INVARIANT SUBSPACES OF ABSTRACT

MULTIPLICATION OPERATORS*

by

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ABSTRACT

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We describe a class of operators on a Banach space $\mathfrak{B}$ whose members behave, in a sense, like multiplication operators, and consequently leave invariant a proper closed subspace of $\mathfrak{B}$. One of the sufficient conditions for an operator to be such an "abstract multiplication" bears a striking resemblence to an assumption made by J. Wermer, who approached the invariant-subspace problem from a very different point of view.

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§1. We want to describe a class of operators on a Banach space $\mathfrak{B}$ whose members behave, in a sense, like multiplication operators, and consequently leave invariant a proper closed subspace of $\mathfrak{B}$ -- that is, they are <u>intransitive</u>. One of the sufficient conditions for an operator to be such an "abstract multiplication" bears a striking resemblance to an assumption made by J. Wermer [1], who approached the invariant-subspace problem from a very different (and rather more sophisticated) point of view. Our comments are presented in the hope that this connection, as well as the more general pattern which appears to emerge, may be more than superficial.

Wermer considered operators whose deviation from being an isometry is limited. More precisely, he assumed that T and T^{-1} are both bounded, and that either

(A) $\|T^n\| = O(e^{|n|^\alpha})$, $n = 0, \pm 1, \ldots$, $0 < \alpha < 1$, and the spectrum of T contains at least two points, or

(B) $\|T^n\| = O(|n|^k)$, for some fixed $k < \infty$.

<u>Such a</u> T <u>is intransitive</u>. If $\alpha = 0$ in (A), or $k = 0$ in (B), $\|T^n\| \leq K$ for all n; the space may then be renormed to make T an isometry, and intransitivity follows from a theorem of Godement [2].

The crucial estimate is that in (A); it is a reformulation of the requirement that

$$(1) \qquad \sum_{n=-\infty}^{\infty} \frac{\log\|T^n\|}{1+n^2} < \infty ,$$

which in turn plays an essential role in Wermer's delicate and highly analytic proof.

We take as model of an intransitive operator not an isometry, but a normal operator on a Hilbert space. Because of the spectral representation theorem, such an operator is basically a multiplication by an L^{∞}-function, acting on an L^2-space. By controlling the deviation of a more general operator from normality, we are able to retain at least those properties of multiplication operators which have a bearing on intransitivity.

The hypotheses are formulated in the language of Banach algebras. Let T be a bounded linear operator on the (complex, separable) Banach space $\mathfrak{B}$, let $\mathfrak{A}$ be the Banach algebra of operators generated by T; denote by $\mathfrak{M}$ the maximal ideal space of $\mathfrak{A}$, and by $A \longrightarrow \tilde{A}$ the Gel'fand homomorphism of $\mathfrak{A}$ to $C(\mathfrak{M})$.

Assume that:

(I) $\mathfrak{A}$ <u>is semi-simple</u>, i.e., $\tilde{A} \equiv 0$ if, and only if, $A = 0$;

(II) $\mathfrak{A}$ <u>is regular</u>, i.e., if $F \subset \mathfrak{M}$ is closed, and $M_o \in \mathfrak{M}-F$, then there is an $A \in \mathfrak{A}$ such that $\tilde{A}(M_o) \neq 0$, and $\tilde{A}(F) = \{0\}$;

(III) <u>for each</u> $A \in \mathfrak{A}$, <u>there is a sequence</u> $\{B_n\} \subset \mathfrak{A}$ <u>converging to</u> A <u>strongly, with the support of each</u> $\tilde{B}_n$

<u>contained in the interior of the support of</u> $\tilde{A}$.

<u>THEOREM</u>: <u>If</u> T <u>satisfies the above hypotheses</u>, <u>then</u> T <u>is intransitive</u>.

§2. These assumptions are quite severe, and we hope to motivate them in a moment. First we give an example of a non-normal operator satisfying (I)-(III) (this illustration actually suggested the present development).

