}_n = \mathfrak{S}_n$ and $\mathfrak{S}_n$ is a maximal Σ_n-reducibility.

2. <u>The Relations</u> "Σ_n in".

The following Theorem 3 gives a characterization of $A \in \Sigma_n^B$, $B \neq \emptyset$ and $B \neq \omega$. A comparison of this characterization for $n = 1$ with Corollary 1 pinpoints the difference between "$A \in \Sigma_1^B$" and "$A \in \Sigma_1^B$ in a positive sense".

Theorem 3. For all sets A and B, $B \neq \emptyset$ and $B \neq \omega$, the following are equivalent:

(1) $A \in \Sigma_n^B$;

(2) there exists a recursive predicate R and recursive functions f,g,h so that if n is odd, then

$$\forall x(x \in A \leftrightarrow \exists x_1 \forall x_2 \ldots \exists x_n [R(x,x1, \ldots, x_n)$$

$$\& \ \forall y_{y<f(x_n)} (g(x_n,y) \in B \ \& \ h(x_n,y) \notin B)]),$$

and if n is even, then

$$\forall x(x \in A \leftrightarrow \exists x_1 \forall x_2 \ldots \exists x_{n-1} \forall x_n [R(x,x_1, \ldots, x_n)$$

$$\vee \ \exists y_{y<f(x_n)} (g(x_n,y) \in B \ \vee \ h(x_n,y) \notin B)]),$$

(3) there exist recursive functions f,g,h so that if n is odd, then

$$\forall x(x \in A \leftrightarrow \exists x_1 \forall x_2 \ldots \exists x_n \forall y_{y<f(x_n)}$$

$$(g(x,y,x_1, \ldots, x_n) \in B \ \& \ h(x_n,y) \notin B)),$$

and if n is even, then

$$\forall x(x \in A \leftrightarrow \exists x_1 \forall x_2 \ldots \exists x_{n-1} \forall x_n \exists y_{y<f(x_n)}$$

$$(g(x,y,x_1, \ldots, x_n) \in B \ \vee \ h(x_n,y) \notin B)).$$

Proof. Suppose $A \in \Sigma_n^B$, $B \neq \emptyset$, $B \neq \omega$, and n is odd. Let $Ch(z) \equiv z$ is characteristic sequence number. (See [3, Chapter $\underset{\sim}{2}$, §1].) For some e, $\forall x(x \in A \leftrightarrow \exists x_1 \forall x_2 \ldots \exists x_n T_n^1(\overline{h}(x_n), e, x, x_1, \ldots, x_{n-1}),$ where h is the characteristic function of the set B.

$$x \in A \leftrightarrow \exists x_1 \forall x_2 \ldots \exists x_n T_n^1(\bar{h}(x_n), e, x, x_1, \ldots, x_{n-1})$$

$$\leftrightarrow \exists x_1 \forall x_2 \ldots \exists x_n [Ch(x_n) \ \& \ \forall y < \ell h(x_n)((x_n)_y = 1$$

$$\leftrightarrow y \in B) \ \& \ T_n^1(x_n, e, x_1, \ldots, x_{n-1})].$$

Let $R(x, x_1, \ldots, x_n) \equiv Ch(x_n) \ \& \ T_n^1(x_n, e, x_1, \ldots, x_{n-1})$.

Then

$$x \in A \leftrightarrow \exists x_1 \forall x_2 \ldots \exists x_n [R(x, x_1, \ldots, x_n)$$

$$\& \ \forall y < \ell h(x_n)((x_n)_y = 1 \leftrightarrow y \in B)].$$

$$\forall y < \ell h(x_n)((x_n)_y = 1 \leftrightarrow y \in B) \leftrightarrow \forall y < \ell h(x_n)((x_n)_y = 1$$

$$\rightarrow y \in B) \ \& \ \forall y < \ell h(x_n)(y \in B \rightarrow (x_n)_y = 1).$$

Let $a \in B$ and $b \notin B$. Define

$$g(x_n, y) = \begin{cases} y, & (x_n)_y = 1 \\ a, & \text{otherwise.} \end{cases}$$

Define

$$h(x_n, y) = \begin{cases} y, & (x_n)_y = 2 \\ b, & \text{otherwise.} \end{cases}$$

$$\forall y < \ell h(x_n)((x_n)_y = 1 \rightarrow y \in B) \leftrightarrow \forall y < \ell h(x_n) g(x_n, y) \in B.$$

Also,

$$\forall y < \ell h(x_n)(y \in B \rightarrow (x_n)_y = 1) \leftrightarrow \forall y < \ell h(x_n) h(x_n, y) \notin B.$$

Thus, $x \in A \leftrightarrow \exists x_1 \forall x_2 \ldots \exists x_n [R(x,x_1,\ldots,x_n)$ & $\forall y < \ell h(x_n) g(x_n,y) \in B$ & $\forall y < \ell h(x_n) h(x_n,y) \notin B]$. Let $f(x_n) = \ell h(x_n)$. Then, $x \in A \leftrightarrow \exists x_1 \forall x_2 \ldots \exists x_n [R(x,x_1,\ldots,x_n)$ & $\forall y < f(x_n)(g(x_n,y) \in B$ & $h(x_n,y) \notin B)$. Hence, for n odd, (1) implies (2).

