ON THE GEOMETRY OF SPHERES IN
SPACES OF CONTINUOUS FUNCTIONS
I and II

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Research Report 71-16

February 1971

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2/17/71

*This work was supported in part by NSF Grant GP 19126.

ON THE GEOMETRY OF SPHERES IN SPACES OF CONTINUOUS FUNCTIONS. I.

by

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1. Introduction

In [16], we introduced certain parameters associated with the inner metric of the unit sphere of a normed space: the inner diameter, the perimeter, and the girth. The first, for instance, is the diameter of the unit sphere, considered as a metric space in its inner metric; the last is the infimum of the lengths of symmetric simple closed curves on the unit sphere. The study of these parameters was continued in several papers. The first significant results for spaces of infinite dimension appear in [21] and [7], where the relationship between the girth and reflexivity is explored. In [8], Harrell and Karlovitz supply further insight into the properties of the girth and discuss the concept of a "flat" Banach space: one that has a curve of length 2 with antipodal endpoints, lying in the unit sphere.

A complete determination of the values of the parameters for L-spaces was made in [19]. The nature of the results encouraged an exploration of other "classical" spaces; in particular, the spaces $C_\sigma(K)$ of continuous real-valued functions on a compact space K, skew with respect to an involutory automorphism σ, appeared to be a likely object of study. These spaces include some congruent to the spaces $C(T)$ and $C_0(T)$ for topological spaces and locally compact spaces T, respectively, and occupy a significant place among spaces whose duals are L-spaces [2; pp. 87-96],[12]. A complete survey of these spaces as to their flatness was carried out in [13]; an example was also found among them showing that the perimeter is not always twice the inner diameter [20].

In the present work, we shall complement the studies just quoted by determining the values of the inner diameter, the perimeter, and the girth of all spaces $C_\sigma(K)$, $C(T)$, and $C_0(T)$. Since these parameters are extrema of distances in the inner metric, and these in turn are infima of the length of curves, the manner of attaining of these extrema, or the failure to do so, is of some interest. Indeed, for the girth, whose value is fixed at 4 for all non-reflexive spaces [21], this question is the one of greatest interest, and to it the major portion of the present Part I is devoted. The forthcoming Part II will deal with the perimeter and the inner diameter.

Sections 2 and 3 introduce the parameters and the function spaces to be studied. Section 4 contains some topological results on the concept of "perfect core" of a space and on scattered spaces (i.e., spaces with empty perfect core). Sections 5 and 7 take up the work in [13] on the spaces $C_\sigma(K)$, showing the conditions on K and σ for various ways in which the girth attains, or fails to attain, its value 4. Sections 6 and 8 translate these conditions for application to spaces $C(T)$ and $C_0(T)$. The relationship of these conditions with the topology of T is particularly interesting, and Section 9 contains a discussion that leaves some questions open, but that is complete at least for metrizable spaces T.

Since L^∞-spaces are congruent to spaces of continuous functions, they could be included in this study. Section 10 contains some results on these spaces, including a necessary condition for a σ-algebra of sets to admit a measure with a prescribed σ-ideal of locally null sets. Thanks are due to C. V. Coffman for permission to include his proof of the crucial Lemma 10.3.

2. Spheres in normed spaces

We generally denote by X a real normed space with $\dim X > 1$. Its norm is $\| \ \|$, its unit ball is $\Sigma(X)$, and its unit sphere, the boundary of $\Sigma(X)$, is $\partial\Sigma(X)$. We write $\Sigma, \partial\Sigma$ if confusion is unlikely. A subspace of X is a linear manifold of X provided with the norm of X. A congruence is an isometric isomorphism of one normed space onto another.

A curve in X is a "rectifiable geometric curve" as defined in [1; pp. 23-26]; for terminological details see [16; p. 61]. The length of a curve c is denoted by $\ell(c)$, and its standard representation in terms of arc-length is $g_c : [0, \ell(c)] \to X$.

We define on $\partial\Sigma$ the inner metric δ_X induced by the norm, by means of $\delta_X(p,q) = \inf\{\ell(c) : c$ a curve from p to q in $\partial\Sigma\}$. This function is studied in detail in [16], where it is shown to be a metric equivalent to the restriction of the norm metric. We denote it by δ if there is no danger of confusion. A trivial result from [16] is that, if Y is a subspace of X and $p, q \in \partial\Sigma(Y)$, then

$$\delta(p,q) \leqq \delta_Y(p,q). \tag{2.1}$$

This inner metric is used in the definition of several metric parameters of $\partial\Sigma$. As in [16; (5.1)], we define

$$
\begin{aligned}
D(X) &= \sup\{\delta(p,q) : p,q\in\partial\Sigma\} \\
M(X) &= \sup\{\delta(-p,p) : p\in\partial\Sigma\} \qquad (2.2)\\
m(x) &= \inf\{\delta(-p,p) : p\in\partial\Sigma\}
\end{aligned}
$$

We call $D(X)$ the _inner diameter_ of $\partial\Sigma$, and $2M(X)$ and $2m(X)$ the _perimeter_ and the _girth_ of Σ, respectively. We recall from [16; Lemma 5.2] that

$$(2.3) \qquad 2 \leqq m(X) \leqq M(X) \leqq D(X) \leqq 4.$$

If the supremum in the definition of $D(X)$ is attained, i.e., if $\delta(p,q) = D(X)$ for some $p,q\in\partial\Sigma$, we say, straining the language a little, that $D(X)$ _is attained_ (_at_ p,q); if there is, in addition, a curve of length $D(X)$ from p to q in $\partial\Sigma$, $D(X)$ is _strongly attained_ (_at_ p,q). Finally, $D(X)$ is _very strongly attained_ if $D(X)$ is attained and if such a curve exists for _each_ pair $p,q\in\partial\Sigma$ with $\delta(p,q) = D(X)$. Similar definitions apply to $M(X)$ and $m(X)$. The fact that two points rather than one appear in the definition of $D(X)$ permits further case distinctions for this parameter; these we forgo. The concepts just defined are significant only when $\dim X$ is infinite, as the next result makes plain.

2.1. _Lemma_. _If_ $\dim X$ _is finite_, _then_ $D(X), M(X), m(X)$ _are all very strongly attained_.

Proof. By compactness; see [16; Theorem 3.3,(b) and Lemma 5.1,(b)].

In discussing these parameters, special significance attaches to the bounds in (2.3). We shall deal with $M(X) = 4$ and $D(X) = 4$ later (Section 11); here we are concerned with $m(X)$ and the number 2. Their relationship is important: thus, $m(X) > 2$ if dim X is finite [16; Theorem 5.5], but $\min\{m(X) : \dim X = n\} = 2 + [\frac{1}{2}n]^{-1}$, n=2,3,... [18]. On the other hand, if X is a Banach space, $m(X) > 2$ implies reflexivity [21]; in the opposite direction, if $m(X) = 2$ is strongly attained, X is non-reflexive [7].

A point $p \in \partial\Sigma$ is a _flat spot of_ X (or _of_ $\partial\Sigma$) if $\delta(-p,p) = 2$. It is a _strictly flat spot_ if there exists a curve of length 2 from $-p$ to p in $\partial\Sigma$. Harrell and Karlovitz [8] call the space X _flat_ if it has a strictly flat spot; we shall call it _very flat_ if it is flat and _every_ flat spot is strictly flat. These concepts are related to the previously introduced terminology: with the use of (2.3), we have the following equivalences:

(i): $m(X) = 2$ _is attained at_ p _if and only if_ p _is a flat spot of_ X;

(ii): $m(X) = 2$ _is strongly attained at_ p _if and only if_ p _is a strictly flat spot of_ X; $m(X) = 2$ _is strongly attained if and only if_ X _is flat_;

(iii): $m(X) = 2$ _is very strongly attained if and only if_ X _has a flat spot, and all flat spots are strictly flat_; i.e., _if and only if_ X _is very flat_.

3. Spaces of continuous functions

All our topological spaces will be Hausdorff spaces; this assumption will not be explicitly repeated.

Let T be a topological space; then $C(T)$ is the Banach space of all bounded real-valued continuous functions with the supremum norm. If T is locally compact, $C_0(T)$ denotes the closed subspace of $C(T)$ consisting of the real-valued continuous functions on T that vanish at infinity.

Let K be a compact space and σ an involutory automorphism of K, i.e., a homeomorphism $\sigma : K \to K$ with $\sigma \circ \sigma = id_K$. We set $K^\sigma = \{t \in K : \sigma t \neq t\}$, the open set of points not fixed by σ. We let $C_\sigma(K)$ denote the closed subspace $\{f \in C(K) : f + (f \circ \sigma) = 0\}$ of $C(K)$; we observe once and for all that

$$(3.1) \qquad f(K \setminus K^\sigma) \subset \{0\}, \quad f \in C_\sigma(K).$$

We recall a well-known device for subsuming the study of $C_0(T)$ for locally compact T and of $C(T)$ for compact T under that of $C_\sigma(K)$. Let T be locally compact. The product space $T \times \{-1,1\}$ (the second factor discrete) is locally compact; it is homeomorphic to the direct sum $T + T$. Let K be $T \times \{-1,1\}$ if T is compact, and its one-point compactification $(T \times \{-1,1\}) \cup \{\infty\}$ otherwise; K is compact. Define $\sigma : K \to K$ by $\sigma(t,j) = (t,-j)$, $t \in T$, $j = \pm 1$, and, if pertinent, $\sigma\infty = \infty$. This is an involutory automorphism of K, and $K^\sigma = T \times \{-1,1\}$.

3.1. Lemma. *If T is locally compact and K,σ are constructed as described, the mapping $f \mapsto \hat{f} : C_0(T) \to C_\sigma(K)$ is a congruence, where $\hat{f}$ is defined by $\hat{f}((t,j)) = jf(t)$, $t \in T$, $j = \pm 1$, and, if pertinent, $\hat{f}(\infty) = 0$.*

Proof. Trivial.

We require two standard constructions. The first concerns the "middle of three" function. For real numbers a,b,c, we let $\mathrm{mid}(a,b,c)$ be the "middle" number: formally, $\mathrm{mid}(a,b,c) = \max\{\min\{a,b\},\min\{b,c\},\min\{c,a\}\}$; the same value is obtained with "max" and "min" interchanged. For any space T, the function $\mathrm{mid} : (C(T))^3 \to C(T)$ is defined by $(\mathrm{mid}(f,g,h))(t) = \mathrm{mid}(f(t),g(t),h(t))$, $f,g,h \in C(T)$, $t \in T$.

3.2. Lemma. *If K is compact and σ an involutory automorphism of K, and if $f \in C_\sigma(K)$, $g \in C(K)$, then $\mathrm{mid}(f,g,-g\circ\sigma) \in C_\sigma(K)$.*

Proof. By direct verification and the fact that mid is symmetric in every pair of arguments.

The second construction refers to a compact space K and an involutory automorphism σ of K. Suppose V is a closed set in K, and that $f_0 \in C(V)$ is given and satisfies

$$(3.2) \qquad -f_0(\sigma t) = f_0(t) \quad \text{for all} \quad t \in V \cap \sigma V.$$

Then there is a unique $f_1 \in C(V \cup \sigma V)$ such that $f_1(t) = -f_1(\sigma t) = f_0(t)$ for all $t \in V$; it obviously satisfies

$\|f_1\| = \|f_0\|$. By the Tietze Extension Theorem, there exists $f_2 \in C(K)$ with $\|f_2\| = \|f_1\| = \|f_0\|$ and $f_2(t) = -f_2(\sigma t) = f_1(t) = f_0(t)$, $t \in V$. We set $f = \frac{1}{2}(f_2 - (f_2 \circ \sigma))$ and find

$$\text{(3.3)} \qquad f \in C_\sigma(K), \quad \|f\| = \|f_0\|, \quad f(t) = f_0(t),\ t \in V.$$

A function f satisfying (3.3) shall be called a <u>skew Tietze extension</u> of f_0; this definition generalizes one given in [13; p]. (Conversely, if $f \in C_\sigma(K)$ is given and f_0 is its restriction to V, f_0 must satisfy (3.2).)

4. Perfect cores, scattered spaces, and continuous functions

The geometric properties of the function spaces introduced in the preceding section depend on topological properties of the underlying domain spaces and automorphisms. Here we deal with those properties that are relevant to the study of the girth.

Every topological space T contains a largest dense-in-itself subset; this set is closed [11; Par. 9]. We denote it by T_p and call it the *perfect core* of T. A space is *perfect* if it coincides with its perfect core; it is *scattered* if its perfect core is empty.

Pełczyński and Semadeni [14] have given several equivalent conditions for a compact space to be scattered; we record one that is fundamental for our work, mainly through its use in [13].

4.1. *Theorem*. (Pełczyński and Semadeni). *A compact space* T *is scattered if and only if there exists* $f \in C(T)$ *such that* $f(T) = [0,1]$.

Proof. [14; Main Theorem, (0) and (3).]

One-half of the proof of this theorem is a consequence of the following lemma, given essentially by Rudin [15].