Let H be the Hilbert space obtained by completing $C_0^\infty([0,1])$ in the norm

$$\|u\|^2 = \int_0^1 |u(x)|^2 + |u'(x)|^2 dx.$$

Thus, H is the first <u>Sobolev space</u> on [0,1]. Now let $\varphi \in C_0^\infty([0,1])$; the map $M_\varphi : u \longrightarrow \varphi \cdot u$ is a bounded operator which, despite its analytic simplicity, is not normal, since in general

$$(M_\varphi u, v) = \int_0^1 \varphi u\bar{v} + (\varphi u)'\bar{v}' \neq \int_0^1 u\overline{\varphi v} + u'(\overline{\varphi v})' = (u, M_\varphi v).$$

Otherwise, M_φ is well behaved. Its <u>spectrum</u> is the range of φ; its norm satisfies $\|M_\varphi\| \leq \sup_{[0,1]} (|\varphi(x)| + |\varphi'(x)|)$.
If we now take φ_0 real-valued, and $T = M_{\varphi_0}$, then the assumptions of the theorem are satisfied: every operator in $\mathfrak{A}$ is of the form M_φ, and such an algebra must be semi-simple (even though some of the smoothness of the generating function may be lost); regularity of $\mathfrak{A}$ is a consequence of Shilov's criterion described below, and

condition (III) can be verified by expressing suitable B_n's as limits of polynomials in φ_o. Finally, since the multiplications in $\mathfrak{A}$ will be defined by functions subject at least to mean smoothness properties, $\mathfrak{A}$ will not contain idempotents, and renorming of H cannot convert T into a normal operator.

Of course, any of the numerous other Sobolev spaces, Hilbert or Banach, would do just as well; more generally, multiplications by suitable bounded functions acting on Hilbert spaces obtained by equipping the domains of certain unbounded operators in $L^2(X,\mu)$ with the graph norm may be expected to illustrate conditions (I)-(III). As far as I know, multiplication operators of this kind do not fall into any of the classes of non-normal operators studied in the literature, but presumably they could and should be described and investigated abstractly. I suspect, and hope to prove, under conditions similar to (I)-(III), and with $\mathfrak{B}$ a Hilbert space, that T may be realized concretely in the form sketched above. If true, this property might be thought of as an extension to T of the finite-dimensional concept of diagonalizability. Assumption (I) then becomes reasonable; it is necessary and sufficient in order that a finite matrix be diagonalizable.

Property (II) -- regularity of $\mathfrak{A}$ -- appears to be common to most multiplication algebras of non-analytic functions. There is a sufficient condition for regularity, due to Shilov [3], which establishes the promised connection with Wermer's theorem:

If sp T *is real, then* $\mathfrak{a}$ *is regular if*

$$\int_0^\infty \frac{\log\|e^{isT}\|}{1+s^2}\, ds < \infty . \tag{2}$$

For self-adjoint T, $\log\|e^{isT}\| \equiv 0$, so (2) is trivially true. In the example above, with φ_o real, $e^{isM\varphi_o} = M_{e^{is\varphi_o}}$, and since

$$\|M_{e^{is\varphi_o}}\| \leq \sup_{[0,1]} (1 + s\,|\varphi_o'(x)|) \leq 1 + s\|M\varphi_o\|,$$

(2) again holds. In this sense, Shilov's criterion restricts the degree of "non-self-adjointness" of T. (Generally, if $T = T_1 + iT_2$, where T_1, T_2 commute, and sp T_1, sp T_2 are both real, then $\mathfrak{a}$ is regular if (2) holds for T_1 and T_2 separately, see [3]; the restriction that φ_o be real is therefore unnecessary).