Define

$$g_1(x,y,x_1,\ldots,x_n) = \begin{cases} g(x_n,y), & R(x,x_1,\ldots,x_n) \\ b & ,\overline{R}(x,x_1,\ldots,x_n) \end{cases}$$

Then, $R(x,x_1,\ldots,x_n)$ & $\forall x < f(x_n) g(x_n,y) \in B \longleftrightarrow \forall x < f(x_n)$ $g_1(x,y,x_1,\ldots,x_n) \in B$. Thus $x \in A \leftrightarrow \exists x_1 \forall x_2 \ldots \exists x_n [\forall x < f(x_n)$ $(g_1(x,y,x_1,\ldots,x_n) \in B$ & $h(x_n,y) \notin B)]$. That is, (2) $\rightarrow$ (3), for n odd. It is clear that (3) $\rightarrow$ (1).

Now, suppose n is even. $A \in \Sigma_n^B$. Thus, for some e, $\forall x(x \in A \longleftrightarrow \exists x_1 \forall x_2 \ldots \exists x_{n-1} \forall x_n \overline{T}_n^1(\overline{h}(x_n),e,x,x_1,\ldots,x_{n-1})$ where h is the characteristic function of B.

$$\begin{aligned} x \in A &\leftrightarrow \exists x_1 \forall x_2 \ldots \exists x_{n-1} \forall x_n \overline{T}_n^1(\overline{h}(x_n),e,x,x_1,\ldots,x_{n-1}) \\ &\leftrightarrow \exists x_1 \forall x_2 \ldots \exists x_{n-1} \forall x_n [Ch(x_n) \;\&\; \forall y < \ell h(x_n)((x_n)_y = 1 \\ &\quad \leftrightarrow y \in B) \rightarrow \overline{T}_n^1(x_n,e,x_1,\ldots,x_{n-1})] \\ &\leftrightarrow \exists x_1 \forall x_2 \ldots \exists x_{n-1} \neg \exists x_n [Ch(x_n) \;\&\; \forall y < \ell h(x_n)((x_n)_{y-1} \\ &\quad \leftrightarrow y \in B) \;\&\; T_n^1(x_n,e,x_1,\ldots,x_{n-1})]. \end{aligned}$$

As for the case n odd, there exists a recursive predicate $R(x,x_1,\ldots,x_n)$ and recursive functions g and h so that

$x \in A \leftrightarrow \exists x_1 \forall x_2 \ldots \exists x_{n-1} \neg \exists x_n [R(x, x_1, \ldots, x_n)$ & $\forall y < \ell h(x_n) g(x_n, y) \in B$ & $\forall y < \ell h(x_n) h(x_n, y) \notin B]$. As before, let $f(x_n) = \ell h(x_n)$. Then,

$$x \in A \leftrightarrow \exists x_1 \forall x_2 \ldots \exists x_{n-1} \forall x_n [\overline{R}(x, x_1, \ldots, x_n) \vee \exists y < f(x_n)(g(x_n, y) \notin B \vee h(x_n, y) \in B)].$$

Interchanging g and h, and $\overline{R}$ and R, we have (1) $\to$ (2). (2) $\to$ (3) is proved as in the case n odd. And, again, it is clear that (3) $\to$ (1). Thus, the proof of Theorem 3 is complete.

Corollary 2. For all sets A and B, $B \neq \emptyset$ and $B \neq \omega$, $A \in \Sigma_n^B$ if and only if:

(1) if n odd, there exists a recursive predicate R and a recursive function f so that

$$\forall x (x \in A \leftrightarrow \exists x_1 \forall x_2 \ldots \exists x_n \forall y_{y < f(x_n)} \exists u \in B \ \exists v \notin B \ R(x, x_1, \ldots, x_n, y, u, v));$$

(2) if n even, there exists a recursive predicate R and a recursive function f so that

$$\forall x (x \in A \leftrightarrow \exists x_1 \ldots \forall x_n \exists y_{y < f(x_n)} \exists u \in B \ \exists v \notin B \ R(x, x_1, \ldots, x_n, y, u, v)).$$

For any two sets A and B, we have shown, in Theorems 1 and 2, that $A \leq_1 B \leftrightarrow A \in \Sigma_1^B$ in a positive sense. Moreover, by Theorem I.2.8, Theorem 2, and Corollary 1, if $B \neq \emptyset$ and $B \neq \omega$, then $A \in \Sigma_1^B$ in a positive sense if and only if there exist f,g, recursive so that $\forall x(x \in A \leftrightarrow \exists y \forall z_{z<f(y)} g(x,y,z) \in B)$. Compare this with the following Corollary 3.

<u>Corollary</u> 3. If $B \neq \emptyset$ and $B \neq \omega$, then $A \in \Sigma_1^B$ if and only if there exist recursive functions f,g and h so that $\forall x(x \in A \leftrightarrow \exists y \forall z_{z<f(y)} (g(x,y,z) \in B \ \& \ h(y,z) \notin B))$.