4.2. *Lemma*. *If* K, J *are compact spaces and* $f : K \to J$ *is continuous and surjective, then* $f(K_p) \supset J_p$. *Consequently a continuous image of a scattered compact space is scattered.*

Proof. By Zorn's Lemma; see [15].

We now describe several auxiliary results concerning scattered and perfect spaces.

4.3. *Lemma*. *If* T *is a perfect compact space and* t_0 *a point in* T, *then* $\{t_0\}$ *is a* G_δ-*set if and only if there exists* $f \in C(T)$ *such that* $f(T \setminus \{t_0\}) = (0,1]$.

Proof. Since T is compact, any f as described must satisfy $f^{-1}(\{0\}) = \{t_0\}$. Since a zero-set is a G_δ-set, the existence of f is obviously sufficient; we must show its necessity.

The closed G_δ-set $\{t_0\}$ is a zero-set; there exists, therefore $h \in C(T)$ with $h(t_0) = 0$, $h(T \setminus \{t_0\}) \subset (0,1]$. Since T is perfect, t_0 is not an isolated point, and there exists a strictly decreasing sequence of numbers (ε_n) with $\varepsilon_0 = 1$ and $\lim_{n \to \infty} \varepsilon_n = 0$, such that $h^{-1}((\varepsilon_n, \varepsilon_{n-1})) \neq \emptyset$, n=1,2,... These non-empty sets are open, hence dense-in-themselves; therefore the compact sets $W_n = h^{-1}([\varepsilon_n, \varepsilon_{n-1}])$, n=1,2,..., are not scattered. By Theorem 4.1 there exists, for k=1,2,..., a function $f_k \in C(W_{2k+1})$ such that

$$(4.1) \qquad f_k(W_{2k+1}) = [\varepsilon_{2k}, \varepsilon_{2k-2}], \quad k=1,2,\ldots$$

Now the set $\{0\} \cup \bigcup_1^\infty [\varepsilon_{2k+1}, \varepsilon_{2k}]$ is closed in $[0,1]$; therefore $W = \{t_0\} \cup \bigcup_1^\infty W_{2k+1}$, its pre-image under h, is closed in K. We define $f' : W \to R$ by

$$(4.2) \qquad f'(t) = \begin{cases} f_k(t) & t\in W_{2k+1}, \quad k=1,2,\ldots \\ 0 & t = t_0. \end{cases}$$

Since each f_k is continuous and the W_{2k+1} are disjoint, f' is continuous except perhaps at $t = t_0$. But, for each $n, t\in W \cap h^{-1}([0,\varepsilon_{2n}))$ implies $t = t_0$ or $t\in W_{2k+1}$ with $k \geqq n$, whence $f'(t) \leqq \varepsilon_{2n-2}$. Thus f' is continuous at $t = t_0$ also, and $f'\in C(W)$. By (4.1) and (4.2),

$$(4.3) \qquad f'(W\setminus\{t_0\}) = \bigcup_1^{\infty}[\varepsilon_{2k},\varepsilon_{2k-2}] = (0,1];$$

by (4.1),(4.2), and the definition of W_n,

$$(4.4) \qquad h(t) \leqq f'(t), \quad t\in W.$$

By the Tietze Extension Theorem, there exists $f''\in C(K)$ such that $f''(K) \subset [0,1]$ and $f''(t) = f'(t)$, $t\in W$. We set $f = \max\{f'',h\}\in C(K)$. By (4.4), we have $f(t) = f'(t)$, $t\in W$; by (4.3), $(0,1] = f'(W\setminus\{t_0\}) \subset f(K\setminus\{t_0\}) \subset [0,1]$; but $f(t) = 0$ implies $h(t) = 0$, i.e., $t = t_0$, so that, in fact, $f(K\setminus\{t_0\}) = (0,1]$, as desired.

4.4. Lemma. *Assume that the space* T *is paracompact, locally compact, and scattered. If* $f\in C(T)$ *satisfies* $cl(f(T)) = [0,1]$, *there exists a countably infinite discrete closed subspace* Q *of* T *such that* $cl(f(Q)) = [0,1]$.

Proof. Since T is paracompact and locally compact, there exists a locally finite cover $\underset{\sim}{G}$ of T such that G is open and $cl\ G$ is compact for each $G\in\underset{\sim}{G}$. Order all closed subintervals of [0,1] with distinct rational endpoints in a sequence (J_n). We construct by induction a sequence (q_n) in T and a sequence (G_n) in $\underset{\sim}{G}$ such that, for $n=1,2,\ldots,$

(4.5) $$q_n \in G_n \setminus \bigcup_{i=1}^{n-1} (\mathrm{cl}\ G_i),$$

(4.6) $$f(q_n) \in J_n.$$

Assume, in fact, q_n, G_n constructed and satisfying (4.5), (4.6) for $n < k$. Now $\bigcup_{i=1}^{k-1} (\mathrm{cl}\ G_i)$ is compact and scattered; by Lemma 4.2, its image under f is compact and scattered, and cannot therefore be dense in the perfect set J_n. But $f(T)$ is dense in J_n by assumption; the existence of $q_k \in T$ and $G_k \in \underset{\sim}{G}$ satisfying (4.5), (4.6) for $n = k$ follows at once, and the construction is complete.

Since $\underset{\sim}{G}$ is locally finite, (4.5) implies that the countably infinite set $Q = \{q_n : n=1,2,\ldots\}$ is discrete and closed in T; and (4.6) implies that $f(Q)$ is dense in $[0,1]$.

In the following lemma, the (completely regular) space T is assumed to be embedded in the canonical way in its Stone-Čech compactification βT.

4.5. Lemma. Assume that the space T is paracompact and scattered. If $t_0 \in T$ has no compact T-neighborhood, then $t_0 \in (\beta T)_p$.

Proof. Let U be a given βT-neighborhood of t_0; the set $V = T \cap \mathrm{cl}_{\beta T} U$ is a T-closed T-neighborhood of t_0 and is consequently paracompact and scattered, but not compact.

There exists, therefore, a T-locally finite class $\underset{\sim}{G}$ of T-open sets such that $V \subset \bigcup \underset{\sim}{G}$, but such that no finite subclass covers V. Since V is scattered, the set Q of V-isolated points is V-dense in V. It is consequently possible to construct sequences (q_n) in Q and (G_n) in $\underset{\sim}{G}$ such that $q_n \in G_n \setminus \bigcup_{i=1}^{n-1} G_i$, n=1,2,... Since $\underset{\sim}{G}$ is T-locally finite, the countably infinite set $Q_0 = \{q_n : n=1,2,\ldots\}$ has no point of T-accumulation in the T-closed set V, and is therefore itself T-discrete -- hence homeomorphic to N -- and T-closed.

Since T is paracompact, it is normal, and therefore $cl_{\beta T} Q_0$ is homeomorphic to βN, which is not scattered. Thus $cl_{\beta T} U$ contains the non-scattered set $cl_{\beta T} Q_0$, and consequently $cl_{\beta T} U \cap (\beta T)_p \neq \emptyset$. Since $(\beta T)_p$ is βT-closed and U is an arbitrary βT-neighborhood of t_0, we conclude that $t_0 \in (\beta T)_p$.

5. Flat spots of $C_\sigma(K)$

In this section, K is a given compact space, σ a given involutory automorphism of K, and the notations $\Sigma, \partial\Sigma, \delta$ refer to the Banach space $X = C_\sigma(K)$. Obviously, $\dim X > 1$ if and only if K^σ has more than one pair of points; this we shall always assume to occur, although in most theorems it will follow from the hypotheses. Our immediate purpose is to characterize the flat spots, if any, of $C_\sigma(K)$.

We begin with a definition and a lemma that will also be required in later sections. If $f \in \partial\Sigma$, (3.1) implies that the set $[-1,1]\backslash f(K^\sigma)$ differs from the open set $[-1,1]\backslash f(K)$ at most by the inclusion of the point 0. It is therefore either a countable (possibly finite or even empty) union of disjoint open intervals, or such a union augmented by the singleton $\{0\}$. Let $\Lambda(f)$ be the length of the longest interval contained in $[-1,1]\backslash f(K^\sigma)$, and set $\Lambda(f) = 0$ if there is none, i.e., if the set is empty or $\{0\}$; thus

$$\Lambda(f) = 0 \text{ if and only if } f(K) = [-1,1]. \tag{5.1}$$

5.1. <u>Lemma</u>. <u>If</u> $f \in \partial\Sigma$, <u>then</u> $\delta(-f,f) \geq 2 + \Lambda(f)$.

<u>Proof</u>. If $\Lambda(f) = 0$, the inequality is trivial. Assume that (a,b) is an interval of length $\Lambda(f) > 0$ in $[-1,1]\backslash f(K^\sigma)$. Let $r \in (a,b)$ be given, and let c be any curve from $-f$ to f in $\partial\Sigma$. Since $\|f-f\| = 0 < 1-r < 2 = \|f+f\|$, there is a point g on c with $\|f-g\| = 1-r$. Since $g \in \partial\Sigma$, there

exists $t_0 \in K^\sigma$ such that $g(t_0) = 1$. Then

$$f(t_0) \geqq g(t_0) - |f(t_0)-g(t_0)| \geqq 1-\|f-g\| = r > a.$$

By the definition of (a,b), we must actually have $f(t_0) \geqq b$, and then

$$\ell(c) \geqq \|g+f\| + \|f-g\| \geqq |g(t_0)+f(t_0)| + 1-r \geqq 1+b+1-r = 2+(b-r).$$

Since r was arbitrarily close to a, and $\ell(c)$ to $\delta(-f,f)$, we indeed have $\delta(-f,f) \geqq 2+(b-a) = 2 + \Lambda(f)$.

<u>Remark</u>. We shall see (Theorem 13.1) that actually $\delta(-f,f) = 2 + \Lambda(f)$ for all $f \in \partial\Sigma$ if K^σ has no isolated points.

To find the flat spots in $C_\sigma(K)$, we next find a special flat point in a special space. On the compact space [-1,1], we define the involutory automorphism $\pi : t \mapsto -t$. The space $C_\pi([-1,1])$ of odd real-valued continuous functions on [-1,1] was used as an auxiliary device in [13] and [20]. In this space we consider the point u given by $u(t) = t$, $t \in [-1,1]$.

5.2. <u>Lemma</u>. <u>The point</u> u <u>is a flat spot of</u> $C_\pi([-1,1])$.

<u>Proof</u>. [20; Lemma 1]. The proof makes use of earlier computations of the girth of finite-dimensional spaces with cube-shaped unit balls [17].

5.3. <u>Theorem</u>. $f \in C_\sigma(K)$ <u>is a flat spot of</u> $C_\sigma(K)$ <u>if and only if</u> $f(K) = [-1,1]$, i.e., <u>if and only if</u> $\Lambda(f) = 0$.

<u>Proof</u>. If f is a flat spot, i.e., if $f \in \partial\Sigma$ and $\delta(-f,f) = 2$, Lemma 5.1 and (5.1) give $\Lambda(f) = 0$, and hence $f(K) = [-1,1]$. Conversely, if $f(K) = [-1,1]$, then $f \in \partial\Sigma$ and the linear mapping $\varphi \mapsto \varphi \circ f : C_\pi([-1,1]) \to C_\sigma(K)$ is well defined and isometric; it defines a congruence of $C_\pi([-1,1])$ onto a subspace Y of $C_\sigma(K)$. Since $\pm f = (\pm u) \circ f$, Lemma 5.2 implies $\delta_Y(-f,f) = 2$. By (2.1) we have $2 \leqq \delta(-f,f) \leqq \delta_Y(-f,f) = 2$; thus equality must hold, and f is a flat spot.

Theorem 5.3 shows that $m(C_\sigma(K)) = 2$ is attained if and only if there exists $f \in C_\sigma(K)$ with $f(K) = [-1,1]$. But this was precisely a necessary and sufficient condition, in the main theorem of [13], for $C_\sigma(K)$ to be flat, i.e., for $m(C_\sigma(K)) = 2$ to be <u>strongly</u> attained. Being attained and being strongly attained are thus equivalent in this case: we now formulate this as part of a general statement that incorporates part of [13; Theorem 5]. For further equivalent conditions we refer to the paper just quoted.

5.4. <u>Theorem</u>. <u>Let</u> K <u>be compact and</u> σ <u>an involutory automorphism for</u> K. <u>Then the following statements are equivalent</u>:

(a): $m(C_\sigma(K)) = 2$ <u>is attained</u>;

(b): $m(C_\sigma(K)) = 2$ <u>is strongly attained</u>; i.e., $C_\sigma(K)$ <u>is flat</u>;

(c): $m((C_\sigma(K))^*) = 2$ <u>is attained</u>;

(d): $m((C_\sigma(K))^*) = 2$ <u>is strongly attained</u>; i.e., $(C_\sigma(K))^*$ <u>is flat</u>;

(e): K^σ <u>is not scattered</u>;

(f): <u>there exists</u> $h \in C_\sigma(K)$ <u>with</u> $h(K^\sigma) = [-1,1]$;

(g): <u>there exists</u> $h \in C_\sigma(K)$ <u>with</u> $h(K) = [-1,1]$.

