A NONLINEAR SPECTRAL THEOREM FOR
ABSTRACT NEMITSKY OPERATORS

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§1. In this paper, the concepts and methods of linear spectral theory on Hilbert space are adapted to the analysis of a class of nonlinear operators. The goal is a representation of such an operator cf> by a (strongly convergent) integral

(1) *(u) = |AE(dA)u;

the domain of integration A should admit a natural interpretation as the spectrum of <&, the values of the 'spectral measure' E are to be idempotent operators, and, when <? is linear, (1) should of course yield the conclusion of the classical spectral theorem. At this stage of my investigation, the class of operators under discussion is small, but also quite basic: it consists of abstract analogues of the Nemitsky operators

(2) $: u(x)—* cp(u(x) ,x) ,

which make up the nonlinear portion of many nonlinear integral equations [l]o When the underlying space X is a subset of E^n, it is possible to give smoothness and growth-conditions on the function cp which guarantee that the operator $ be continuous, bounded, etc., on $L^2(X)$([l,2])o Regularity questions aside, the crucial property from our more abstract point of view is that the value of (<J>u)(x) depends on u only through the value of u(x); this means that enough projections (characteristic functions) in the Hilbert space commute with <£ to allow a decomposition of the space. More precisely, we make the

Definition 1: Let H be. a. real Hilbert space. An operator $ from H to H JLs called an abstract Nemitsky operator provided that

(i) $ is continous and bounded (takes bounded sets into bounded sets);

(ii) there exists a maximal abelian algebra G of bounded, linear symmetric operators on H (abbreviated: m.a.a.s.), such that for every projection *P&G,*

(3) $$P\Phi = \Phi P;$$

(iii) for every ueH, the function of A defined by.

$$*_{U}(A) = *(Au)$$

is uniformly continuous on every set G' *a* G which is bounded in the operator norm.

In addition, we shall always assume that $ has been normalized to make <J>(0) = 0.

The basic properties of these operators are discussed in §2; in particular, the somewhat surprising condition (iii) will be motivated. It is proved in Theorem 1 that, corresponding to every Nemitsky operator <£, H can be realized concretely as a space $L^2(x)$ on which * acts as a functional operator (2). Condition (3iii) ensures that the generating function cp(c,x) will be continuous in c for a.e. x ; this restriction on cp is a standard one ([1,2]). The discovery of the rather delicate connection between the continuity of cp(*,x) and the uniformity property (3iii) was made by V. J. Mizel in his study of additive functionals [3]; the arguments presented in his paper are crucial to the proof of Theorem l_o

In view of the active research on additive functionals, as evidenced by [3] and the referencescited there, we prove (Theorem 2) an abstract Hilbert-space version of Mizel's basic representation theorem. Namely, if a real-valued function 0 on H satisfies additivity and uniformity properties analogous to those of Def. 1, then 8 has a concrete representation on some $L^2(x)$,

(4) 9 : u->J 0(u(x) ,x)dn, $ueL^2(X)$.
X

If the Nemitsky operator <$> is the gradient of a function 0 on H, then 9 will be shown to satisfy the hypotheses of Theorem *2,* so that both <£ and its potential can be represented in the standard forms (2), (4). If $ is not only a gradient, but also Gateaux differentiable, and if the function u—><J>^{r}(u) is bounded from H to the space 8(H) of continuous linear operators on H, a necessary and sufficient condition for <£ to be an abstract Nemitsky operator is that the derivatives be mutually commuting: $'(u)$'(v) = *'(v) *• (u) , for all u,veH (Theorem 3).

It is interesting to note that the integral representation (1) can be obtained even when <3? is not a gradient (concretely, this is generally the case for vector-valued Nemitsky operators [2]); since a linear operator is a gradient precisely when it is symmetric, it appears that the availability of the algebra G is more important to our spectral theorem than the notion of symmetry attached to gradients. (As stated before, the majority of gradient operators is still excluded).