An attempt to estimate $\log\|e^{isT}\|$ via the series expansion of e^{isT} will quickly lead to the imposition of Wermer's growth condition on $\|T^n\|$; however, (1) and (2) are not quite equivalent since (1) requires sp T to be located on the unit circle, while (2) is still satisfied by the operator $M\varphi_o$ of our example.

Condition (III) is the least satisfactory, since its consequences for T are unexplored. It is needed in the proof of our theorem to guarantee that certain closed ideals in $\mathfrak{a}$ are intersections of maximal ideals; examples from harmonic analysis indicate that this latter property cannot be expected to follow from (I) and (II) alone. It is conceivable, however, that (III), being an assertion about an algebra of operators, is derivable from the other two assumptions.

§3. Proof of the Theorem

In what follows, the letters u,v,w will denote elements of the Banach space $\mathfrak{B}$, and A,B will denote operators from $\mathfrak{A}$. (We exclude the trivial case $T = \lambda 1$.)

DEFINITION: *Let* $I_u = \{A\in\mathfrak{A}; Au = 0\}$. ($I_u$ *is a closed ideal.*) *The support of* u *is the set*

$$[u] = \{M\in\mathfrak{M}; I_u \subset M\}.$$

The support of $A\in\mathfrak{A}$ *is the set*

$$[A] = \text{closure}\{M\in\mathfrak{M}; \widetilde{A}(M) \neq 0\}.$$

$[A]^o$ *is the interior of* $[A]$.

(If $u\in\mathfrak{B}$ *were* a function on $\mathfrak{M}$, $M\in[u]$ would mean that $u(M) \neq 0$ if $\widetilde{A}(M) \neq 0$ implies $Au \neq 0$.)

The following propositions express those properties of $[u]$ and $[A]$ which are to be expected of multiplication operators.

(α) $$[u] = \emptyset \text{ iff } u = 0.$$

Proof: If $u = 0$, then $I_u = \mathfrak{A}$, so I_u is contained in no maximal ideal, and $[u] = \emptyset$. Conversely, if I_u is contained in no maximal ideal, then -- being itself an ideal -- it must coincide with $\mathfrak{A}$. In particular, $u = 1u = 0$.

(β) $$[Au] \subseteq [u].$$

<u>Proof</u>: Let $M \in [Au]$. If $Bu = 0$, then $0 = ABu = B(Au)$, whence $B \in M$. So $I_u \subset M$, and $M \in [u]$.

(γ) $$[Au] \subseteq [A].$$

<u>Proof</u>: Suppose to the contrary that there is an $M_o \in [Au]$ such that $\tilde{A}(M) = 0$ on a neighborhood $\mathfrak{G}$ of M_o. Since $\mathfrak{A}$ is regular, there is a $B \in \mathfrak{A}$ such that $\tilde{B}(M_o) \neq 0$, but $\tilde{B} \equiv 0$ off $\mathfrak{G}$. Then $\tilde{A}\tilde{C} \equiv 0$, and by semi-simplicity, $AC = 0$. Hence $CAu = 0$, and $C \in I_{Au} \subset M_o$. But this is impossible, since $\tilde{C}(M_o) \neq 0$.

(δ) $Au = 0$ <u>if, and only if</u> $[u] \subseteq \{M; \tilde{A}(M) = 0\}$.

[Remark: The "if" part of (δ) is equivalent to the assertion: $I_u = \bigcap_{M \in [u]} M$, meaning that I_u is the intersection of the maximal ideals containing it. See §2 above.]

<u>Proof</u>: By (III), there is a sequence $\{B_n\} \subset \mathfrak{A}$, such that $B_n u \longrightarrow Au$ and $[B_n] \subset [A]^o$. For each n, $B_n u = 0$, since on the one hand $[B_n u] \subseteq [u]$ (by (β)), and on the other $[B_n u] \subseteq [B_n]$ (by (γ)); but $[B_n] \cap [u] = \emptyset$ by hypothesis, so $[B_n u] = \emptyset$, and by (α), $B_n u = 0$. Hence $Au = 0$.