<u>Proof</u>. Statements (b),(d),(e),(f),(g) are equivalent by [13; Theorem 5]. The implications (b) $\rightarrow$ (a) and (d) $\rightarrow$ (c) are trivial. Theorem 5.3 shows that (a) implies (g). To prove that (c) implies (d) we observe that $(C_\sigma(K))^*$ is an L-space (cf. [12]), and refer to the study of these spaces in [19]: it follows from [19; Theorem 8] that an L-space that is not flat is congruent to some $\ell^1(A)$, and [19; Theorem 7] shows that when $m(\ell^1(A)) = 2$ (i.e., when A is infinite), it is not attained. This gives the desired implication.

6. Flat spots of other function spaces

Theorems 5.3 and 5.4 permit us to obtain analogous results for normed spaces congruent to some $C_\sigma(K)$. We base our selection on the work in [13; Section 6], but leave the study of L^∞-spaces for later (Section 10). We present the results in tabular form. Concerning the terminology, we recall that a topological space is *basically disconnected* if the closure of every co-zero set is open; extremally disconnected spaces are basically disconnected.

6.1. *Theorem. Let* T *be a completely regular space.*

If X *is*	*and* T *is*	*then* $f \in X$ *is a flat spot of* X *if and only if*
$C_0(T)$	*locally compact*	$\lvert f\rvert(K) \cup \{0\} = [0,1]$.
$C(T)$	*pseudocompact (in particular, compact)*	$\lvert f\rvert(K) = [0,1]$
$C(T)$	(*no restriction*)	$\mathrm{cl}(\lvert f\rvert(K)) = [0,1]$.

Proof. If T is locally compact, we define K,σ as in Lemma 3.1. By that lemma, f is a flat spot of $C_0(T)$ if and only if $\hat{f}$ is a flat spot of $C_\sigma(K)$. By Theorem 5.3, this is equivalent to $f(T) \cup -f(T) \cup \{0\} = \hat{f}(K) = [-1,1]$; but this is in turn equivalent to $\lvert f\rvert(K) \cup \{0\} = [0,1]$. The conclusion for compact T follows from this, since $C(T) = C_0(T)$ and $f(T)$ is compact.

If T is completely regular and embedded in the usual way in its Stone-Čech compactification βT, the mapping

$f \mapsto f' : C(T) \to C(\beta T)$ is a congruence, where f' is the unique continuous extension of f to βT. Using the conclusion for a compact space as applied to βT, we conclude that f is a flat spot of $C(T)$ if and only if f' is a flat spot of $C(\beta T)$, i.e., if $\mathrm{cl}(|f|(T)) = |f'|(\beta T) = [0,1]$. If T is pseudocompact, its image $|f|(T)$ is pseudocompact in R, hence compact. This completes the proof.

6.2. *Theorem. Let* T *be a completely regular space. The statements*

(a): $m(X) = 2$ *is attained*;

(b): $m(X) = 2$ *is strongly attained*; i.e., X *is flat*;

(c): $m(X^*) = 2$ *is attained*;

(d): $m(X^*) = 2$ *is strongly attained*; i.e., X^* *is flat*,

are equivalent for $X =$	*where* T *is*	*and hold if and only if*
$C_0(T)$	*locally compact*	T *is not scattered*; *or equivalently, there exists* $h \in X$ *with* $h(T) = [0,1]$;
$C(T)$	*compact*	
	pseudocompact	*there exists* $h \in X$ *with* $h(T) = [0,1]$;
	not pseudocompact	(*always*)
	paracompact (*in particular, metrizable*)	T *is not both compact and scattered*;
	basically disconnected	T *is infinite*.

Proof. Theorem 5.4 and [13; Theorems 6,7,8,9].

__Remark__. In [13] the unsatisfactory nature of the condition for pseudocompact T was noted, and it was pointed out that the condition is satisfied even for certain scattered and locally compact pseudocompact spaces.

7. Strictly flat spots of $C_\sigma(K)$

In this section we again assume that K is a given compact space and σ a given involutory automorphism of K, and the notations $\Sigma, \partial\Sigma, \delta$ refer to $X = C_\sigma(K)$. Our purpose is to characterize the strictly flat spots of $C_\sigma(K)$, if any; we obtain a description similar in nature to Theorem 5.3, but necessarily more complicated.

Theorem 5.4 shows that, if $C_\sigma(K)$ has a flat spot, it must have a strictly flat spot. It should not be inferred, however, that each flat spot is necessarily strictly flat. Indeed, the "test point" u in Lemma 5.2 is a flat spot of $C_\pi([-1,1])$, but is not strictly flat; and indeed a different "test point" of $C_\pi([-1,1])$ was shown in [13; Lemma 3] to be a strictly flat spot. The question raised here, which is the question, "For what K,σ is $m(C_\sigma(K)) = 2$ very strongly attained?", leads to considerable topological ramifications; we shall explore here some of those we found most significant.

7.1. Theorem. *$f \in C_\sigma(K)$ is a strictly flat spot of $C_\sigma(K)$ if and only if there exists a compact $A \subset K^\sigma$ such that $f(A) = f(K^\sigma) = f(K) = [-1,1]$ and $f(A \cap \sigma A) \subset \{-1,1\}$.*

Proof. Since a strictly flat spot is a flat spot, Theorem 5.3 allows us to stipulate, as we do, that $f(K) = [-1,1]$.

The condition is necessary: By the definition, there exists a curve c of length 2 from $-f$ to f in $\partial\Sigma$; we parametrize it, for convenience, by $g : [-1,1] \to C_\sigma(K)$, defined by $g(s) = g_c(s+1)$, $s \in [-1,1]$. Thus $g(\pm 1) = \pm f$.

Define the set

$$A = \{t \in K : (g(s))(t) = 1 \text{ for some } s \in [-1,1]\}.$$

Obviously, $A \subset K^{\sigma}$. We claim that A is closed in K, hence compact. Suppose $t_0 \in K \setminus A$; since the function $s \longmapsto (g(s))(t_0) : [-1,1] \to [-1,1]$ is continuous and does not take the value 1, there exists ρ, $0 < \rho < 1$ such that $(g(s))(t_0) \leqq \rho^2$ for all $s \in [-1,1]$. Since $g : [-1,1] \to C_{\sigma}(K)$ is continuous, the set $\{g(s) : s \in [-1,1]\}$ is compact, hence equicontinuous in $C(K)$; there exists therefore a neighborhood U of t_0 such that $t \in U$ implies $|(g(s))(t)-(g(s))(t_0)| \leqq \rho(1-\rho)$ for all $s \in [-1,1]$, and consequently $(g(s))(t) \leqq \rho < 1$ for all $s \in [-1,1]$. Therefore $U \subset K \setminus A$, and $K \setminus A$ is open; hence A is closed.

Let $s \in [-1,1]$ be given. Since $\|g(s)\| = 1$, there exists $t_s \in K$ such that $(g(s))(t_s) = 1$; this implies $t_s \in A$. Since s is arc-length on the curve c (up to a constant shift) we find, as in the proof of [13; Theorem 5],

$$\begin{aligned} s = 1-(1-s) &\leqq 1-\|g(1)-g(s)\| = 1-\|g(s)-f\| \leqq 1-(1-f(t_s)) = \\ &= f(t_s) = (1+f(t_s))-1 \leqq \|g(s)+f\|-1 = \|g(s)-g(-1)\|-1 \leqq \\ &\leqq (s+1)-1 = s, \end{aligned}$$

so that $s = f(t_s) \in f(A)$. Since $s \in [-1,1]$ was arbitrary and $f(K) = [-1,1]$, we find $f(A) = f(K^{\sigma}) = [-1,1]$, as required.

If $t_1 \in A \cap \sigma A$, there exist $s, s' \in [-1,1]$ such that $(g(s))(t_1) = 1$, $(g(s'))(t_1) = -(g(s'))(\sigma t_1) = -1$. Then

$$2 = |(g(s'))(t_1)-(g(s))(t_1)| \leqq \|g(s')-g(s)\| \leqq |s'-s| \leqq 2.$$

Thus equality holds, $\{s,s'\} = \{-1,1\}$, and $f(t_1) = (g(1))(t_1) = \pm 1$. Thus $f(A \cap \sigma A) \subset \{-1,1\}$ (an empty intersection is not excluded).

<u>The condition is sufficient</u>: As a preliminary step, we consider in R^2 the compact set $\Omega = \{(\xi,\eta) : |\xi| + |\eta| \leqq 1\}$, and define the continuous function $\varphi : [0,1] \times \Omega \rightarrow R$ by

$$\varphi(s,\xi,\eta) = \mathrm{mid}(\xi,s,-s) + \mathrm{mid}(\eta,1-s,-(1-s)).$$

For all values of the arguments φ obviously satisfies

$$\varphi(s,-\xi,-\eta) = -\varphi(s,\xi,\eta) \tag{7.1}$$

$$|\varphi(s,\xi,\eta)| \leqq |\xi|+|\eta| \leqq 1 \tag{7.2}$$

$$\varphi(0,\xi,\eta) = \eta \qquad \varphi(1,\xi,\eta) = \xi. \tag{7.3}$$

For each $s \in [0,1]$ we have $(s,1-s) \in \Omega$ and

$$\varphi(s,s,1-s) = s + (1-s) = 1. \tag{7.4}$$

Finally, a simple but tedious computation shows that the piecewise linear function φ satisfies

$$|\varphi(s',\xi,\eta)-\varphi(s,\xi,\eta)| \leqq |s'-s|, \quad s,s' \in [0,1], \ (\xi,\eta) \in \Omega. \tag{7.5}$$

We are now ready to carry out the proof proper. We define $h_0 \in C(A)$ by $h_0(t) = 1-|f(t)|$, $t \in A$. Since $f(A \cap \sigma A) \subset \{-1,1\}$, we have $h_0(A \cap \sigma A) \subset \{0\}$. We construct a skew Tietze extension $h_1 \in C_\sigma(K)$ of h_0, and define $h = \mathrm{mid}(h_1, 1-|f|, -(1-|f|))$; by Lemma 3.2, $h \in C_\sigma(K)$, and

(7.6) $$|f| + |h| \leqq 1$$

(7.7) $$h(t) = 1-|f(t)|, \quad t \in A.$$

In this paragraph, upper signs in $\pm$ correspond throughout, as do lower. We define $g^-, g^+ : [0,1] \to C_\sigma(K)$ by setting

(7.8) $$(g^{\pm}(s))(t) = \varphi(s, \pm f(t), h(t)), \quad s \in [0,1], \quad t \in K,$$

as we may, in view of (7.6) and (7.1). From (7.8) and (7.2), $\|g^{\pm}(s)\| \leqq 1$, $s \in [0,1]$. Given $s \in [0,1]$, there exists, by the assumption, $t_s^{\pm} \in A$ such that $\pm f(t_s^{\pm}) = s$; by (7.8), (7.7), and (7.4), $\|g^{\pm}(s)\| \geqq (g^{\pm}(s))(t_s^{\pm}) = \varphi(s,s,1-s) = 1$. Therefore equality holds, and $g^{\pm}(s) \in \partial\Sigma$. Further, (7.8) and (7.3) imply $g^{\pm}(0) = h$, $g^{\pm}(1) = \pm f$. Finally, (7.8) and (7.5) yield $\|g^{\pm}(s') - g^{\pm}(s)\| \leqq |s'-s|$, $s, s' \in [0,1]$.

Thus g^- and g^+ are Lipschitzian and are parametrizations of curves of length $\leqq 1$ each, from h to $-f$ and from h to f, respectively, in $\partial\Sigma$. Combining end-to-end the two curves, the former traversed backwards, we obtain a curve of length $\leqq 2$ from $-f$ to f in $\partial\Sigma$; since the endpoints are antipodes, the length must be precisely 2. We conclude that f is a strictly flat spot.

7.2. Corollary. $C_\sigma(K)$ is very flat if and only if K^σ is not scattered and there exists, for every $f \in C_0(K)$ with $f(K) = [-1,1]$, a compact set $A \subset K^\sigma$ such that $f(A) = [-1,1]$ and $f(A \cap \sigma A) \subset \{-1,1\}$.

Proof. Theorems 5.4, 5.3, and 7.1.

It is often convenient to be able to assume that K is perfect. We now show that, as far as the issues discussed in this section are concerned, this implies no loss of generality.

Assume that K^σ is not scattered; this is necessary if $C_\sigma(K)$ is to have any flat spots at all (Theorem 5.4). The perfect core K_p of K is compact and obviously invariant under σ; for typographical reasons we denote the restriction of σ to K_p, an involutory automorphism of K_p, again by σ. Since K^σ is open in K, we find that $(K_p)^\sigma = K_p \cap K^\sigma = (K^\sigma)_p$, the non-empty perfect core of K^σ. For each $f \in C_\sigma(K)$ we denote by f_p its restriction to K_p and find that $f \mapsto f_p : C_\sigma(K) \to C_\sigma(K_p)$ is a linear contraction. We write $\partial\Sigma_p = \partial\Sigma(C_\sigma(K_p))$.