Turning now to the representation (1), we shall motivate it by a formal discussion of the concrete case, and defer precise statements to §3» Thus, let (2) induce a suitably regular operator on $L^2(X, d\i)$, and define:

A = essential range of q̃>(c,x) = cp(c,x)/c as function from !R X X to E ;

A' (x) = essential range of c̃p(«,x) as a function on IR ;

A(x) = A fl A^1 (x) U {0} ; M_A = (u_6L^2(X); $(u) = au some measurable function a, with a(x) < A and a(x)∊Λ(x) for a.e. x], for AeIR » When '4? is linear, cp(c,x) = *l(x)c*, then A is precisely the usual spectrum of <£, and M_Λ consists of those functions in L^2 which vanish wherever *t* (x) J> A, so that M_Λ is the range of the spectral projection E((-oo,A)) associated with <I>. In a limiting sense, then, we may think of the range of E(dA) in (1) as consisting of those u(x) for which $(u) = Au, and of (1) as a superposition of such eigenfunctions.

In §3, these ideas are reformulated in the abstract setting, without (explicit) intervention of φ; hopefully, they will thereby become relevant in the study of more general operators. At this point, we only remark that the appropriate description of the 'spectrum' Λ is not simply an analogue of the usual definition given in the linear case (although the result is then the same); in particular, the set of λ's for which $(\lambda I-\varphi)^{-1}$ fails to exist as a differentiable operator on H (the spectrum according to Neuberger [4]) does not necessarily coincide with Λ. For example, when $H = \mathbb{R}$, and $\varphi(c) = c^3 + c^2$, then $\Lambda = [-\frac{1}{4}, \infty)$, but the inverse of $-\frac{1}{3} - \varphi$ does not have a finite derivative everywhere. It might be quite useful to have a description of Λ in terms of properties of the resolvent.

§2. In this section, we develop some implications of Definition 1. Throughout, (X, S, μ) is a finite measure space, $L^2(X)$ consists of the real, square-integrable functions on X, and H is a real, separable Hilbert space with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$.

<u>Definition 2</u>: A real-valued function φ on $\mathbb{R} \times X$ is a <u>Caratheodory</u> function if

(i) $\varphi(\cdot,x)$ is continous on $\mathbb{R}$ for a.e. x;

(ii) $\varphi(c,\cdot)$ is measurable on X for every $c \in \mathbb{R}$.

We always assume that $\varphi(0,x) \equiv 0$.

Definition 3: Let φ be a Caratheodory function. If the operator Φ defined by $(\Phi u)(x) = \varphi(u(x),x)$ maps $L^2(X)$ into itself, and is continuous and bounded, then it is called a (concrete) Nemitsky operator. We shall also write: $\Phi u = \varphi \circ u$,

Krasnosel'skii [1] shows that when $X \subset \mathbb{R}^n$, the continuity and boundedness of Φ are automatic as soon as it is known that $\Phi : L^2 \to L^2$. This property, in turn, is equivalent to the estimate $|\varphi(c,x)| \leq a(x) + b|c|$, $a \in L^2(X)$, $b \geq 0$. Another, rather subtle, characteristic of Φ was identified by Mizel [3]:

Lemma 1: *Let Φ be a concrete Nemitsky operator. Then Φ, considered as a map from $L^\infty(X)$ to $L^2(X)$, is uniformly continuous on bounded subsets of $L^\infty(X)$.* The proof requires only minor modifications of the arguments given in [3, p.454-5], and will be omitted.