3. <u>The Σ_1-reducibility</u> $\leq_{rm}$.

We consider in this section the effect of eliminating the bounded quantifier in the definition of $\mathfrak{I}_1$.

<u>Definition</u> 5. $A \leq_{rm} B \longleftrightarrow$ there exists a recursive function f so that $\forall x(x \in A \leftrightarrow \exists y f(x,y) \in B)$.

<u>Theorem</u> 4.

(1) $\leq_{rm}$ is a Σ_1-reducibility relation;

(2) $A \leq_m B \rightarrow A \leq_{rm} B \rightarrow A \leq_{\mathfrak{s}_1} B$;

(3) $(A \leq_{rm} \emptyset \rightarrow A = \emptyset) \ \& \ (A \leq_{rm} \omega \rightarrow A = \omega)$;

(4) $B \neq \emptyset \ \& \ B \neq \omega \rightarrow (A \in \Sigma_1 \rightarrow A \leq_{rm} B)$;

(5) $A \leq_{rm} B \ \& \ B \in \Sigma_1 \rightarrow A \in \Sigma_1$;

(6) $\leq_r \not\subseteq \leq_{rm}$.

<u>Proof</u>. The proofs follow immediately from the definition. We will present the proof of (4). Suppose $A \in \Sigma_1 \ \& \ A \neq \emptyset$. Let $a \in B$ and $b \notin B$. Define

$$f(x,y) = \begin{cases} a, & R(x,y) \\ b, & \overline{R}(x,y), \end{cases}$$

where $x \in A \leftrightarrow \exists y R(x,y)$. Then, $x \in A \leftrightarrow \exists y f(x,y) \in B$. Suppose $A = \emptyset$. Choose $b \notin B$. Define $f(x,y) = b$, all x and y. Then, $x \in A \leftrightarrow \exists y f(x,y) \in B$.

<u>Corollary</u> 4. $A \leq_{rm} B \not\rightarrow A \leq_m B$.

<u>Proof</u>. Let $A \in \Sigma_1$ so that A is not recursive. Then $A \leq_m B$ only if B is not recursive. Thus, (4) above is not true for $\leq_m$.

We show now that $A \leq_{\mathcal{g}_1} B \not\rightarrow A \leq_{rm} B$. Thus, the bounded quantifier in the definition of $\mathfrak{I}_1$, Theorem I.2.8 and Corollary 1 cannot be eliminated.

<u>Lemma</u> 2. Let $f(x) = x^2 + 1$ and $g(x) = (x+1)^2$. Then $\forall x \forall y (x>0 \ \& \ y>0 \rightarrow f(x) \neq g(y))$.

<u>Proof</u>. If $x^2 + 1 = (y+1)^2$, then $(y+1)^2 - x^2 = 1$. $(y+1+x)(y+1-x) = 1$. Thus, $y + 1 + x = -1$ and $y + 1 - x = 1$, or $y + 1 + x = 1$ and $y + 1 - x = 1$. Thus $y = -2$ and $x = 0$, or $x = y = 0$.

<u>Lemma</u> 3. There exist functions α and β so that:

(1) $\forall x(\alpha(x) = 0$ or $\alpha(x) = 1)$, $\forall x(\beta(x) = 0$ or $\beta(x) = 1)$;

(2) $\forall x(\alpha(x) = 0 \leftrightarrow \beta(x^2+1) = \beta((x+1)^2) = 0)$, $\forall x(\beta(x) = 0 \leftrightarrow \alpha(x^2+1) = \alpha((x+1)^2) = 0)$;

(3) there is no partial recursive function h so that $\alpha(x) = 0 \leftrightarrow \exists y \beta(h(x,y)) = 0$;

(4) there is no partial recursive function h so that $\beta(x) = 0 \leftrightarrow \exists y \alpha(h(x,y)) = 0$.

Proof. Let $f(x) = x^2 + 1$ and $g(x) = (x+1)^2$. For each natural number, define $C(x)$ inductively by:

(i) $x \in C(x)$;

(ii) $y \in C(x) \rightarrow f(y) \in C(x) \wedge g(y) \in C(x)$;

(iii) $C(x)$ is the smallest set satisfying clauses (i) and (ii).

We define functions α and β by induction. This construction differs from the constructions in [3] in that at stage $s+1$ not only are initial segments α_{s+1} and β_{s+1} defined, but, for each $x < \ell h(\alpha_{s+1})$ so that $(\alpha_{s+1})_x = 1$, and for each $x < \ell h(\beta_{s+1})$ so that $(\beta_{s+1})_x = 1$, α and β are defined on $C(x)$, so that (2) is satisfied, as follows: If $y \in C(x)$ and $\alpha(y) = 0$, then $\beta(f(y)) = \beta(g(y)) = 0$. If $y \in C(x)$ and $\beta(y) = 0$, then $\alpha(f(y)) = \alpha(g(y)) = 0$. Thus, at stage $s+1$, infinitely many values of α are defined.

Condition (3) is equivalent to the following (3'):

(3') $\forall e \exists x \{[\alpha(x) = 0 \ \& \ \forall y(\{e\}(x,y) \text{ defined} \rightarrow \beta(\{e\}(x,y)) = 1)]$ or $[\alpha(x) = 1 \ \& \ \exists y \beta(\{e\}(x,y)) = 0]\}$.