7.3. Theorem. Assume that K^σ is not scattered. If $f' \in \partial\Sigma_p$, there exists $f \in \partial\Sigma$ such that $f' = f_p$. If $f \in \partial\Sigma$, then f is a flat spot of $C_\sigma(K)$ if and only if f_p is a flat spot of $C_\sigma(K_p)$, and f is a strictly flat spot of $C_\sigma(K)$ if and only if f_p is a strictly flat spot of $C_\sigma(K_p)$.

Proof. If $f' \in \partial\Sigma_p$, we have $f' \in C(K_p)$, $-f'(\sigma t) = f'(t)$, $t \in K_p$, and $\|f'\| = 1$. A skew Tietze extension f of f' will satisfy $f \in C_\sigma(K)$ and $\|f\| = 1$, i.e., $f \in \partial\Sigma$, as well as $f_p = f'$.

Assume that $f \in \partial\Sigma$. Then $f(K) = [-1,1]$ if and only if $f_p(K_p) = f(K_p) = [-1,1]$, by Lemma 4.2. It follows by Theorem 5.3 that f is a flat spot of $C_\sigma(K)$ if and only if f_p is a flat spot of $C_\sigma(K_p)$.

Assume still that $f \in \partial\Sigma$. If $A \subset K^\sigma$ is a compact set with $f(A) = [-1,1]$ and $f(A \cap \sigma A) \subset \{-1,1\}$, its perfect core satisfies $A_p \subset (K^\sigma)_p = (K_p)^\sigma$, $f_p(A_p) \doteq f(A_p) = [-1,1]$ (by Lemma 4.2), and $f_p(A_p \cap \sigma A_p) = f(A_p \cap \sigma A_p) \subset f(A \cap \sigma A) \subset$ $\subset \{-1,1\}$. Conversely, if $A' \subset (K_p)^\sigma \subset K^\sigma$ is a compact set such that $f_p(A) = [-1,1]$, $f_p(A' \cap \sigma A') = \{-1,1\}$, we have $f(A') = [-1,1]$, $f(A' \cap \sigma A') \subset \{-1,1\}$. It follows by Theorem 7.1 that f is a strictly flat spot of $C_\sigma(K)$ if and only if f_p is a strictly flat spot of $C_\sigma(K_p)$.

7.4. Corollary. $C_\sigma(K)$ _is very flat if and only if_ K^σ _is not scattered and_ $C_\sigma(K_p)$ _is very flat_.

Proof. Theorems 5.4 and 7.3.

The results in this section make it easy to construct spaces K and automorphisms σ such that $C_\sigma(K)$ is very flat, as well as others for which this is not the case. A useful, though extremely special, sufficient condition is given in the following theorem. Here the terms "simple arc" and "simple closed curve" are taken in their topological sense, as denoting homeomorphic images of a closed interval and a circle, respectively.

7.5. Theorem. *If every point of* K^σ *lies on some* σ-*invariant simple closed curve in* K, *then* $C_\sigma(K)$ *is very flat*.

Proof. It is assumed that K^σ is not empty. Any σ-invariant simple closed curve is perfect and must lie in K^σ; thus K^σ is not scattered. Let $f \in C_\sigma(K)$ satisfy $f(K) = [-1,1]$ and find $t_0 \in K^\sigma$ such that $f(t_0) = 1$. By the assumption there is a simple arc $A \subset K^\sigma$ connecting t_0 and σt_0 such that $A \cap \sigma A = \{\sigma t_0, t_0\}$. Now A is compact and connected, and $\{-1,1\} = \{f(\sigma t_0), f(t_0)\} \subset f(A) \subset \subset f(K) \subset [-1,1]$; we infer $f(A) = [-1,1]$. Further, $f(A \cap \sigma A) = \{f(\sigma t_0), f(t_0)\} = \{-1,1\}$. The conclusion follows from Corollary 7.3.

7.6. Scholium. The simplest instance of the application of Theorem 7.5 is obtained by letting K and σ be a circle and reflection in its centre. Other examples in which K is a perfect compact subset of a finite- or infinite-dimensional real Hilbert space and σ is reflection in a closed subspace of co-dimension greater than 1 are available in profusion: they include all the cases in which K is a convex reflection-invariant compact set with a core point, or (in the finite-dimensional case) the boundary of such a set. In all these cases, $C_\sigma(K)$ is very flat.

It is equally easy, on the other hand, to give examples of a similar nature for which $C_\sigma(K)$ is not very flat. This is the case, for instance, for $C_\pi([-1,1])$: the flat spot u clearly does not satisfy the conditions of Theorem 7.1. In this case, of course, the function space is congruent to $C_0((0,1])$; it is, however, almost as easy to give an example that is not "decomposable" in this way. Let K, for instance, be a circle with a pair of equal radial segments attached at opposite points, and let σ be the reflection in the centre. Then the condition of Corollary 7.2 is clearly not satisfied.

7.7. Problem. Characterize all perfect compact sets K in a real Hilbert space H that are invariant under the reflection σ in a closed subspace H_0 of H with $\operatorname{codim} H_0 > 1$ and are such that $C_\sigma(K)$ is very flat. What about the special case $\dim H = 2$, $H_0 = \{0\}$, K connected?

When we come to enquire, in Section 9, what spaces $C_0(T)$ or $C(T)$ are very flat, we shall see that the topological problems that arise are of a quite different nature.

8. Strictly flat spots of other function spaces

8.1. Theorem. *Let T be a locally compact space. Then $f \in C_0(T)$ is a strictly flat spot of $C_0(T)$ if and only if $|f|(T) = [0,1]$ and there exist compact sets $A_{-1}, A_1 \subset T$ such that*

$$(8.1) \qquad -f(A_{-1}) \cup f(A_1) = [-1,1] \text{ and } |f|(A_{-1} \cap A_1) \subset \{1\}.$$

If $f(T) \subset [0,1]$, condition (8.1) is equivalent to

$$(8.2) \qquad f(A_{-1}) = f(A_1) = [0,1] \text{ and } f(A_{-1} \cap A_1) \subset \{1\}.$$

If T is compact, the conclusion holds with $C_0(T)$ replaced by $C(T)$.

Proof. We construct K,σ, and the congruence $f \mapsto \hat{f}$: $C_0(T) \to C_\sigma(K)$ as in Lemma 3.1. We consider the relations

$$(8.3) \qquad A_j = \{t \in T : (t,j) \in A\},\ j = \pm 1, \text{ and } A = \{(t,j) \in T \times \{-1,1\} : t \in A_j\}$$

between a pair of sets A_{-1}, A_1 in T and a set A in $T \times \{-1,1\} = K^\sigma$ as defining either in terms of the other; they are obviously compatible. Since the mappings $t \mapsto (t,j) : T \to T \times \{-1,1\}$ are topological embeddings, A_{-1}, A_1 are compact if and only if A is compact.

For every $f \in C_0(T)$, the definition of $\hat{f}$ and (8.3) imply

$$(8.4) \qquad -f(A_{-1}) \cup f(A_1) = \{jf(t) : t \in A_j, j = \pm 1\} = \\ = \{jf(t) : (t,j) \in A\} = \hat{f}(A)$$

$$|f|(A_{-1} \cap A_1) = \{ |f(t)| : (t,-1),(t,1) \in A\} = \{ |f(t)| : (t,1) \in A \cap \sigma A\} = \\ = |\hat{f}|(A \cap \sigma A).$$

Now f is a strictly flat spot of $C_0(T)$ if and only if $\hat{f}$ is a strictly flat spot of $C_\sigma(K)$. By Theorem 7.1 this is the case if and only if there is a compact set $A \subset K^\sigma = T \times \{-1,1\}$ such that $\hat{f}(A) = \hat{f}(K^\sigma) = [-1,1]$ and $\hat{f}(A \cap \sigma A) \subset \{-1,1\}$. By (8.4) with the preceding discussion, this is equivalent to the existence of compact sets A_{-1}, $A_1 \subset T$ satisfying (8.1), together with $|f|(T) = |\hat{f}|(K^\sigma) = [0,1]$.

The conclusion for $f \geqq 0$ follows by inspection, since $f(A_{-1}), f(A_1)$ are compact. The conclusion for compact T follows since $C_0(T) = C(T)$.

8.2. <u>Theorem</u>. <u>Let</u> T <u>be a normal space</u>. <u>Then</u> $f \in C(T)$ <u>is a strictly flat spot of</u> $C(T)$ <u>if and only if</u> $cl(|f|(T)) = [0,1]$ <u>and there exist closed sets</u> A_{-1}, $A_1 \subset T$ <u>such that</u>

$$(8.5) \qquad -cl(f(A_{-1})) \cup cl(f(A_1)) = [-1,1] \ \underline{\text{and}} \ |f|(A_{-1} \cap A_1) \subset \{1\}.$$

<u>Proof</u>. We consider T embedded in βT, with the extension mapping $f \mapsto f' : C(T) \to C(\beta T)$, a congruence. Obviously $|f|' = |f'|$, $f \in C(T)$.

If $cl(|f|(T)) = [0,1]$, we have $|f'|(\beta T) = cl(|f|(T)) = [0,1]$. If A_{-1}, $A_1 \subset T$ are closed sets satisfying (8.5), we can use the Tietze Extension Theorem to find $h \in C(T)$ with $h(t) = j(1-|f(t)|)$, $t \in A_j$, $j = \pm 1$. If h' is the extension of h to βT, we define A'_{-1}, $A'_1 \subset \beta T$ by $A'_j = \{t \in \beta T : h'(t) = j(1-|f'(t)|)\}$, $j = \pm 1$. These sets are compact. Obviously $|f'|(A'_{-1} \cap A'_1) \subset \{1\}$. Since $A_j \subset A'_j$, $j = \pm 1$, (8.5) yields $[-1,1] \supset -f'(A'_{-1}) \cup f'(A'_1) \supset -cl(f(A_{-1})) \cup cl(f(A_1)) = [-1,1]$, so that equality holds. By Theorem 8.1 applied to βT, f', and A'_{-1}, A'_1 we conclude that f' is a strictly flat spot of $C(\beta T)$; hence f is a strictly flat spot of $C(T)$.

Conversely, suppose that f is a strictly flat spot of $C(T)$, whence f' is a strictly flat spot of $C(\beta T)$. By Theorem 8.1, $cl(|f|(T)) = |f'|(T) = [0,1]$, and there exist compact sets A'_{-1}, $A'_1 \subset \beta T$ satisfying $-f'(A'_{-1}) \cup f'(A'_1) = [-1,1]$, $|f'|(A'_{-1} \cap A'_1) \subset \{1\}$. By a construction similar to that in the first part of the proof, there exists $h' \in C(\beta T)$ such that $h'(t) = j(1-|f'(t)|)$ for all $t \in A'_j$, $j = \pm 1$. We set $B'_j = \{t \in \beta T : |f'(t)| + 2jh'(t) > 1\}$ and $A_j = cl_T(B'_j \cap T)$, $j = \pm 1$. Then the sets A_{-1}, $A_1 \subset T$ are T-closed; since the B'_j are βT-open, $B'_j \cap T$ is dense in B'_j, and therefore

$$(8.6) \qquad cl(f(A_j)) = cl(f'(B'_j)), \quad j = \pm 1.$$

On the other hand, $t \in A'_j \setminus B'_j$ implies $1 \geqq |f'(t)| + 2jh'(t) = 2-|f'(t)|$, so that $|f'(t)| = 1$. Therefore $f'(B'_j) \cup \{-1,1\} \supset f'(A'_j)$, $j = \pm 1$. Combining this with (8.6), we find

$$\begin{aligned}[-1,1] &= -f'(A'_{-1}) \cup f(A'_1) \subset -f'(B'_{-1}) \cup f'(B'_1) \cup \{-1,1\} \subset \\ &\subset -cl(f'(B'_{-1})) \cup cl(f'(B'_1)) \cup \{-1,1\} \subset [-1,1];\end{aligned}$$

equality must hold, and this implies the first half of (8.5). Finally, $A_j \subset cl_{\beta T}B'_j \subset \{t\in\beta T : |f'(t)| + 2jh'(t) \geqq 1\}$, $j = \pm 1$, so that $t\in A_{-1} \cap A_1$ implies $2|f(t)| = 2|f'(t)| = = |f'(t)| - 2h'(t) + |f'(t)| + 2h'(t) \geqq 1 + 1 = 2$, whence $|f(t)| = 1$. This implies the second half of (8.5).

Remark. It is possible to extend Theorem 8.2 to any completely regular T at the cost of requiring A_{-1},A_1 to be zero-sets. We omit a description of the modifications required in the proof.

8.3. Corollary.

If T is	locally compact	compact	paracompact
then the function space X =	$C_0(T)$	$C(T)$	$C(T)$
is very flat if and only if T is	not scattered	not scattered	not both compact and scattered
and there exist, for every $f\in X$ with	$\|f\|(T)\cup\{0\}=[0,1]$	$\|f\|(T)=[0,1]$	$cl(\|f\|(T)) = [0,1]$,
sets $A_{-1},A_1 \subset T$			
that are	compact	compact	closed
and satisfy	(8.1)	(8.1)	(8.5).