Corollary 1: *For any $u \in L^2(X)$, the map $a \to \Phi(au)$ from $L^\infty(X)$ to $L^2(X)$ is uniformly continuous on bounded subsets of $L^\infty(X)$.*

Proof: Fix $M > 0$, $\varepsilon > 0$. Since Φ is continuous, there is a $\delta > 0$ such that $\|\Phi(v)\| < \varepsilon/3$ whenever $\|v\| < \delta$. Now put $E_N = \{x : |u(x)| \leq N\}$, and choose N so large that $\|\chi_{X-E_N} u\| < \delta/M$. Then for any $a \in L^\infty(X)$ with $\|a\|_\infty \leq M$, $\|a\chi_{X-E_N} u\| < \delta$, and so

(*) $$\|\Phi(a\chi_{Y-F_N} u) - \Phi(b\chi_{Y-F_N} u)\| < 2\varepsilon/3$$

whenever $\|a\|_\infty, \|b\|_\infty \le M$. On the other hand, $u\chi_{E_N}$ is bounded, so by Lemma 1 there is $\delta_1 > 0$ such that

(**) $$\|\Phi(a\chi_{E_N} u) - \Phi(b\chi_{E_N} u)\| < \varepsilon/3$$

when $\|a-b\|_\infty < \delta_1$, and $\|a\|_\infty, \|b\|_\infty \le M$. (*) and (**) together with additivity of Φ establish the uniform continuity, q.e.d.

Comparison of Corollary 1 and Definition 1 explains the significance of property (3iii); (3iii) will necessarily be satisfied if an abstract Nemitsky operator can be realized by means of a Caratheodory function. We do not know, incidentally, whether or not (3iii) is already a consequence of (3i) and (3ii).

Theorem 1: Let $ be a Nemitsky operator on H. Then there exist a measure space (X,B,|j) , *an* isometric isomorphism K from H onto $L^2(X)$, and a concrete Nemitsky operator $\hat{\$}$ on $L^2(X)$, such that

K<f>(u) = 8(Ku), for all ueHc

Proof: The existence of (X,R,fi) and of the isometry K is a consequence of the structure theorem for G ([5]). Moreover, G can be identified with the algebra L°° (X) of multiplications on $L^2(X)$: for every AeG, there is a unique aeL°°(X) , such that KAK"¹ $\hat{U}$ = aC, $\hat{u}$eL²(X); this map is onto L°° (X) , and $||A||_H = ||a||_{\infty}$. Under the isomorphism, $\hat{\&}$ induces an operator <b on $L^2(X)$. We show that $ is a Nemitsky operator.

The regularity properties of $ assume the following form for <&:

(i) ' $ is continuous and bounded;

(ii) ' for every measurable *E cz X,* and any $\hat{u}$e$L^2(X)$,

$$\hat{\Phi}(\chi_E \hat{u}) = \chi_E \hat{\Phi}(\hat{u}) ;$$

(iii)' Φ, as map from $L^{2}(X)$ to $L^{2}(X)$, is uniformly-continuous on bounded subsets of $L^{\infty}(X)$.

(Actually, (3iii) implies that Φ satisfies the conclusion of Corollary 1, and (iii)' in turn follows from this stronger assertion),

Next, for rational c, define $\bar{c} \in L^{2}(X)$ by $\bar{c}(x) = c$, and put

$$\varphi_{c}(x) = \Phi(\bar{c})(x).$$

$\varphi_{c}(x)$ will be extended to a Caratheodory function by use of the following lemma:

<u>Given any</u> $\eta > 0$, <u>there is a measurable set</u> S_{η} <u>such that</u>

(A) $\mu(T - S_{\eta}) < \eta$,

(B) <u>for each pair of numbers</u> $M, \epsilon > 0$, <u>there is a</u> $\delta = \delta(\epsilon, M) > 0$ <u>such that for rational</u> h, h', <u>we have</u>

$$h, h' \in [-M, M] \text{ and } |h - h'| < \delta \Rightarrow \sup_{x \in S_{\eta}} |\varphi_{h}(x) - \varphi_{h'}(x)| \leq \epsilon .$$