Condition (4) is equivalent to the following (4'):

(4') $\forall e \exists x \{[\beta(x) = 0 \ \& \ \forall y(\{e\}(x,y) \text{ defined} \rightarrow \alpha(\{e\}(x,y)) = 1)]$ or $[\beta(x) = 1 \ \& \ \exists y \alpha(\{e\}(x,y)) = 0]\}$.

Stage 0. Define $\alpha_0 = \beta_0 = 1$.

Stage $s + 1$. By induction hypothesis α_s and β_s are already defined. Also, the following conditions are satisfied:

(5) $\forall x[\alpha(x)$ defined & $\alpha(x) = 0 \rightarrow \beta(f(x))$ is defined and $\beta(g(x))$ is defined & $\beta(f(x)) = \beta(g(x)) = 0]$.

(6) $\forall x[\beta(x)$ defined & $\beta(x) = 0 \rightarrow \alpha(f(x))$ is defined and $\alpha(g(x))$ is defined and $\alpha(f(x)) = \alpha(g(x)) = 0]$.

(7) $\forall x[\alpha(f(x))$ defined & $\alpha(f(x)) = 0 \rightarrow [\beta(x)$ defined & $\alpha(g(x))$ defined & $(\beta(x) = 0 \leftrightarrow \alpha(g(x)) = 0)]]$.

(8) $\forall x[\alpha(g(x))$ defined & $\alpha(g(x)) = 0 \rightarrow [\beta(x)$ defined & $\alpha(f(x))$ defined & $(\beta(x) = 0 \leftrightarrow \alpha(f(x)) = 0)]]$.

(9) $\forall x[\beta(f(x))$ defined & $\beta(f(x)) = 0 \rightarrow [\alpha(x)$ defined & $\alpha(g(x))$ defined & $(\alpha(x) = 0 \leftrightarrow \alpha(g(x)) = 0)]]$.

(10) $\forall x[\beta(g(x))$ defined & $\beta(g(x)) = 0 \rightarrow [\alpha(x)$ defined & $\beta(f(x))$ defined & $(\alpha(x) = 0 \leftrightarrow \beta(f(x)) = 0]]$.

$s = 2e$. α_{2e+1} and β_{2e+1} shall be defined at this stage so that (3') is true at e for all extensions of α_{2e+1} and β_{2e+1}.

Case 1. $\exists x[(\alpha(x)$ has not been defined or $(\alpha(x)$ has been defined & $\alpha(x) = 0))$ & $\forall y[\{e\}(x,y)$ defined $\longrightarrow \beta(\{e\}(x,y)$ defined & $\beta(\{e\}(x,y)) = 1]]$.

Note. $\alpha(x)$ defined includes both the case $x < \ell h(\alpha_{2e})$ and $x \geq \ell h(\alpha_{2e})$ where $\alpha(x)$ is defined at some stage $\leq 2e$. $\alpha(x) = 0$ includes the case $(\alpha_{2e})_x = 1$.

Let a be the least x satisfying the hypothesis of case 1.

Suppose $\alpha(a)$ is already defined and $\alpha(a) = 0$. Then (3') is already satisfied at e. If $\alpha(\ell h(\alpha_{2e}))$ is already defined, then define

$$\alpha_{2e+1} = \alpha_{2e} p_{\ell h(\alpha_{2e})}^{\alpha(\ell h(\alpha_{2e}))+1}$$

and $\beta_{2e+1} = \beta_{2e}$. It is clear that α_{2e+1} and β_{2e+1} satisfy the induction hypotheses (5)-(10). If $\alpha(\ell h(\alpha_{2e}))$ is not defined, then define $\alpha_{2e+1} = \alpha_{2e} \cdot p_{\ell h(\alpha_{2e})}^{2}$ and $\beta_{2e+1} = \beta_{2e}$. α_{2e+1} and β_{2e+1} satisfy (5)-(10).

Suppose $\alpha(a)$ has not been defined. If $a \neq f(b)$ and $a \neq g(b)$, for any b, then define

$$\alpha_{2e+1} = \alpha_{2e} \cdot \prod_{\ell h(\alpha_{2e}) \leq x < a} p_x^{h(x)} \cdot p_a^1,$$

where $h(x) = \alpha(x) + 1$, if $\alpha(x)$ is already defined, and $h(x) = 2$, otherwise. Define $\beta_{2e+1} = \beta_{2e}$. Then, (3') is satisfied at e by α_{2e+1} and β_{2e+1}. α_{2e+1} is defined so that $\alpha(a) = 0$. Therefore, define values of α and β on $C(a)$ by the rules:

$y \in C(a)\ \&\ \alpha(y) = 0 \rightarrow \beta(f(y)) = \beta(g(y)) = 0$, and $y \in C(A)\ \&\ \beta(y) = 0 \rightarrow \alpha(f(y)) = \alpha(g(y)) = 0$. Then, α_{2e+1} and β_{2e+1} satisfy (5)-(10).