Proof. Theorems 6.2, 6.1, 8.1, and 8.2. Note that (8.1) requires $0\in f(T)$, whence $|f|(T) = [0,1]$, when T is locally compact.

We have an analogue of Theorem 7.3. Instead of deriving it afresh from Theorems 6.1 and 8.1, we prefer to reduce it to its prototype. Let T be locally compact and not scattered;

then T_p is a closed, hence locally compact, and non-empty set. Let $f \mapsto f_p : C_0(T) \to C_0(T_p)$ be the restriction mapping. With K,σ constructed as in Lemma 3.1, we find $(K_p)^\sigma = (K^\sigma)_p = T_p \times \{-1,1\}$; therefore K_p,σ (restricted to K_p) correspond to T_p under the construction of Lemma 3.1. With these constructions we find that $\hat{f}_p = (f_p)^\wedge$ for all $f \in C_0(T)$. The desired theorem then follows at once from Theorem 7.3.

8.4. Theorem. Assume that T is locally compact and not scattered. If $f' \in \partial\Sigma(C_0(T_p))$, there exists $f \in \partial\Sigma(C_0(T))$ such that $f' = f_p$. If $f \in \partial\Sigma(C_0(T))$, then f is a flat spot of $C_0(T)$ if and only if f_p is a flat spot of $C_0(T_p)$, and f is a strictly flat spot of $C_0(T)$ if and only if f_p is a strictly flat spot of $C_0(T_p)$. If T is compact, the conclusion holds with $C_0(T)$, $C_0(T_p)$ replaced by $C(T)$, $C(T_p)$, respectively.

8.5. Corollary. If T is locally compact, then $C_0(T)$ is very flat if and only if T is not scattered and $C_0(T_p)$ is very flat. If T is compact, then $C(T)$ is very flat if and only if T is not scattered and $C(T_p)$ is very flat.

Proof. Theorems 6.2 and 8.3.

9. *Very flat and not very flat function spaces*

In this section we investigate conditions on T that ensure or preclude that $C(T)$ or $C_0(T)$ is very flat. We begin with some negative results that are applications of Lemma 4.3.

9.1. *Lemma.* *If* T *is compact and if there exists* $t_0 \in T_p$ *such that* $\{t_0\}$ *is a* G_δ*-set -- at least relative to* T_p *-- then* $C(T)$ *is not very flat.*

Proof. On account of Corollary 8.5 we may assume without loss that T is perfect and $\{t_0\}$ is a G_δ-set. By Lemma 4.3, there exists $f \in C(T)$ with $f(T \setminus \{t_0\}) = (0,1]$. Since T is compact, $f(T) = [0,1]$. By Corollary 8.3 and Theorem 8.1, if $C(T)$ were very flat there would exist compact sets A_{-1}, $A_1 \subset T$ satisfying (8.2). But then $0 \in f(A_j)$, $j = \pm 1$, whence $t_0 \in A_{-1} \cap A_1$ and $0 = f(t_0) \in f(A_{-1} \cap A_1)$, in contradiction to (8.2).

9.2. *Theorem.* *Assume that* T *is completely regular and first countable.* *If* T *is paracompact and not locally compact, or if* T *is not scattered, then* $C(T)$ *is not very flat.*

Proof. In both cases it is enough to prove that $T \cap (\beta T)_p \neq \emptyset$: indeed, βT is first countable at any $t_0 \in T \cap (\beta T)_p$ (see [5; 9.7]), so that $\{t_0\}$ is a G_δ-set in βT; since $C(T)$ is congruent to $C(\beta T)$, the conclusion follows from Lemma 9.1 applied to $C(\beta T)$.

If T is paracompact and not locally compact, the existence of $t_0 \in T \cap (\beta T)_p$ follows from Lemma 4.5. If T is not scattered, we have $T \cap (\beta T)_p \supset T_p \neq \emptyset$.

9.3. Theorem. Assume that T is locally compact. If T_p is compact and first countable, or if T_p is not compact but paracompact, then $C_0(T)$ is not very flat.

Proof. If T_p is empty, $C_0(T)$ is not even flat (Theorem 6.2). If T_p is compact and non-empty, it is enough, on account of Corollary 8.5, to show that $C(T_p)$ is not very flat; if T_p is also first countable, this follows from Theorem 9.2, since the perfect non-empty space T_p is not scattered.

For the remaining case we may assume without loss, on account of Corollary 8.5, that T is perfect, locally compact, and paracompact but not compact. By [3; Theorem XI.7.3], T has a partition into closed-and-open σ-compact sets. Since T is not compact, at least one of these sets is not compact, or else the partition is not finite; in either case, T contains a closed-and-open σ-compact non-compact set P. Since P is open, it is perfect; since it is not compact, its one-point compactification $P \cup \{\infty\}$ is (compact and) perfect. Since P is σ-compact, the singleton $\{\infty\}$ is a G_δ-set in $P \cup \{\infty\}$. By Lemma 4.3 applied to $\{\infty\}$ in $P \cup \{\infty\}$, there exists $f_0 \in C_0(P)$ such that $f_0(P) = (0,1]$. We now define $f \in C_0(T)$ by $f(t) = f_0(t)$, $t \in P$, and $f(T) \in T \setminus P$ (the latter set may be empty); this definition is sound, since P is closed-and-open. Also, $f(T) \cup \{0\} = [0,1]$. Assume by contradiction that $C_0(T)$ is very flat. By Corollary 8.3 and Theorem 8.1 there must exist a compact set $A_1 \subset T$ such that $f(A_1) = [0,1]$

(forget about A_{-1}!). Now $A_1 \cap P$ is compact, so $f(A_1 \cap P) = f_0(A_1 \cap P)$ is a compact set in $(0,1]$, hence in $[\varepsilon,1]$ for some $\varepsilon > 0$. But then $[\varepsilon,1] \cup \{0\} \supset f(A_1 \cap P) \cup$ $\cup f(A_1 \setminus P) = f(A_1) = [0,1]$, a contradiction.

We next give a sufficient condition for $C(T)$ to be very flat.

9.4. Theorem. If T is paracompact, locally compact, and scattered, but not compact, then $C(T)$ is very flat.

Proof. We apply Corollary 8.3, as we may, since T is paracompact but not compact. If $f \in C(T)$ satisfies $cl(|f|(T)) = [0,1]$, there exists, by Lemma 4.4, a countably infinite discrete closed subspace Q of T such that $|f|(Q)$ is dense in $[0,1]$ or, equivalently, such that $-f(Q) \cup f(Q)$ is dense in $[-1,1]$. It is then clear that Q can be partitioned into disjoint sets Q_{-1}, Q_1, both of course still discrete and closed, such that $-f(Q_{-1}) \cup f(Q_1)$ is still dense in $[-1,1]$. It follows that f satisfies (8.5) with $A_j = Q_j$, $j = \pm 1$. By Corollary 8.3, $C(T)$ is very flat.

Before we give further instances of affirmative results, we observe that Theorems 9.2, 9.3, and 9.4 include a complete survey of $C_0(T)$ and $C(T)$ for paracompact first countable T; this includes all metrizable spaces.

9.5. Theorem. Assume that T is paracompact and first countable, in particular metrizable. Then $C(T)$ is very flat if and only if T is locally compact and scattered, but *not compact. If T is also locally compact, then* $C_0(T)$ *is never very flat.*

Proof. Consider first $C(T)$. The condition is sufficient, by Theorem 9.4, even without first countability. Assume, conversely, that $C(T)$ is very flat; by Theorem 9.2, T must be locally compact and scattered; and, by Theorem 6.2, T cannot then be compact.

As for $C_0(T)$ when T is locally compact: since the subset T_p is closed, it is both paracompact and first countable, so that $C_0(T)$ cannot be very flat, on account of Theorem 9.3.

Remark. If A is any infinite set, Theorem 9.5 shows that $\ell^\infty(A)$ is very flat, since a discrete space is metrizable, locally compact, and scattered, and an infinite one is not compact. By contrast, $c_0(A)$ (or $\ell_0^\infty(A)$) is not even flat (and $m(c_0(A)) = 2$ is not attained at all), by Theorem 6.2.

We conclude this section by giving another sufficient condition for $C(T)$, and our lone sufficient condition for $C_0(T)$.

9.6. *Theorem. Let* T *be the product of an uncountable family of compact spaces, each with more than one point, and* t_0 *a point in* T. *Then* $C(T)$ *and* $C_0(T\setminus\{t_0\})$ *are both very flat.*

Proof. Since T contains a subspace homeomorphic to a Cantor cube, it is not scattered; the same is true of

$T\{t_0\}$. We deal with $C(T)$ first; the parts in square brackets serve to prepare the proof for $C_0(T\setminus\{t_0\})$.

We assume $T = \prod_{\iota\in I} T_\iota$, I uncountable, each T_ι compact with at least two points. Let $f \in C(T)$ be given with

$$(9.1) \qquad |f|(T) = [0,1]$$

[and with $f(t_0) = 0$]. Now each function in $C(T)$ depends only on a countable set of co-ordinates (see [4]; the proof consists in observing that the functions in $C(T)$ with this property constitute a subalgebra satisfying the assumptions of the Stone-Weierstrass Theorem). This means that there exists a partition of I into disjoint non-empty sets I', I'', the latter uncountable, such that, with $T' = \prod_{\iota\in I'} T_\iota$, $T'' = \prod_{\iota\in I''} T_\iota$, we have -- modulo standard identifications -- $T = T' \times T''$ and

$$(9.2) \qquad f(t',t'') = f'(t') \text{ for some } f' \in C(T) \text{ and all } (t',t'') \in T' \times T''.$$

Since I'' is not empty [nor a singleton] we may choose distinct points $t''_{-1}, t''_1 \in T''$ [and both distinct from t''_0, where $t_0 = (t'_0, t''_0)$]. We set $A_j = \{(t', t''_j) : t' \in T'\}$, $j = \pm 1$; these are compact sets in T [in $T\setminus\{t_0\}$]. They are disjoint, and (9.1), (9.2) imply $-f(A_{-1}) \cup f(A_1) = -f'(T') \cup f'(T') = [-1,1]$. By Corollary 8.3, $C(T)$ is very flat.

Let now $f_0 \in C_0(T\{t_0\})$ be given, with $|f_0|(T\setminus\{t_0\}) \cup \{0\} =$

$= [0,1]$. Define $f \in C(T)$ by setting $f(t) = f_0(t)$, $t \neq t_0$, and $f(t_0) = 0$. Then $|f|(T) = [0,1]$. With A_{-1}, A_1 constructed for this f as precedingly, we find that these disjoint compact sets in $T \setminus \{t_0\}$ satisfy $-f_0(A_{-1}) \cup f_0(A_1) = -f(A_{-1}) \cup f(A_1) = [-1,1]$. By Corollary 8.3, $C_0(T \setminus \{t_0\})$ is very flat.

Remark. Lest it should be thought that the condition of Theorem 9.6 is sufficient because of the great "thickness" of the compact space T, we shall give in Example 10.7 a very "thick" compact space T such that $C(T)$ is not very flat.

10. L^∞-spaces

The last type of space congruent to some $C_\sigma(K)$ that we shall investigate as to its being very flat is L^∞-spaces. We shall see that the "usual" L^∞-spaces are indeed very flat.

Let S be a non-empty set, $\underset{\sim}{S}$ a σ-subalgebra of the algebra of subsets of S, and $\underset{\sim}{S}_0$ a σ-ideal of $\underset{\sim}{S}$. The triple $(S,\underset{\sim}{S},\underset{\sim}{S}_0)$ is given. A function $f : S \to R$ is <u>measurable</u> if $f^{-1}(U)\in\underset{\sim}{S}$ for every open set $U \subset R$, and a measurable function is <u>null</u> if $f^{-1}(R\backslash\{0\})\in\underset{\sim}{S}_0$. Two measurable functions are <u>equivalent</u> if their difference is null. If f is a measurable function and $E\in\underset{\sim}{S}$, the <u>essential image of</u> E <u>under</u> f is the set $\text{ess}\, f(E) = \{r\in R : f^{-1}(U) \cap E\notin\underset{\sim}{S}_0$ for all open $U \subset R$ such that $r\in U\}$; it is a closed set in R, and is the same for equivalent functions and for sets that differ by an element of $\underset{\sim}{S}_0$. It may thus be called the essential image under the equivalence class of f of the element of the quotient algebra $\underset{\sim}{S}/\underset{\sim}{S}_0$ represented by E. A measurable function f is <u>essentially bounded</u> if $\text{ess}\, f(R)$ is bounded.

The space $L^\infty(S,\underset{\sim}{S},\underset{\sim}{S}_0)$ is defined as the linear space of all equivalence classes of essentially bounded measurable functions, with the norm $\|f\| = \max\{|r| : r\in \text{ess}\, f(R)\}$; here, as in common practice, we strain the language by inessential confusion of an equivalence class with some representative of it. This space is a Banach space; we shall generally write L^∞ for it if the triple $(S, \underset{\sim}{S}, \underset{\sim}{S}_0)$ is understood. We quote the fundamental representation theorem for L^∞-spaces in the form required for our applications.