This lemma is due to Mizel [3, pp.458-9]; his proof, which applies almost verbatim to the situation considered here, relies heavily on properties (i)'-(iii)' of Φ. Now select a sequence $\eta_{m} \to 0$, put $S = \bigcup_{m=1}^{\infty} S_{\eta_m}$, and define

$$\varphi(c, x) = \begin{cases} \lim\limits_{\substack{h \to c \\ (h \text{ rational})}} \varphi_{h}(x) & \text{for } x \in S \\ 0 & \text{for } x \in X - S \end{cases}$$

It follows from Mizel's lemma that φ is a Caratheodory function. Furthermore, $\hat{\Phi}(s) = \varphi \circ s$ whenever s is a simple function with rational values. If $u \in L^2(X)$ and $u \geq 0$, then u is the pointwise and norm limit of such simple functions, $s_n \to u$ a.e. and in $L^2(X)$. By continuity of φ,

(5) $$\varphi \circ s_n \to \varphi \circ u \quad \text{a.e.}$$

In addition, the sequence of integrals

$$\int |\varphi \circ s_n(x)|^2 \, d\mu$$

is uniformly absolutely continuous ([3, p.452]), so that by Vitali's theorem [6, p.134], $\varphi \circ u$ is in $L^2(X)$, and

(6) $$\|\varphi \circ s_n\|_{L^2} \to \|\varphi \circ u\|_{L^2} .$$

Because the measure space is finite, (5) and (6) combined yield

$$\|\varphi \circ s_n - \varphi \circ u\|_{L^2} \to 0 .$$

On the other hand, $\varphi \circ s_n = \hat{\Phi}(s_n)$, and

$$\|\hat{\Phi}(s_n) - \hat{\Phi}(u)\|_{L^2} \to 0 .$$

Hence $\hat{\Phi}(u) = \varphi \circ u$ a.e. . The extension to arbitrary $u \in L^2(X)$ is immediate. Thus $\hat{\Phi}$ is a Nemitsky operator, q.e.d.

A very similar theorem holds for additive functionals on H:

<u>Theorem 2;</u> <u>Let</u> θ <u>be a real-valued functional on</u> H, <u>with</u> $\theta(0) = 0$, <u>and suppose that</u>

i) 8 JLS, continuous,

ii) there exists a m.a.a.s. C such that whenever A,BeG, and AB = 0, then 8 (Au+Bu) = 8(Au) + 8(Bu),ueH.

iii) for every ueH, the function $9_U(A)$ = S(Au) is uniformly continuous on bounded subsets of G.

Then H may be realized as an L^2 *(X,B,\i),* and 8 as a ~~functional~~ 9 ~~oji~~ $\hat{L}^2$ (X) ~~of the form~~

$$(8) \qquad \hat{9}(\hat{u}) = \int_X *(\hat{u}(x),x)d|i, \quad \hat{u}\in L^2(X)$$

where vf is a Caratheodory function on I x X.

Proof; The representation space is obtained just as in Theorem 1. The functional $\hat{8}$ induced by 9 on L^2(X) has the following properties: it is continuous, additive on functions of disjoint support, and uniformly continuous on bounded subsets of L°° (X). Mizel's theorem 2 in [3] characterizes precisely such functionals on L^-spaces, and in the special case p = *2,* assures the existence of the representation (8), q.e.d»

Suppose now that a Nemitsky operator * on H is the gradient of a functional 9,

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}[S(u+\epsilon v) - 0(u)] = (\Phi(u),v), \quad u,v\in H.$$

Then <I>, of course, has a concrete representation $\hat{<I>}$, and on the basis of the known properties of $<\hat{£}$ ([2]), it is to be expected that the potential 8 will be given by an expression of the form (8) on the same representation space. That this is indeed the case follows from Theorem 2 and

Lemma 2; Let the Nemitsky operator $ be the gradient of 8. Then 0 satisfies the hypotheses of Theorem 2 (the algebra G needed there coincides with that associated with <f>) .