Suppose $\exists b\ a = g(b)$. By clause (6), $\beta(b)$ is not defined or, $\beta(b)$ is defined and $\beta(b) = 1$. (In fact, if the latter, then $b < \ell h(\beta_{2e})$.) Also, by (7), $\alpha(f(b))$ is not defined or, $\alpha(f(b))$ is defined and $\alpha(f(b)) = 1$. Define

$$\alpha_{2e+1} = \alpha_{2e} \cdot \prod_{\ell h(\alpha_{2e}) \leq x < a} p_x^{h(x)} \cdot p_a^1,$$

where $h(x) = \alpha(x) + 1$, if $\alpha(x)$ is already defined, and $h(x) = 2$, otherwise. (Then, in particular, $\alpha(f(b)) = 1$, since $f(b) < g(b)$.) If $\beta(b)$ is defined, define $\beta_{2e+1} = \beta_{2e}$; if not define

$$\beta_{2e+1} = \beta_{2e} \cdot \prod_{\ell h(\beta_{2e}) \leq x \leq b} p_x^{h(x)},$$

where $h(x) = \beta(x) + 1$, if $\beta(x)$ is already defined, and $h(x) = 2$, otherwise. Also, define values of α and β on $C(a)$ as described above. Then (3') is satisfied at e for α_{2e+1} and β_{2e+1}, and α_{2e+1} and β_{2e+1} satisfy (5)-(10).

Suppose $\alpha(a)$ is not defined and $\exists b\ a = f(b)$. By clause (6), $\beta(b)$ is not defined or, $b < \ell h(\beta_{2e})$ and $(\beta_{2e})_b = 2$. Also, by clause (8), $\alpha(g(b))$ is not defined or, $\alpha(g(b))$ is defined and $\alpha(g(b)) = 1$. Since $a = f(b) < g(b)$, $g(b)$ is not defined. Define

$$\alpha_{2e+1} = \alpha_{2e} \cdot \prod_{\ell h(\alpha_{2e}) \leq x < a} p_x^{h(x)} \cdot p_a^1 \cdot \prod_{a < x \leq g(b)} p_x^{h(x)},$$

where $h(x)$ is defined as before. If $b < \ell h(\beta_{2e})$, define $\beta_{2e+1} = \beta_{2e}$; otherwise define

$$\beta_{2e+1} = \beta_{2e} \cdot \prod_{\ell h(\beta_{2e}) \leq x \leq b} p_x^{h(x)},$$

where, again, $h(x)$ is defined as before. Define values α and β on $C(a)$ in the usual manner. Then, α_{2e+1} and β_{2e+1} satisfy (3') at (e), and satisfy (5)-(10).

Case 1 of stage $2e + 1$ is now complete.

<u>Case</u> 2. $\forall x[(\alpha(x)$ has not been defined or $\alpha(x) = 0) \rightarrow \exists y[\{e\}(x,y)$ defined & $(\beta(\{e\}(x,y))$ not defined or $\beta(\{e\}(x,y)) = 0)]]$.

Let a be the least x so that, for all y, $a \neq f(y)$ and $a \neq g(y)$, and so that $\alpha(a)$ is not yet defined. Let b be the least y satisfying the consequent of case 2 at $x = a$.

Suppose $\beta(\{e\}(a,b))$ is defined and $\beta(\{e\}(a,b)) = 0$. Then, define

$$\alpha_{2e+1} = \alpha_{2e} \cdot \prod_{\ell h(\alpha_{2e}) \leq x \leq a} p_x^{h(x)},$$

$h(x) = \alpha(x) + 1$, if $\alpha(x)$ defined, $h(x) = 2$, otherwise. Define $\beta_{2e+1} = \beta_{2e}$. Then, (5)-(10) hold, and (3') is satisfied at e for all extensions of α_{2e+1} and β_{2e+1}.

Suppose $\beta(\{e\}(a,b))$ is not defined. Also, suppose $\{e\}(a,b) \neq f(c)$ and $\{e\}(a,b) \neq g(c)$, for any c. First, define

$$\beta_{2e+1} = \beta_{2e} \cdot \prod_{\ell h(\beta_{2e}) \leq x < \{e\}(a,b)} p_x^{h(x)} \cdot p^1_{\{e\}(a,b)},$$

where $h(x) = \beta(x) + 1$, if $\beta(x)$ is already defined, and $\beta(x) = 2$, otherwise. Secondly, define

$$\alpha_{2e+1} = \alpha_{2e} \cdot \prod_{\ell h(\alpha_{2e}) \leq x \leq a} p_x^{h(x)},$$

where $h(x) = \alpha(x) + 1$, if $\alpha(x)$ is already defined, and $h(x) = 2$, otherwise. In particular, $\alpha(a) = 1$ and $\beta(\{e\}(a,b)) = 0$. Thus (3') is satisfied by α_{2e+1} and β_{2e+1} at e. Since $\beta(\{e\}(a,b))$ has been defined so that $\beta(\{e\}(a,b)) = 0$, define the necessary values of α and β on $C(\{e\}(a,b))$ as before. That is, $(y \in C(\{e\}(a,b)) \;\&\; \beta(y) = 0) \rightarrow \alpha(f(y)) = \alpha(g(y)) = 0$, and $(y \in C(\{e\}(a,b)) \;\&\; \alpha(y) = 0) \rightarrow \beta(f(y)) = \beta(g(y)) = 0$. Then (5)-(10) are satisfied also.