10.1. Theorem. *If $\tilde{S}$ is the Stone space of the σ-complete quotient algebra $\underset{\sim}{S}/\underset{\sim}{S}_0$, $\tilde{S}$ is compact and basically disconnected, and there exists a positively-preserving congruence $f \mapsto \tilde{f} : L^\infty \to C(\tilde{S})$ such that, for every $E \in \underset{\sim}{S}$ and the closed-and-open set $\tilde{E} \subset \tilde{S}$ corresponding to the elements of $\underset{\sim}{S}/\underset{\sim}{S}_0$ represented by E, we have* $\tilde{f}(\tilde{E}) = \text{ess } f(E)$, $f \in L^\infty$.

Proof. [22; pp. 206-207]. That T is basically disconnected follows from [22; pp. 85-86] and [5; Theorem 16.17].

The following result appears in [13; Theorem 10].

10.2. Theorem. *L^∞ is flat unless it is finite-dimensional.*

Proof. Theorems 10.1 and 6.2.

To formulate our main theorem we need to discuss measures on $\underset{\sim}{S}$. If $(S, \underset{\sim}{S}, \mu)$ is a measure space, a set $E \in \underset{\sim}{S}$ is *locally* μ-*null* (cf. [9]) if $\mu(E \cap F) = 0$ for all $F \in \underset{\sim}{S}$ with $\mu(F) < \infty$ or, equivalently, if $F \in \underset{\sim}{S}$, $F \subset E$ implies $\mu(F) = 0$ or $\mu(F) = \infty$ (we shall use the latter formulation); these sets constitute a σ-ideal of $\underset{\sim}{S}$. The triple $(S, \underset{\sim}{S}, \underset{\sim}{S}_0)$

is said to admit the measure μ if $(S,\underset{\sim}{S},\mu)$ is a measure space and $\underset{\sim}{S}_0$ is precisely the σ-ideal of locally μ-null sets. Our aim is to show that L^∞ is very flat provided $(S,\underset{\sim}{S},\underset{\sim}{S}_0)$ admits some measure. The argument is presented in three steps. The first result is due to C. V. Coffman (private communication).

10.3. Lemma. (Coffman). *Let $\underset{\sim}{B}$ be the class of Borel sets in $[0,1]$, and let $([0,1],\underset{\sim}{B},\nu)$ be a finite measure space such that $\nu(J) > 0$ for every interval $J \subset [0,1]$. Then there exists a Borel set D such that $\nu(J \cap D) > 0$ and $\nu(J \setminus D) > 0$ for every interval $J \subset [0,1]$.*

Proof. We consider the Banach space $L^1(\nu)$. For given $r,s,0 \leqq r < s \leqq 1$, we define $Z_{rs} = \{\varphi \in L^1(\nu) : \varphi\chi_{[r,s]} \geqq 0\}$. It is obvious that Z_{rs} is a closed set in $L^1(\nu)$; we claim that it is nowhere dense. Indeed, let $\varphi \in Z_{rs}$ and $\varepsilon > 0$ be given. Since $\varphi \in L^1(\nu)$ there exists $\delta \leqq \frac{1}{2}\varepsilon$ such that $\int_B |\varphi| d\nu < \frac{1}{2}\varepsilon$ for every Borel set B with $\nu(B) < \delta$. We choose a positive integer n so great that $n\delta > \nu([r,s])$, and a set of n pairwise disjoint intervals contained in $[r,s]$; there must be one, say J, such that, under the assumption, $0 < \nu(J) \leqq n^{-1}\nu([r,s]) < \delta$. We set $\psi = \varphi - (1 + \varphi)\chi_J \in L^1(\nu)$. Then $\|\varphi-\psi\| = \int_J (1+\varphi) d\nu \leqq \nu(J) + \int_J \varphi d\nu < \delta + \frac{1}{2}\varepsilon \leqq \varepsilon$; but $\int_J \psi d\nu = -\int_J d\nu = -\nu(J) < 0$, so that $\psi \notin Z_{rs}$.

The set $\bigcup\{-Z_{rs} \cup Z_{rs} : r,s \text{ rational}, 0 \leqq r < s \leqq 1\}$

is of first category in $L^1(\nu)$; there exists, therefore, an element $\varphi_0 \in L^1(\nu)$ that does not lie in this set. We define the Borel set $D = \varphi_0^{-1}([0,\infty))$ -- it is defined modulo ν-null sets, so we pick some arbitrary representative -- and we claim that D satisfies the conclusion. Let J be an interval, and choose rationals $r,s, 0 \leqq r < s \leqq 1$ such that $[r,s] \subset J$. If $\nu(J \cap D) = 0$, it follows that φ_0 is negative ν-a.e. on J, a fortiori on $[r,s]$; certainly $\varphi_0 \chi_{[r,s]} \leqq 0$, whence $\varphi_0 \in -Z_{rs}$, a contradiction; if $\nu(J \setminus D) = 0$, it follows that J, and a fortiori $[r,s]$, is ν-essentially contained in D, so that $\varphi_0 \chi_{[r,s]} \geqq 0$, whence $\varphi_0 \in Z_{rs}$, another contradiction. Our claim is thus established.

Remark. In [9; (18.31)] it is proposed to prove this lemma, for the special case in which ν is Lebesgue measure, by means of an explicit construction of D.

10.4. Lemma. *Assume that* $(S, \underset{\sim}{S}, \underset{\sim}{S}_0)$ *admits a measure. If* $f : S \longrightarrow R$ *is a measurable function such that* ess $f(S) = [0,1]$, *there exist disjoint sets* $A_{-1}, A_1 \in \underset{\sim}{S}$ *such that* ess $f(A_{-1}) =$ ess $f(A_1) = [0,1]$.

Proof. 1. We shall first prove the lemma under the additional assumption that $(S, \underset{\sim}{S}, \underset{\sim}{S}_0)$ admits a σ-*finite* measure μ; then $\underset{\sim}{S}_0$ is the class of μ-null elements of $\underset{\sim}{S}$. We perform the usual trick and observe that there exists $\varphi \in L^1(\mu)$ such that $\varphi > 0$ μ-a.e. (a "Freudenthal unit" of $L^1(\mu)$; see [2; p. 107]); then $(S, \underset{\sim}{S}, \underset{\sim}{S}_0)$ admits the finite measure μ' defined by $\mu'(E) = \int_E \varphi d\mu$, $E \in \underset{\sim}{S}$, since $\mu'(S) = \|\omega\| < \infty$ and the μ'-null elements of $\underset{\sim}{S}$ are precisely the μ-null ones, i.e., the elements of $\underset{\sim}{S}_0$. There is thus no loss in assuming, as we shall, that $(S,\underset{\sim}{S},\mu)$ is a <u>finite</u> measure space and that $\underset{\sim}{S}_0$ is the class of μ-null elements of $\underset{\sim}{S}$.

Let f be given as in the statement. The assumption on f is equivalent to

$$(10.1) \qquad \mu(f^{-1}(R\setminus[0,1])) = 0, \quad \mu(f^{-1}(J)) > 0 \text{ for every interval } J \subset [0,1].$$

On $\underset{\sim}{B}$, the class of Borel sets of $[0,1]$, we define the measure $\nu(B) = \mu(f^{-1}(B))$, $B \in \underset{\sim}{B}$. This measure is finite: $\nu([0,1]) = \mu(S) < \infty$. For every interval $J \subset [0,1]$, (10.1) implies $\nu(J) > 0$. We now choose $D \in \underset{\sim}{B}$ as given in Lemma 10.3, and define $A_1 = f^{-1}(D)$, $A_{-1} = f^{-1}([0,1]\setminus D)$; A_{-1} differs by a μ-null set from $S\setminus A_1$. These sets are disjoint; for every interval $J \subset [0,1]$ we have $\mu(f^{-1}(J) \cap A_{-1}) = \mu(f^{-1}(J\setminus D)) = \nu(J\setminus D) > 0$ and $\mu(f^{-1}(J) \cap A_1) = \mu(f^{-1}(J \cap D)) = \nu(J \cap D) > 0$. Thus $[0,1] \subset \text{ess } f(A_j) \subset \text{ess } f(S) = [0,1]$, $j = \pm 1$, so that equality holds.

2. We return to the general case, and assume that $(S,\underset{\sim}{S},\mu)$ is a measure space and that $\underset{\sim}{S}_0$ is the class of locally μ-null elements of $\underset{\sim}{S}$. For the given f, (10.1) is replaced by

$$f^{-1}(R\setminus[0,1]) \in \underset{\sim}{S}_0, \quad f^{-1}(J) \notin \underset{\sim}{S}_0 \text{ for every interval } J \subset [0,1].$$

For given $r,s, 0 \leqq r < s \leqq 1$, the fact that $f^{-1}([r,s])$ is not locally μ-null implies the existence of $E_{rs} \in \underset{\sim}{S}$ such that $E_{rs} \subset f^{-1}([r,s])$ and $0 < \mu(E_{rs}) < \infty$. Set $S' = \bigcup\{E_{rs} : r,s, \text{rational}, 0 \leqq r < s \leqq 1\}$. Then $S' \in \underset{\sim}{S}$, and $E_{rs} \subset f^{-1}([r,s]) \cap S'$ for all rational $r,s, 0 \leqq r < s \leqq 1$, so that $\text{ess } f(S') = \text{ess } f(S) = [0,1]$. We set $\underset{\sim}{S}' = \{E \in \underset{\sim}{S} : E \subset S'\}$ and let μ' be the restriction of μ to $\underset{\sim}{S}'$. Then $(S', \underset{\sim}{S}', \mu)$ is a σ-finite measure space, and $\underset{\sim}{S}_0 \cap \underset{\sim}{S}'$ is the class of μ'-null elements of $\underset{\sim}{S}'$. If f' is the restriction of f to S', we still have $\text{ess } f'(S') = \text{ess } f(S') = [0,1]$.

We may then apply the first part of the proof to $(S', \underset{\sim}{S}', \underset{\sim}{S}_0 \cap \underset{\sim}{S}')$ and f', and find disjoint sets A_{-1}, $A_1 \in \underset{\sim}{S}' \subset \underset{\sim}{S}$ such that $\text{ess } f(A_j) = \text{ess } f'(A_j) = [0,1]$, $j = \pm 1$, as required in the conclusion.

10.5. <u>Theorem</u>. <u>If</u> $(S, \underset{\sim}{S}, \underset{\sim}{S}_0)$ <u>admits a measure, then</u> $L^\infty(S, \underset{\sim}{S}, \underset{\sim}{S}_0)$ <u>is very flat unless it is finite-dimensional</u>.

<u>Proof</u>. We assume that L^∞ is infinite-dimensional (equivalently, that $\underset{\sim}{S}/\underset{\sim}{S}_0$ is infinite). By Theorem 10.2, L^∞ is flat. It remains to show that every flat spot of L^∞ is a strictly flat spot.

Let f be a flat spot of L^∞, and consider $E = f^{-1}((-\infty, 0)) \in \underset{\sim}{S}$, specified up to sets in $\underset{\sim}{S}_0$. Since multiplication by $\chi_{S \setminus E} - \chi_E$ is a congruence of L^∞ onto itself, there is no loss in assuming, as we shall, that $f \geqq 0$.

We now use the positivity-preserving congruence $L^{\infty} \to C(\tilde{S})$ of Theorem 10.1. We find that $\tilde{f}$ is a flat spot of $C(\tilde{S})$ and that $\tilde{f} \geqq 0$. By Theorem 6.1, $\tilde{f}(\tilde{S}) = [0,1]$, so that ess $f(S) = [0,1]$.

By Lemma 10.4 there exist disjoint sets $A_{-1}, A_1 \in \underset{\sim}{S}$ such that ess $f(A_j) = [0,1]$, $j = \pm 1$. If $\tilde{A}_j$ is the closed-and-open set in $\tilde{S}$ corresponding to the element of $\underset{\sim}{S}/\underset{\sim}{S}_0$ represented by A_j, $j = \pm 1$, we find that the compact sets $\tilde{A}_{-1}, \tilde{A}_1$ are disjoint and, by Theorem 10.1, $\tilde{f}(\tilde{A}_j) = [0,1]$, $j = \pm 1$. By Theorem 8.1, $\tilde{f}$ is a strictly flat spot of $C(\tilde{S})$, so that f is a strictly flat spot of L^{∞}.

10.6. Corollary. If Y is an infinite-dimensional abstract L-space, then $X = Y^*$ is very flat.

Proof. (cf. [13; Corollary 11]). By Kakutani's Representation Theorem [2; pp. 107-108], Y is congruent to $L^1(\mu)$ for a "localizable" or "decomposable" measure space $(S,\underset{\sim}{S},\mu)$, so that Y^* is congruent to $L^{\infty}(S,\underset{\sim}{S},\underset{\sim}{S}_0)$, where $\underset{\sim}{S}_0$ is the class of (locally) μ-null elements of $\underset{\sim}{S}$ [9; (19.25),(20.20)]. Since $(S,\underset{\sim}{S},\underset{\sim}{S}_0)$ admits the measure μ, the conclusion follows from Theorem 10.5.