Proof: The additivity property (7ii) of 0 is a consequence of the basic formula

$$e(u) = \int_0^1 (\$(tu),u)dt$$

(see [2]). Indeed^ if A and B are projections, then (7ii) follows immediately from (3ii). The general case may be reduced to this, since whenever AB = 0, there are projections P,QeG such that PA = A, QB = B, and PQ = 0.

Now let ueH be fixed, and let K > 0. Since <£ is bounded, there exists a constant K' such that

(9) ||*(Au)|| ≤; K',

whenever ||A|| £ K. Combining (9) with Lagrange's formula ([2]), we find

|0(Au) - 9(Bu)| = |9(Bu+ [Au-Bu]) - 8(Bu)| £

£ ||*(Bu + T[Au-Bu])|| • j|Au-Bu|j, 0 < T < 1

£K'||u||||A-B||,

which implies (7iii). q.e.d.

Finally, under strong smoothness assumptions, gradient Nemitsky operators can be characterized more intrinsically.

Theorem 3; Let $ be a continuous gradient operator on H, and suppose that the Gateaux derivative $'(u) exists at each ueHo Denote by G^ the family f<$>' (u) ;ueH}. Then:

(A) <I> has property (3ii) if, and only if, CI_o is abelian;

(B) if the map u -> $' (u) from H to G is bounded, and G_o jL§ abelian, then $ jLs ja Nemitsky operator.

Remark: If $ is a concrete Nemitsky operator given by the function cp(c,x), then, formally, $'(u) is multiplication by >I^(u(x),x), where ^ = ^ cp. This relationship is usually proved ([2]) under the assumption that \I> is itself a Caratheodory function; from this one can deduce, for X $c3R^n$, that the family G_o must be uniformly bounded.

Proof: Recall that the $' (u) , as derivatives of a gradient, are necessarily symmetric,, Thus, if G_o is abelian, it is contained in a m_oa.a.s. G. Let PeG be a projection Q = I - P, and let ueH be $fixed_o$ if we put $(v) = <£(Pv + Qu) , then *' (v) = cj?'(Pv + Qu)Q = Q<£>' (Pv + Qu) . Hence 0 = P^1 (v) = = (PΨ) ' (v) ; in particular, P^(0) = P>£(u), or

(10) P$P(u) = P$(u) .

Next, differentiating the operators <£>(Pu) and P$(Pu) with respect to u, we find that the derivatives are identical, and since $(0) = 0,

(11) *(Pu) * P$(Pu) „

(10 and (11)together show that $ commutes with projections.

The uniform continuity condition (3iii), is obtained just as in Theorem 2, and assertion (B) follows. The converse of (A) is an easy consequence of the definition of $'. q.e.d.

§3. In the last section, we discussed the analogue for Nemitsky operators of the linear spectral representation theorem (which states that a symmetric operator may be realized as a multiplication operator on an L^2-space); now we turn to the spectral <u>integral</u> for $. First, we translate the definitions made in §1 into Hilbert space language.

<u>Definition 2</u>; The <u>operator spectrum</u> of <£ is the set

<£ = {AeG ; Au = <£(u) for some ueH} .

<u>Definition 3</u>; The <u>spectrum</u> of Φ is the set

A= [A; Aesp A for some Ae£}.

(By 'sp A' we denote the usual spectrum of the symmetric operator A).

<u>Definition 4</u>; For -coev < \i < +co, put $£_{\nu\mu}$ = {AeX;v <£ sp A < \x).

<u>Definition 5</u>; Denote by $M_{\nu\mu}$ -oo < v < \x < oo, the set of ueH for which there is an $Ae£_{\nu\mu}$ satisfying <£(u) = Au, and

(12) if PeG is a projection, and Pu = 0, then PA = 0 „

<u>Remark</u>: 1. In §1, we introduced the set $M_{\wedge}$ in the concrete case; its relation to the above definition is just: M^ = $M_{\wedge}$ - M_{v}.