Suppose $\beta(\{e\}(a,b))$ is not defined and $\exists c \{e\}(a,b) = g(c)$. By clause (5), $\alpha(c)$ is not defined or, $\alpha(c)$ is defined and $\alpha(c) = 1$. Also, by (9), $\beta(f(c))$ is not defined or, $\beta(f(c))$ is defined and $\beta(f(c)) = 1$. Firstly, define

$$\beta_{2e+1} = \beta_{2e} \cdot \prod_{\ell h(\beta_{2e}) \leq x < \{e\}(a,b)} p_x^{h(x)} \cdot p_{\{e\}(a,b)}^{1},$$

where $h(x) = \beta(x) + 1$, if $\beta(x)$ is already defined, and $\beta(x) = 2$, otherwise. (Then, in particular, $\beta(f(c)) = 1$, since $f(c) < g(c)$.) Secondly, define values of α and β on $C(\{e\}(a,b))$ in the usual manner. Now we want to extend α_{2e} so that $\alpha(a)$ is defined, $\alpha(a) = 1$, $\alpha(c)$ is defined, and $\alpha(c) = 1$. $c < g(c) = \{e\}(a,b)$. Thus, $c \notin C(\{e\}(a,b))$. Hence $\alpha(c)$ is still undefined, or $\alpha(c)$ is defined and $\alpha(c) = 1$. a was chosen so that, for all x, $a \neq f(x)$ and $a \neq g(x)$. Thus $\alpha(a)$ is still undefined. Define

$$\alpha_{2e+1} = \alpha_{2e} \cdot \prod_{\ell h(\alpha_{2e}) \leq x \leq \max\{a,b\}} p_x^{h(x)},$$

where $h(x) = \alpha(x) + 1$, if $\alpha(x)$ is defined, and $\alpha(x) = 2$, otherwise. $\alpha(a) = 1$ and $\beta(\{e\}(a,b)) = 0$, thus (3') is satisfied at e. Also (5)-(10) are satisfied by this α_{2e+1} and β_{2e+1}. (The only important clause in this case is (9), which still holds, since $\beta(g(c)) = 0$, but $\beta(f(c)) = \alpha(e) = 1$.)

Finally, suppose $\beta(\{e\}(a,b))$ is not defined and $\exists c\{e\}(a,b) = f(c)$. By clause (5), $\alpha(c)$ is not defined or, $\alpha(c)$ is defined and $\alpha(c) = 1$. Also, by (10), $\beta(g(c))$ is not defined or, $\beta(g(c))$ is defined and $\beta(g(c)) = 1$. Since $f(c)$ is not defined and $f(c) < g(c)$, $g(c)$ is not defined. Firstly, define

$$\beta_{2e+1} = \beta_{2e} \cdot \prod_{\ell h(\beta_{2e}) \leq x < \{e\}(a,b)} p_x^{h(x)} \cdot p_{\{e\}(a,b)}^{1} \cdot \prod_{\{e\}(a,b) < x \leq g(c)} p_x^{h(x)},$$

where $h(x)$ is defined as before. In particular β_{2e+1} is defined so that $\beta(\{e\}(a,b)) = \beta(f(c)) = 0$ and $\beta(g(c)) = 1$. Secondly, define the necessary values of α and β on $C(\{e\}(a,b))$. Now we want to extend α_{2e} so that $\alpha(a)$ is defined, $\alpha(a) = 1$, $\alpha(c)$ is defined, and $\alpha(c) = 1$. Proceed exactly as in the previous paragraph. Then α_{2e+1} and β_{2e+1} are obtained so that (3') at e and (5)-(10) are satisfied.

Case 2 of stage $2e + 1$ is now complete.

<u>$s = 2e + 1$</u>. α_{2e+2} and β_{2e+2} shall be defined at this stage so that (4') is true at e for all extensions of α_{2e+2} and β_{2e+2}. Stage $2e + 2$ is the same <u>mutatis</u> <u>mutandis</u> as stage $2e + 1$.

Define α and β by $\alpha(x) = (\alpha_{\mu s[x<\ell h(\alpha_s)]})_x \dot{-} 1$, and $\beta(x) = (\beta_{\mu a[x<\ell h(\beta_a)]})_x \dot{-} 1$.

Clearly, α and β satisfy (3') and (4') and therefore (3) and (4). By induction clauses (5) and (6), $\alpha(x) = 0 \rightarrow \beta(x^2+1) = \beta((x+1)^2) = 0$, and $\beta(x) = 0 \rightarrow \alpha(x^2+1) = \alpha((x+1)^2) = 0$. By clauses (7)-(10), the converses are also true. Thus α and β satisfy clause (2).

The proof of Lemma 3 is complete.

<u>Theorem</u> 5. There exist sets A and B so that $A \leq_r B$, $A \leq_{\mathcal{S}_1} B$, and $A \not\leq_{rm} B$. In fact, the $\mathcal{S}_1$-degrees of A and B are identical and the rm-degrees of A and B are incomparable.