The condition that $(S,\underset{\sim}{S},\underset{\sim}{S}_0)$ admit a measure cannot be dropped entirely in Theorem 10.5, as the following example shows.

10.7. Example. Let $\underset{\sim}{B}$ be the σ-algebra of Borel sets in $[0,1]$, and $\underset{\sim}{B}_0$ the σ-ideal of Borel sets of first category. By a remark of Kelley [10; p. 1172], the quotient algebra $\underset{\sim}{B}/\underset{\sim}{B}_0$ is isomorphic to the complete algebra of regularly closed subsets of [0,1]. Therefore the Stone space of this quotient algebra is homeomorphic to the Stone space of the latter algebra; this is the "projective envelope" or "Gleason space" of [0,1] (see [6]). It is an extremally disconnected space G with a continuous surjective mapping $g : G \to [0,1]$ that is irreducible; this means that there is no closed proper subset $A \subset G$ with $g(A) = [0,1]$. A fortiori, there do not exist closed sets $A_{-1}, A_1 \subset G$ with $g(A_{-1}) = g(A_1) = [0,1]$, $g(A_{-1} \cap A_1) \subset \{1\}$. By Theorems 6.1 and 8.1, g is a flat spot of $C(G)$, but not a strictly flat spot. Hence neither $C(G)$, nor the congruent space $L^\infty([0,1], \underset{\sim}{B}, \underset{\sim}{B}_0)$, is very flat.

10.8. Conjecture. (Converse of Theorem 10.5). If $L^\infty(S, \underset{\sim}{S}, \underset{\sim}{S}_0)$ is very flat, then $(S, \underset{\sim}{S}, \underset{\sim}{S}_0)$ admits a measure.

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ON THE GEOMETRY OF SPHERES IN SPACES OF CONTINUOUS FUNCTIONS II

by

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Introduction.

In a paper with the same general title [8], we began the study of certain metric parameters of the unit spheres in spaces of continuous functions. The spaces considered were $C(T)$, $C_0(T)$ (for locally compact T), and most especially $C_\sigma(K)$, the space of continuous real-valued functions on a compact space K, skew with respect to an involutory automorphism σ of K. This study is part of a survey of such parameters for the "classical" spaces, begun in [6] and continued in [2] and [7]. In Part I we dealt with the properties of the girth; here we shall deal with the perimeter and the inner diameter of the unit sphere.

We refer to the Introduction of [8] for a more detailed explanation. We consider that paper and the present one as a single work, continuing the numbering of the sections and quoting freely from Part I. However, only the contents of Sections 2 and 3, as well as Lemmas 5.1 and 5.2 and Theorem 5.3, will be required for the understanding of the present Part II. This does not apply to the summary in the final section.

In Sections 11 and 12 we supply required preliminary material to complement Sections 2 and 3, respectively. In Sections 13 and 14 we determine the values of the perimeter and the inner diameter, respectively, and the manner of attaining of the extrema defining them, for all the spaces under consideration. Section 15 summarizes in tabular form most of the results obtained in Parts I and II.

11. Spheres in normed spaces, continued

Sections 5-10 dealt with the girth of a space of continuous functions and with the manner in which it is or is not attained. From this point on, we turn our attention to the remaining two parameters introduced in Section 2, the inner diameter and the perimeter. The present section and the next complement the material in Sections 2 and 3, respectively, with a view to discussing these parameters.

We return to the context of Section 2: X is a given real normed space, generally with $\dim X > 1$, and the notations $\Sigma, \partial\Sigma, \delta$ refer to this space unless another is indicated. We need a few miscellaneous bits of terminology and information.

Several of the curves we shall be constructing are 'polygonal'. A curve in X obtained by joining end-to-end straight-line segments, each traversed once from one endpoint to the other, is a polygon; the endpoints of these segments, in the order of their traversing, are the consecutive vertices of the polygon.

If X', X'' are real normed spaces, $X' \oplus X''$ denotes the normed space that is, algebraically, the outer direct sum of X' and X'', with the norm $\|x' \oplus x''\| = \max\{\|x'\|, \|x''\|\}$. If $X = X' \oplus X''$, and primes are added to $\Sigma, \partial\Sigma, \delta$ to indicate reference to these various spaces, we have $\Sigma = \Sigma' \oplus \Sigma''$

and $\partial\Sigma = (\partial\Sigma'\oplus\Sigma'') \cup (\Sigma'\oplus\partial\Sigma'')$. The following lemma refers to this situation.

11.1 Lemma. *Assume* $X = X' \oplus X''$ *and* $\dim X' > 1$. *If* $p,q\in\partial\Sigma$, *where* $p = p' \oplus p''$, $q = q' \oplus q''$, *and* $p',q'\in\partial\Sigma'$, *then* $\delta(p,q) \leq \max\{\delta'(p',q'),\|q''-p''\|\}$.

Proof. By the assumption, $\|p''\|,\|q''\| \leq 1$. Let c' be a curve from p' to q' in $\partial\Sigma'$, and set $\ell = \ell(c')$. Define $g : [0,1] \to X$ by $g(r) = g_{c'}(\ell r) \oplus (rq''+(1-r)p'')$, $r\in[0,1]$. Then $g(0) = p$, $g(1) = q$, $\|g(r)\| = \max\{\|g_{c'}(\ell r)\|, \|rq'' + (1-r)p''\|\} = 1$, and $\|g(r')-g(r)\| = \max\{\|g_{c'}(\ell r')-g_{c'}(\ell r)\|, |r'-r|\,\|q''-p''\|\} \leq |r'-r|\max\{\ell,\|q''-p''\}$, $r,r'\in[0,1]$. Therefore g parametrizes a curve of length not exceeding $\max\{\ell,\|q''-p''\|\}$ from p to q in $\partial\Sigma$, so that $\delta(p,q) \leq \max\{\ell,\|q''-p''\|\}$. Since ℓ was arbitrarily close to $\delta'(p',q')$, the conclusion follows.

Remark. Among closely related results we mention, without detailing the proof, the following: *if* X' *is flat, then* X *is flat; if, in addition,* X *is very flat, then* X' *is also very flat*.

We take up again the discussion, begun in Section 2, of the significance of the bounds in (2.3) for the parameters $D(X)$, $M(X)$, $m(X)$. As for the lower bound 2, we merely add a few trivial remarks that are immediate consequences of the definitions.

11.2. Lemma. *$M(X) = 2$ if and only if every point of $\partial\Sigma$ is a flat spot. If this is the case, $M(X)$ is attained; and then $M(X)$ is [very] strongly attained if and only if $m(X) = 2$ is [very] strongly attained (i.e., X is [very] flat). $M(X) = 2$ is very strongly attained if and only if every point of $\partial\Sigma$ is a strictly flat spot.*

As for $D(X) = 2$: although spaces with this property exist [6], we shall not meet any instance in the body of this paper.

Before examining the upper bound 4 in (2.3), we state as a lemma a part of the proof of [5; Theorem 1].

11.3. Lemma. *If $p, q \in \partial\Sigma$, then $2\delta(p,q) \leqq \delta(-p,p) + 4$. If equality holds, there is a plane curve of length $\delta(p,q)$ from p to q in $\partial\Sigma$. In any case, $2D(X) \leqq M(X) + 4$.*

Proof. Let Y be a two-dimensional subspace of X containing p and q (it is unique if $p \pm q \neq 0$). By (2.1) and [3; Theorem 4.2],

$$\delta(-p,q) + \delta(p,q) \leqq \delta_Y(-p,q) + \delta_Y(q,p) = L(Y) \leqq 4, \tag{11.1}$$

where $L(Y)$ is half the length of the simple closed curve describing $\partial\Sigma(Y)$. Therefore

$$\delta(-p,p) \geqq \delta(p,q) - \delta(-p,q) \geqq 2\delta(p,q) - 4; \tag{11.2}$$

this proves the first part of the conclusion. The last part follows by taking suprema. If equality holds throughout (11.2), it must hold throughout (11.1); in particular, $\delta(p,q) = \delta_Y(p,q)$, and the shorter arc of $\partial\Sigma(Y)$ from p to q is the required plane curve.

As in [5], we say that $p \in \partial\Sigma$ is a <u>pole of</u> X (or <u>of</u> $\partial\Sigma$) if $\delta(-p,p) = 4$. The following proposition is a restatement of several results and remarks in [5].

11.4. <u>Theorem</u>. $D(X) = 4$ <u>if and only if</u> $M(X) = 4$. $D(X) = 4$ <u>is attained at</u> p,q <u>if and only if</u> $q = -p$ <u>and</u> p <u>is a pole</u>. $M(X) = 4$ <u>is attained at</u> p <u>if and only if</u> p <u>is a pole</u>. <u>The following statements are equivalent</u>:

(a): X <u>has a pole</u>;

(b): $D(X) = 4$ <u>is attained</u>;

(c): $D(X) = 4$ <u>is very strongly attained</u>;

(d): $M(X) = 4$ <u>is attained</u>;

(e): $M(X) = 4$ <u>is very strongly attained</u>;

(f): <u>Either</u> X <u>is congruent to</u> $R \oplus Z$ <u>for some</u> normed space Z, or there exists a convex cone V with apex 0 and a point a in X such that $\Sigma = (a-V) \cap (-a+V)$.

<u>Proof</u>. The first statement follows from (2.3) and Lemma 11.3. For the second, assume $\delta(p,q) = 4$; equality must hold in (11.1) with $\delta(-p,q) = 0$, i.e., $q = -p$; the conclusion follows, as does the next statement, from the definition of pole.
If p is a pole of X, Lemma 11.3 with $q = -p$ implies the existence of a plane curve of length 4 from $-p$ to p in $\partial\Sigma$. The equivalence of (a),(b),(c),(d),(e) follows. Statements (a)

and (f) are equivalent by [5; Theorem 4.1 and Corollary 4.2].

Remark. For our purposes, statement (f) will be used only as a sufficient condition for the existence of poles; this is the "easy" part of the proof of [5; Theorem 4.1]. We note that $D(X) = M(X) = 4$ does not in general imply the existence of a pole [5; Example 4.3].

12. Spaces of continuous functions, continued

We proceed with the study of the spaces introduced in Section 3, and especially of $C_\sigma(K)$ for a compact space K and an involutory automorphism σ of K. The crucial topological question concerning K and σ in the study of the girth was whether K^σ is scattered or not. Here we are concerned, instead, with the existence of certain closed-and-open sets in K.

A non-empty closed-and-open set $B \subset K$ is a σ-block if $B \cap \sigma B = \emptyset$; this implies $B \subset K^\sigma$. There are two extreme kinds of σ-block. Firstly, a singleton $\{t_0\}$ is a σ-block if and only if $t_0 \in K^\sigma$ is an isolated point of K or, equivalently, of K^σ; we remark that reference to a point of K^σ being isolated is unambiguous, since K^σ is open. At the other extreme, a σ-block splits K^σ if $B \cup \sigma B = K^\sigma$; if such a σ-block exists, we say that K^σ is split. The significance of these extreme cases will soon be made clear. Of course, both types of σ-block may be present simultaneously.

12.1. Lemma. *Assume that B is a σ-block, set $K' = K\backslash(B \cup \sigma B)$, and let σ' be the restriction of σ to K'. The linear mapping $f \mapsto f_B \oplus f' : C_\sigma(K) \to C(B) \oplus C_{\sigma'}(K')$ is a congruence, where f_B, f' are the restrictions of f to B, K', respectively. In particular, if $t_0 \in K^\sigma$ is an isolated point and $K' = K\backslash\{t_0, \sigma t_0\}$, with σ' as before, $C_\sigma(K)$ is congruent to $R \oplus C_{\sigma'}(K')$. If B splits K^σ, then $C_\sigma(K)$ is congruent to $C(B)$.*

Proof. Verification from the definitions. For the last statement, use (3.1); here the obvious convention $C(\emptyset) = \{0\}$ is used.

When dealing with $C_0(T)$ for a locally compact space T, the pertinent conditions are given as follows.

12.2. *Lemma. Assume that* T *is locally compact, and construct* K, σ *as in Lemma* 3.1. *Then* K *has a* σ-*block if and only if* T *has a compact component;* K^σ *has an isolated point if and only if* T *has an isolated point;* K^σ *is split if and only if* T *is compact.*

Proof. Under the construction of K and σ, the existence of a σ-block of K is equivalent to the existence of a non-empty compact open set in T. This implies the existence of a compact component of T. Conversely, suppose A is a compact component of T, and let V be a compact neighborhood of A. Then A is a component of V, and hence [1; Par. 47,II] the intersection of a class of sets closed-and-open in V. Since $A \subset \operatorname{int} V$, there is a finite class of these sets whose intersection B satisfies $A \subset B \subset \operatorname{int} V$. It is clear that B is compact, and also open in $\operatorname{int} V$, hence in T. The last two parts of the statement follow by direct verification.