2. Without additional conditions - such as raono-tonicity - which make the behavior of 0 more predictable,

the $M_{\nu\mu}$ are likely to be very complicated sets. For example, if $H = \mathbb{R}$ and $\Phi(c) = \sin c$, then $M_{\nu\mu} = \{c; \nu c \le \sin c \le \mu c\}$; i.e., $M_{\nu\mu}$ is a collection of intervals, and as ν and μ vary, these intervals may merge or separate.

3. In the function representation, (12) states that support $\hat{u}$ = support $\hat{A}$. These definitions are not vacuous:

<u>Lemma 3</u>: i) If $\nu_1 \le \nu < \mu \le \mu_1$, then $M_{\nu\mu} \subset M_{\nu_1\mu_1}$;

ii) $\lim_{\mu\to-\infty} [\lim_{\nu\to-\infty} M_{\nu\mu}] = \{0\}$;

iii) $\lim_{\substack{\nu\to-\infty \\ \mu\to+\infty}} M_{\nu\mu}$ is dense in H.

<u>Proof</u>: i) follows immediately from the definition. The remainder of the proof is most efficiently carried out in that L^2-representation of H in which G corresponds to multiplications by the L^∞-functions; we denote the images of $u \in H$, $A \in G$, by $\hat{u}(x)$ and $\hat{A}(x)$, respectively.

Fix $u \in H$, and put $\Phi(u) = v$. Define $E_n = \{x; |\hat{u}(x)| \le n, |\hat{v}(x)| \le n\}$, and put $\hat{P}_n = \chi_{E_n}$. P_n is then a projection in G, and with the notation $u_n = P_n u$, $v_n = P_n v$, (3ii) implies

$$\Phi(u_n) = v_n.$$

Moreover, again by (3ii), support $\hat{v}_n \subset$ support $\hat{u}_n$. Thus, we define $\hat{A}_n = \hat{P}_n \cdot \hat{v}_n/\hat{u}_n \in L^\infty$, then sp A_n is bounded, so that $A_n \in \mathcal{E}_{\nu_n\mu_n}$ for some $-\infty < \nu_n < \mu_n < +\infty$, and

$$Pu_n = 0 \Rightarrow PA_n = 0, \text{ for projections } P \in G.$$

Consequently, $u_n \in M_{\nu_n \mu_n}$, and as $n \to \infty$, $u_n \to u$. This proves (iii). Assertion (ii) follows from the fact that to any nonzero $u \in M_{\nu\mu}$, $-\infty < \nu < \mu < +\infty$, there corresponds precisely one $A \in C$ with the properties demanded by Definition 5. Indeed, suppose that $\Phi(u) = Au = A'u$; then (12) implies that

$$\text{(13)} \qquad \text{support } \hat{A} = \text{support } \hat{A'} = \text{support } \widehat{\Phi(u)}.$$

and so $Au = A'u$ means $\hat{A} = \hat{A'}$ a.e., or $A = A'$, q.e.d.

<u>Remark</u>; Observe that only property (3ii) of Φ was needed in the proof. While continuity of Φ <u>will</u> play a role later, the uniform continuity condition (3iii) is unnecessary, and the spectral theorem (1) will be obtained for operators which may not have a spectral representation by a Caratheodory function.

Next, we introduce the 'spectral projections' E_μ which enter into (1). Until further notice, $u \in H$ is a fixed element.

<u>Definition 6</u>; $P_{\nu\mu} = \{P \in G \mid P$ is a projection, $Pu \in M_{\nu\mu}\}$.

$$p = \inf\{\|Pu-u\|;\ P \in P_{\nu\mu}\}.$$

Observe that $P_{\nu\mu} \neq \emptyset$ for $\nu < 0 < \mu$, since then $0 \in P_{\nu\mu}$; furthermore, p exists since the numbers $\|Pu-u\|$ are bounded below by 0. We will show that there is a unique $P \in P_{\nu\mu}$ for which the inf is attained.