<u>Proof</u>. Apply Lemma 3 to obtain functions α and β. Let $A = \{x|\alpha(x) = 0\}$ and $B = \{x|\beta(x) = 0\}$. Then, there exist recursive functions f and g so that $\forall x(x \in A \leftrightarrow f(x) \in B \,\&\, g(x) \in B)$, and $\forall x(x \in B \leftrightarrow f(x) \in A \,\&\, g(x) \in A)$. Thus $A \leq_r B$. (Also, $B \leq_r A$.) By the definition of $\mathcal{S}_1$, $A \leq_{\mathcal{S}_1} B$ and $B \leq_{\mathcal{S}_1} A$. On the other hand, by Lemma 3, $A \not\leq_{rm} B$ and $B \not\leq_{rm} A$.

It is also interesting to notice that for two sets A and B, the existence of recursive functions f and g so that $\forall x(x \in A \leftrightarrow f(x) \in B \,\&\, g(x) \in B)$ does not imply $A \leq_m B$.

By Theorem I.2.2 (10), $A \leq_r B$ does not imply $A \leq_{\mathcal{S}_1} B$. Also, by Theorem 4 (6), $A \leq_r B$ does not imply $A \leq_{rm} B$. Theorem 5 gives an example of sets A and B so that $\underset{\sim}{d}(A) = \underset{\sim}{d}(B)$, $\underset{\sim}{d}_{\mathcal{S}_1}(A) = \underset{\sim}{d}_{\mathcal{S}_1}(B)$, and $\underset{\sim}{d}_{rm}(A) | \underset{\sim}{d}_{rm}(B)$. Is there a set A so that $\underset{\sim}{d}_{\mathcal{S}_1}(A) = \underset{\sim}{d}_{\mathcal{S}_1}(\overline{A})$ and $\underset{\sim}{d}_{rm}(A) | \underset{\sim}{d}_{rm}(\overline{A})$? This question is open. Notice that by the following argument Lemma 3 cannot be used to obtain such a set A. Suppose there exist recursive functions f and g so that $x \in A \leftrightarrow f(x) \notin A$ & $g(x) \notin A$ and $x \in \overline{A} \leftrightarrow f(x) \in A$ & $g(x) \in A$. Then, $x \in A \rightarrow f(x) \notin A$. Also $f(x) \notin A \rightarrow x \in A$, because $x \notin A \rightarrow f(x) \in A$. Thus $A \leq_m B$, which implies $A \leq_{rm} B$.

However, we have already established (Theorem I.2.9) the weaker result that there exists a set A so that A and $\overline{A}$ are $\mathcal{S}_1$-incomparable, from which it follows that A and $\overline{A}$ are also rm-incomparable.

4. <u>The Reducibility</u> $\mathcal{S}_1 \cap \mathcal{P}_1$.

We consider in this final section the reducibility relation $\mathcal{S}_1 \cap \mathcal{P}_1$. This reducibility is of some interest since it is easily defined and, as the next theorem shows, is between many-one reducibility and relative recursiveness.

<u>Theorem</u> 6.

(1) $\mathcal{S}_1 \cap \mathcal{P}_1 \subsetneq \{(A,B) | A \leq_r B\}$.

(2) $\{(A,B) | A \leq_m B\} \subsetneq \mathcal{S}_1 \cap \mathcal{P}_1$.

Proof.

(1) Follows from Theorems I.2.5, I.2.6, and I.2.2(2).

(2) Clearly $A \leq_m B \rightarrow A \leq_{\mathcal{S}_1} B \mathbin{\&} A \leq_{\mathcal{P}_1} B$. Lemma 3 and Theorem 5 give us sets A and B so that $x \in A \leftrightarrow f(x) \in B \mathbin{\&} g(x) \in B$. Thus $A \leq_{\mathcal{S}_1} B$ and $A \leq_{\mathcal{P}_1} B$. On the other hand, A and B are constructed so that $A \not\leq_m B$.

Let R^X denote a number theoretic predicate recursive uniformly in X, where X is a set variable. By a theorem of Nerode [1, Theorem 11], A is truth-table reducible to B ($A \leq_{tt} B$) if and only if there exists such an R^X so that $\forall x(x \in A \leftrightarrow R^B(x))$. $\leq_m$ and $\leq_1$ can be expressed in this form. $A \leq_m B$ if and only if $\forall x(x \in A \leftrightarrow f(x) \in B)$ for some recursive function f, and $A \leq_1 B$ if and only if $\forall x(x \in A \leftrightarrow f(x) \in B)$ for some one-one recursive function f. In either case, $f(x) \in X$ is such an R^X. We will say that a subrecursive reducibility $\mathfrak{R}$ is defined by predicates R^X if for all A and B, $A \mathfrak{R} B$ is and only if there exists R^X so that $\forall x(x \in A \leftrightarrow R^B(x))$ and $\forall C, D[\forall x(x \in C \leftrightarrow R^D(x)) \rightarrow C \mathfrak{R} D]$.