Conversely, let $M \in u$; since $Au = 0$, there is a $B \in \mathfrak{A}$ such that $BAu = 0$ and $B \notin M$ (for otherwise, $\mathfrak{A} \subset M$). From $BAu = 0$ follows $BA \in M$. So $0 = \widetilde{BA}(M) = \tilde{B}(M)\tilde{A}(M)$, and $\tilde{B}(M) \neq 0$ requires that $\tilde{A}(M) = 0$. q.e.d.

(ε) <u>If</u> $[v] \subseteq [u]$, <u>and</u> $Au = 0$, <u>then</u> $Av = 0$.

<u>Proof</u>: By (δ), $[u] \subseteq \{M;\tilde{A}(M) = 0\}$; since $[v] \subseteq [u]$, the direct part of (δ) gives $Av = 0$.

The theorem can now be stated in somewhat greater generality. <u>Let</u> $u \in \mathfrak{B}$. <u>The set</u> $\mathfrak{B}_u = \{v \in \mathfrak{B}; [v] \subseteq [u]\}$ <u>is a closed linear subspace of</u> $\mathfrak{B}$, <u>and is invariant under</u> $\mathfrak{A}$.

<u>Proof</u>: Invariance is immediate from (β). To see that $\mathfrak{B}_u$ is a subspace, note first that obviously $[v] = [\alpha v]$, for any complex α. Next, let $v, w \in \mathfrak{B}_u$, and (if $v + w \neq 0$) let $M \in [v + w]$. For any $A \in I_u$, we have, by (ε), $Av = Aw = 0$, hence $A(v + w) = 0$, and so $A \in M$. Therefore $M \in [u]$, and $[v + w] \subseteq [u]$, which by definition means $v + w \in \mathfrak{B}_u$. To show that $\mathfrak{B}_u$ is closed, let $w_n \in \mathfrak{B}_u$, $v_n \rightarrow v$. Let $M \in [v]$ (again excluding the trivial case $v = 0$), and let $A \in I_u$. By (ε), $Av_n = 0$, and so $Av = 0$. As above, we conclude that $[v] \subseteq [u]$, so that $v \in \mathfrak{B}_u$. q.e.d.

The only remaining question is whether at least one of these subspaces $\mathfrak{B}_u$ is proper. There are two possibilities. If there are distinct points $M_1, M_2 \in \mathfrak{M}$, contained in $[u_1]$, $[u_2]$, respectively ($u_1 = u_2$ is allowed), then there is an $A \in \mathfrak{A}$ such that $\tilde{A}(M_1) \neq 0$, while $\tilde{A} \equiv 0$ on a neighborhood of M_2. Put $u = Au_1$; then $u \neq 0$, while $u_2 \notin \mathfrak{B}_u$, so that $\mathfrak{B}_u$ is proper. The other possibility is that there is a single $M_o \in \mathfrak{M}$ such that $[u] = \{M_o\}$ for all $u \in \mathfrak{B} - \{0\}$. This actually cannot happen: because $\mathfrak{A}$ is semi-simple and $T \neq \lambda 1$, $\mathfrak{M}$ contains at least two points. There is then an $A \in \mathfrak{A}$, such that $\tilde{A} \not\equiv 0$, but $\tilde{A} \equiv 0$ on a neighborhood of M_0. So we would have $Au = 0$ for all $u \in \mathfrak{B}$, and yet $A \neq 0$. This contradiction proves the assertion, and with it, the theorem.

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REFERENCES

1. J. Wermer, "The existence of invariant subspaces", Duke Math. J. 19(1952), 615-622.

2. R. Godement, "Théorèmes taubériens et théorie spectrale", Ann. Sci. École Norm. Sup. (3)64(1947), 119-138.

3. G. Shilov, "On regular normed rings", Trudy Matem. In-ta. im. V. A. Steklova 21(1947), 1-118.

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