Lemma 4; Let ^{p};Q^{eP}$_{vu}$ • Then P + Q - PQ = R∈P$_{vu}$ •

Proof: By assumption, there are A,Be£$_{\nu\mu}$ such that

a) APu=$(Pu)

b) BQu=$(Qu)

Write Q = PQ + (I-P)Q; then b) becomes

BPQu + B(l-P)Qu = $(PQu) + $((I-P)Qu).

Multiplying by (I-P), and using (3ii) and commutativity, we get

c) B(I-P)Qu = $((I-P)Qu).

Let C = AP + B(I-F)Q, and add a) and c) to obtain:

C(Ru) = $(Ru).

Now APe£..., B(I-P)Qe£..., and the product of AP with B(I-P)Q is zero; therefore v <[sp C < |i and Ce£$_{\nu\mu}$» Since A,B have property (12), so does C. Thus RueM$_{\nu\mu}$, q.e.d.

Corollary 1: If p = ||Pu-u|| = ||Qu-u||, then P = Q.

Proof; Put R = P + Q - PQ_0 Then ReP$_{\nu\mu}$, and if P ^ *Q,* ||Ru-u|| < *p,* which contradicts the definition of p.

Corollary 2; if |jpu-u!| > *p,* then there is an ReP$_{\nu\mu}$ such that PR = *P,* and |JRu-u|| < ||PU-U||O

Proof: By definition of *p,* there is a *QeP*$_{\nu\mu}$ such that ||Qu-u|| < ||PU-U||. Put R=P + Q-PQ; R has the desired properties.

Corollary 3: If p = ||Pu-u|| < ||Qu-u||, then PQ = Q.

Proof: Obvious.

Corollary 4; There exists a $P_{\nu\mu} e P_{\nu\mu}$, such that p = ||P^U - u||.

Proof: For each n, pick $P_n eS:>_{vu}$ such that $||P_n u — u|| < P + "^1>$ and $P_n P_{n-1} = P_{n-1}$ (by Cor. 2)„ It is easy to see that the limit $P_{V\wedge} = \lim_{n\to\infty} P_n$ exists, and that $||P_{VM>} u-u|| £\_ p$.

Now by assumption, there are $A_n e£_{v|j}$. such that

$A_n P_n u = \$(P_n u)$. With $Q_n = P_n - P_{n+1}$^ we also have $A_n Q_n u = \$(Q_n u)$ •

Since the Q_n are disjoint, and $A_n e£_{\nu\mu}$, the series $\sum^{CD} A_n Q_n$ converges and defines an operator AeG, with v ^ sp A < u. Now $AP_n u = \sum^n A_i Q_i u = E^* (Q_\pm u) = \$(\sum^n Q_\pm u) = \Phi(P_n u)$, and

$$AP_{\nu\mu} u = \lim_n AP_n u = \lim_n \Phi(P_n u) = \Phi(P_{\nu\wedge} u),$$

the last equality holding by (3i)„ (12) now follows easily: if R is a projection in $C_{,}$, and $RP^{\nu\mu}u = 0$, then $RP_n u = 0$ and thus $RA_n = 0$. But then RA = 0, and so $P_{v\wedge i} e P_{Hr}$ By definition of P> $||P_{v\wedge} u - u|| \geq P$> and th« corollary is proved.

Corollary 5: If $v_1 < v$, then $P_{v_1 ii} P_{v|j} = P_{vn}$.

This follows from Cor. 3, and gives an unambiguous meaning to

Definition 7: For each ueH, let $P^{\nu\mu}(u)$ denote the projection constructed above, and define an operator

$E_\mu : H \to H$ by

$$E_\mu u = \lim_{\nu \to -\infty} P_{\nu,\mu}(u)u.$$

The following properties of E_μ are easily verified:

Lemma 5; (i) $E_\mu^2 = E_\mu$;

(ii) $E_\nu E_\mu = E_{\min\{\mu,\nu\}}$;

(iii) for each $u \in H$, $\lim_{\mu \to \infty} E_\mu u = u$, and

$\lim_{\mu \to -\infty} E_\mu u = 0$;

(iv) if $-\infty < \mu \le \inf \Lambda$, then $E_\mu = 0$;

if $\sup \Lambda < \mu < +\infty$, then $E_\mu = 1$;

(v) if $\nu < \mu$, then $(E_\mu - E_\nu)u \in \mathcal{E}_{\nu\mu}$,

for each $u \in H$;

(vi) the function $\mu \to E_\mu u$ is left-continuous.