Lemma 4. $\exists A[A \leq_{\mathcal{S}_1} \overline{A} \mathbin{\&} A \not\leq_{\mathcal{P}_1} \overline{A}]$.

Proof. Choose $A \in \Sigma_1$ so that $A \notin \Pi_1$. Then $A \leq_{\mathcal{S}_1} B$, all B. Thus, $A \leq_{\mathcal{S}_1} \overline{A}$. $\overline{A} \in \Pi_1$, so $A \leq_{\mathcal{P}_1} \overline{A} \rightarrow A \in \Pi_1$. Thus, $A \not\leq_{\mathcal{P}_1} \overline{A}$.

Theorem 7. $A \leq_{tt} B$ does not imply $A \leq_{\mathcal{S}_1 \cap \mathcal{P}_1} B$.

Proof. The proof follows from Lemma 4 since $A \leq_{tt} \overline{A}$ for all A.

Theorem 8. $A \leq_{\mathcal{S}_1 \cap \mathcal{P}_1} B$ does not imply $A \leq_{tt} B$.

Proof. There exist recursively enumerable sets A and B so that $\underset{\sim}{d}(A) = \underset{\sim}{d}(B)$ and $A \not\leq_{tt} B$ (see [2, §9.6]). $A \in \Sigma_1$, hence $A \leq_{\mathcal{S}_1} B$. Since $B \in \Sigma_1$, $B \in \Pi_1^X \rightarrow B \leq_r X$. $B \leq_r X \rightarrow A \leq_r X \rightarrow A \in \Pi_1^X$. Thus, $A \leq_{\mathcal{P}_1} B$. Therefore $A \leq_{\mathcal{S}_1 \cap \mathcal{P}_1} B$.

Definition 6. $\mathfrak{S}_1 = \{R^X(x) \mid R^X(x)$ is uniformly recursive in X and $\forall B \forall C [B \in \Sigma_1^C \rightarrow R^B \in \Sigma_1^C]\}$. $\mathfrak{S}_2 = \{R^X(x,y) \mid R^X(x,y)$ is uniformly recursive in X and $\forall B \forall C [B \in \Sigma_1^C \rightarrow R^B \in \Sigma_1^C\}$.

Theorem 9. Suppose $B \neq \emptyset$ and $B \neq \omega$. Then $A \leq_{\mathcal{S}_1} B \leftrightarrow$ there exists $R^X(x,y) \in \mathfrak{S}_2$ so that $\forall x(x \in A \rightarrow \exists y R^B(x,y))$.

Proof. It is immediate from the definition of $\mathcal{S}_1$ that the right hand side implies the left hand side.

Suppose $A \leq_{\mathcal{S}_1} B$. By Theorem I.2.8, there are recursive functions f and g so that $\forall x(x \in A \leftrightarrow \exists y \forall z_{z<f(y)} g(x,y,z) \in B)$. Define $R^X(x,y) \equiv \forall z_{z<f(y)} g(x,y,z) \in X$, $R^X \in \mathfrak{S}_2$. This completes the proof.

Theorem 10. $\mathfrak{S}_1 = \{R^X(x,x) \mid R^X(x,y) \in \mathfrak{S}_2\}$.

Proof. Obviously, $R^X(x,y) \in \mathfrak{S}_2$ implies $R^X(x,x) \in \mathfrak{S}_1$. If $R^X(x) \in \mathfrak{S}_1$, define $R^X(x,y) \equiv R^X(x)$, then $R^X(x,y) \in \mathfrak{S}_2$ and $R^X(x) \equiv R^X(x,x)$.

<u>Open Questions</u>.

1. By Theorem 8, $\mathcal{S}_1 \cap \mathcal{P}_1$ is not defined by predicates R^X uniformly recursive in X. If $\forall x(x \in A \leftrightarrow R^B(x))$ and $R^X \in \mathcal{O}_1$, is $A \leq_{\mathcal{S}_1 \cap \mathcal{P}_1} B$? By definition of $\mathcal{S}_1$, $\forall x(x \in A \leftrightarrow R^B(x))$ and $R^X \in \mathcal{O}_1$ implies $A \leq_{\mathcal{S}_1} B$, therefore it is sufficient to show $A \leq_{\mathcal{P}_1} B$.

2. Is $\mathcal{S}_1 \cap \mathcal{P}_1$ a maximal proper Σ_0-reducibility?

<u>Remark</u>. $\mathcal{S}_2 \cap \mathcal{P}_2 \not\subseteq \{(A,B) \mid A \leq_r B\}$. Choose A and B so that $A \not\leq_r B$ but so that for some recursive R, $\forall x(x \in A \leftrightarrow x^2 \in B \;\&\; \forall z\, R(x,z)))$.

Bibliography

[1] Nerode, Anil, "General Toplogy and Partial Recursive Functionals", Summaries of Talks presented at the Summer Institute for Symbolic Logic, Cornell University, 1957, 247-251.

[2] Rogers, Hartley, Jr., _Theory of Recursive Functions and Effective Computability_, McGraw-Hill, New York, 1967.

[3] Selman, Alan L., "Arithmetical Reducibilities, I", Department of Mathematics, Carnegie-Mellon University RR 71-6, 1971.