We are now ready to establish the integral formula (1).
Let $u \in H$ be fixed, and let $\varepsilon > 0$ be given. Choose N so large that $\|\Phi((E_N - E_{-N})u) - \Phi(u)\| < \frac{\varepsilon}{2}\|u\|$; this is possible by (iii) of Lemma 5, and by continuity of Φ.
Now subdivide the interval $[-N,N]$, $-N = \mu_0 < \mu_1 < \ldots < \mu_n = N$, making $|\mu_k - \mu_{k-1}| < \delta$ for $k = 1, \ldots, n$. Then observe that

$$\sum_{k=1}^{n} \ldots_{\mu_k} \ldots_{\mu_{k-1}} \qquad \sum_{k=1}^{n} \ldots_{\mu_k} \ldots_{\mu_{k-1}} \qquad \sum_{k=1}^{n} A_k u_k,$$

where $\lambda_k \in [\mu_{k-1},\mu_k]$ by Lemma 5, (v), and $P_k u = (E_{\mu_k} - E_{\mu_{k-1}})u$.

Thus, if $\lambda_k \in [\mu_{k-1},\mu_k]$,

$$(14) \qquad \sum_{k=1}^{n} \lambda_k P_k u - \Phi((E_N - E_{-N})u) = \sum_{k=1}^{n} (\lambda_k I - A_k) P_k u.$$

Because the $P_k u$ are pairwise orthogonal, and because sp $A_k \subset [\mu_{k-1},\mu_k]$, we have the estimates

$$\|\lambda_k I - A_k\| \le |\mu_k - \mu_{k-1}| < \frac{\varepsilon}{2},$$

$$\Big\|\sum_{k=1}^{n} \lambda_k P_k u - \Phi((E_N - E_{-N})u)\Big\| \le \frac{\varepsilon}{2}\Big\|\sum_{k=1}^{n} P_k u\Big\| \le \frac{\varepsilon}{2}\|u\|.$$

Hence, for sufficiently large N and sufficiently fine subdivisions of $[-N,N]$,

$$(15) \qquad \Big\|\sum_{k=1}^{n} \lambda_k (E_{\mu_k} - E_{\mu_{k-1}})u - \Phi(u)\Big\| < \varepsilon\|u\|.$$

The sum in (15) may naturally be interpreted as a Stieltjes-sum approximating $\int_{-\infty}^{+\infty} \lambda E(d\lambda)u$. We have therefore proved:

Theorem 4. _Let_ Φ _be an operator on_ H _which satisfies_ (3i),(3ii). _Then there exists a family_ $\{E_\mu\}$ _of idempotent operators, with the properties listed in Lemma_ 5, _such that for each_ $u \in H$,

$$(1) \qquad \Phi(u) = \int_{-\infty}^{\infty} \lambda E(d\lambda)u;$$

the integral is to be taken in the Stieltjes-sense indicated in equ. (15).

Remarks. 1) The claim made at the beginning of this paper, to the effect that the integral (1) need only be extended over Λ, may be established by replacing $E_{\mu} u$ with $P_{\mu}(u)u$, (cf. Def. 7) and studying the properties of $P_{\mu}(u)$ as a resolution of the identity.

2. At this point, it is not clear whether the integral (1) can be used to construct a functional calculus; the basic difficulty lies with the non-linearity and discontinuity of the 'projections' E_{μ} »

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