12.3. *Theorem. The following function spaces have poles:* $C(T)$ *for every topological space* T; $C_\sigma(K)$ *for every compact space* K *and involutory automorphism* σ *such that* K^σ *is split or has an isolated point;* $C_0(T)$ *for every locally compact* *space* T *that is compact or has an isolated point. If* X *is any one of these function spaces,* $D(X) = 4$ *and* $M(X) = 4$ *are both very strongly attained.*

Proof. For given T, $\Sigma(C(T)) = (1-C_+) \cap (-1+C_+)$, where $C_+ = \{f \in C(T) : f \geq 0\}$ is the positive cone. By Theorem 11.4, $C(T)$ has a pole. The same conclusion then follows trivially for $C_0(T)$ if T is compact, and from Lemma 12.1 for $C_\sigma(K)$ if K^σ is split. If K^σ has an isolated point, Lemma 12.1 and Theorem 11.4 imply that $C_\sigma(K)$ has a pole, and if T has an isolated point the same conclusion then follows for $C_0(T)$ by Lemmas 3.1 and 12.2. The conclusions concerning $D(X)$ and $M(X)$ follow from Theorem 11.4.

In further investigation of the inner diameter and the perimeter, this theorem allows us to disregard spaces $C(T)$ (and also L^∞-spaces, in view of Theorem 10.1). When considering $C_\sigma(K)$ and $C_0(T)$ we may assume, when expedient, that K^σ is not split and T is not compact, and that K^σ and T lack isolated points, i.e., are dense-in-themselves.

13. The perimeter

Until further notice, K is a given compact space, σ a given involutory automorphism of K, and the notations Σ, $\partial\Sigma$, δ refer to the space $X = C_\sigma(K)$. We take up the computation of $\delta(-f,f)$ for $f\in\partial\Sigma$, begun in Lemmas 5.1 and 5.2 and Theorem 5.3, in terms of the number $\Lambda(f)$ defined there.

13.1. Theorem. Assume that K^σ is dense-in-itself. If $f\in\partial\Sigma$, then $\delta(-f,f) = 2 + \Lambda(f)$.

Proof. 1. Let $f\in\partial\Sigma$ be given, and set $\Lambda = \Lambda(f)$. If $\Lambda = 0$, Theorem 5.3 indeed gives $\delta(-f,f) = 2$. We may therefore assume that $\Lambda > 0$. In view of Lemma 5.1, it is sufficient to prove $\delta(-f,f) \leqq 2 + \Lambda$. We shall do this by constructing, for given $\varepsilon > 0$, a point $g\in\partial\Sigma$ with

(13.1) $\quad g(t_m) = f(t_m) = 1$ for some $t_m\in K^\sigma$,

(13.2) $\quad \|g-f\| \leqq \frac{1}{2}\Lambda + \varepsilon$,

(13.3) $\quad g(K) = [-1,1]$.

By (13.1), the straight-line segment with endpoints f, g lies entirely in $\partial\Sigma$. (13.2) then implies $\delta(g,f) \leqq \frac{1}{2}\Lambda + \varepsilon$. Theorem 5.3 and (13.3) yield $\delta(-g,g) = 2$. We conclude that $\delta(-f,f) \leqq \delta(-f,-g) + \delta(-g,g) + \delta(g,f) \leqq 2 + \Lambda + 2\varepsilon$. Since ε was arbitrarily small, the conclusion follows. The remainder of the proof consists in the construction of g.

2. Since $\|f\| = 1$, we may choose $t_m\in K^\sigma$ such that $f(t_m) = 1$.

By the definition of Λ there exists a finite set Q with $\{-1,1\} \subset Q = -Q \subset f(K^\sigma) \subset [-1,1]$, and such that the distance between consecutive points is never greater than Λ. (One possible construction: include in Q, firstly, -1 and 1; secondly, the odd integral multiples of $\frac{1}{2}\Lambda$ that lie in $f(K^\sigma)$; lastly, for each odd integral multiple of $\frac{1}{2}\Lambda$ that lies in $[-1,1]\backslash f(K^\sigma)$, the endpoints of the component of this set in which it lies.) Let $\varepsilon > 0$ be less than one-third the least distance between distinct points of Q, but otherwise arbitrary.

We conclude from these properties of Q and ε that there exists a positive integer n and points $t_i \in K^\sigma$, $i = 1,\dots,n$, such that

(13.4) $$1 = f(t_m) > f(t_1) > \dots > f(t_n) \geqq 0,$$

(13.5) $$3\varepsilon < 1 - f(t_1) \leqq \Lambda,\ 3\varepsilon < f(t_{i-1}) - f(t_i) \leqq \Lambda,\ i=2,\dots,n.$$

(13.6) $$f(t_n) = 0 \quad \text{or} \quad \varepsilon < \tfrac{3}{2}\varepsilon < f(t_n) \leqq \tfrac{1}{2}\Lambda$$

(here $Q = f(\{t_m, t_1, \dots, t_n, \sigma t_1, \sigma t_m\}))$. The set $f^{-1}((1-\varepsilon,1])$ is open and contains t_m. Since t_m is not an isolated point, we may choose $t_0 \in K^\sigma$ such that

(13.7) $$1 \geqq f(t_0) > 1 - \varepsilon,\quad t_0 \neq t_m;$$

we may then replace (13.5) by

(13.8) $$2\varepsilon < f(t_{i-1}) - f(t_i) \leqq \Lambda,\quad i = 1,\dots,n.$$

Let U_i be an open neighborhood of t_i such that

(13.9) $$f(U_i) \subset f(t_i) + (-\epsilon,\epsilon), \ i = 0,\ldots,n;$$

Assume in addition, as we may by (13.7), that $t_m \notin U$; also, if $f(t_n) = 0$, that $U_n \cap \sigma U_n = \emptyset$. It follows from the construction and from (13.6) and (13.8) that the sets $U_0,\ldots,U_n,\sigma U_0,\ldots,\sigma U_n$ are pairwise disjoint; let V be the complement of their union in K; then V is compact and $t_m \in V$.

Let V_i be a compact neighborhood of t_i such that $V_i \subset U_i$, $i = 0,\ldots,n$. Since each V_i is a neighborhood in the dense-in-itself set K^σ, it is not scattered. By Theorem 4.1 there exist functions $v_i \in C(V_i)$, $i=0,\ldots,n$, such that

(13.10) $$\begin{aligned} v_0(V_0) &= [\tfrac{1}{2}f(t_0) + \tfrac{1}{2}f(t_1),1] \\ v_i(V_i) &= [\tfrac{1}{2}f(t_i) + \tfrac{1}{2}f(t_{i+1}),\tfrac{1}{2}f(t_i) + \tfrac{1}{2}f(t_{i-1})], \\ &\qquad\qquad i = 1,\ldots,n-1, \\ v_n(V_n) &= [0,\tfrac{1}{2}f(t_n) + \tfrac{1}{2}f(t_{n-1})]. \end{aligned}$$

From (13.6),(13.7),(13.8) it follows that

(13.11) $$v_i(V_i) \subset [-1,1] \cap (f(t_i) + [-\tfrac{1}{2}\Lambda,\tfrac{1}{2}\Lambda]), \ i = 0,\ldots,n.$$

Consider the pairwise disjoint compact sets $V,V_0,\ldots,V_n$, and their union W; we define $w_0 \in C(W)$ by

(13.12) $$w_0(t) = \begin{cases} 0 & t \in V \\ v_i(t)-f(t) & t \in V_i \qquad i = 0,\ldots,n. \end{cases}$$

We observe that $-w_0(\sigma t) = w_0(t) = 0$ for $t \in W \cap \sigma W = V$; by (13.9) and (13.11), $\|w_0\| \leqq \frac{1}{2}\Lambda + \varepsilon$. We construct a skew Tietze extension w of w_0, and set $g = \mathrm{mid}(f+w,1,-1)$. Certainly $g \in C_\sigma(K)$, by Lemma 3.2, and $\|g\| \leqq 1$. Now $g - f = \mathrm{mid}(w,1-f,-1-f)$, so that $\|g-f\| \leqq \|w\| = \|w_0\| \leqq \frac{1}{2}\Lambda + \varepsilon$, and (13.2) is verified. Since $t_m \in V$, we have $g(t_m) = f(t_m) - 0 = 1$, and (13.1) holds. On V_i, g coincides with $\mathrm{mid}(v_i,1,-1) = v_i$ (by (13.11)); therefore, using the fact that $g \in C_\sigma(K)$, we conclude from (13.10) that

$$[-1,1] \supset g(K) \supset \left(-\bigcup_0^n v_i(V_i)\right) \cup \left(\bigcup_0^n v_i(V_i)\right) =$$

$$= [-1,0] \cup [0,1] = [-1,1].$$

Equality must hold, so that $g \in \partial\Sigma$ and (13.3) is verified.

13.2. <u>Theorem. Assume that K^σ is dense-in-itself and not split. If $f \in \partial\Sigma$, then $\delta(-f,f) \leqq 3$, with equality if and only if $f(K) = \{-1,0,1\}$. When equality holds, there is a curve of length 3 from $-f$ to f in $\partial\Sigma$.</u>

<u>Proof</u>. For every $\varepsilon > 0$ the set $f(K^\sigma) \cap (-\varepsilon,\varepsilon)$ is not empty; for otherwise $f^{-1}([\varepsilon,1])$ is a σ-block that splits K^σ. Therefore $\Lambda(f) \leqq 1$, with equality exactly when $f(K^\sigma) = \{-1,0,1\}$; but this last condition is equivalent to $f(K) = \{-1,0,1\}$, since we have just ruled out $f(K^\sigma) = \{-1,1\}$. Theorem 13.1 then yields the conclusion, except for the existence of the curve. This existence will follow from the more general result given in Theorem 14.3, but we give a direct proof here in view of its simplicity.

Assume that $f(K) = \{-1,0,1\} = f(K^\sigma)$. Set $B = f^{-1}(\{1\})$, $K' = f^{-1}(\{0\})$. Then B is a σ-block and, by Lemma 12.1, there is a congruence $C_\sigma(K) \rightarrow C(B) \oplus C_{\sigma'}(K')$ under which f corresponds to $1 \oplus 0$. Since B is not a singleton, there exists $h \in C(B)$ with $\{0,1\} \subset h(B) \subset [0,1]$; since $K'^{\sigma'} = K' \cap K^\sigma \neq \emptyset$, there exists $g \in \partial\Sigma(C_{\sigma'}(K'))$. It is then verified by inspection that the polygon with consecutive vertices $-1 \oplus 0$, $(h-1) \oplus g$, $h \oplus g$, $1 \oplus 0$ is a curve of length 3 from $-1 \oplus 0$ to $1 \oplus 0$ in $\partial\Sigma(C(B) \oplus C_{\sigma'}(K'))$; under the congruence, it corresponds to the required curve of length 3 from $-f$ to f in $\partial\Sigma$.

13.3. <u>Theorem</u>. <u>If</u> K <u>has no</u> σ-<u>block, then</u> $M(C_\sigma(K)) = 2$ <u>is strongly attained; and it is very strongly attained if and only if</u> $m(C_\sigma(K)) = 2$ <u>is very strongly attained</u>. <u>If</u> K <u>has a</u> σ-<u>block, then</u> $M(C_\sigma(K))$ <u>is very strongly attained; it is equal to</u> 3 <u>if</u> K^σ <u>is dense-in-itself and not split, and equal to</u> 4 <u>otherwise</u>.

<u>Proof</u>. If $f \in \partial\Sigma$ and $\Lambda(f) > 0$, there exists $a \in [-1,1] \backslash f(K)$. Then $f^{-1}((|a|,1])$ is a σ-block. Consequently, if K has no σ-block, $\Lambda(f) = 0$ for all $f \in \partial\Sigma$; by Theorem 5.3, all $f \in \partial\Sigma$ are flat spots. By Theorem 5.4, $m(C_\sigma(K)) = 2$ is strongly attained. The conclusion of the present theorem for this case then follows from Lemma 11.2.

If K^σ is not dense-in-itself or is split, $M(C_\sigma(K)) = 4$ is very strongly attained, by Theorem 12.3. Assume then that K^σ is dense-in-itself and not split. By Theorem 13.2, $M(C_\sigma(K)) \leq 3$; and, according to the same theorem, equality will hold, with $M(C_\sigma(K)) = 3$ very strongly attained, provided there exists $f \in C_\sigma(K)$ with $f(K) = \{-1,0,1\}$. But if B is a σ-block of K, $f = \chi_B - \chi_{\sigma B}$ satisfies these conditions, since B does not split K^σ.

We can translate the last two theorems into statements about $C_0(T)$, by means of Lemmas 3.1 and 12.2.

13.4. Theorem. *Assume that* T *is locally compact and dense-in-itself, but not compact. If* $f \in \partial\Sigma(C_0(T))$, *then* $\delta_{C_0(T)}(-f,f) \leqq 3$, *with equality if and only if* $|f|(T) = \{0,1\}$. *When equality holds, there is a curve of length* 3 *from* $-f$ *to* f *in* $\partial\Sigma(C_0(T))$.

13.5. Theorem. *Assume that* T *is locally compact. If* T *has no compact component, then* $M(C_0(T)) = 2$ *is strongly attained; and it is very strongly attained if and only if* $m(C_0(T)) = 2$ *is very strongly attained. If* T *does have a compact component, then* $M(C_0(T))$ *is very strongly attained; it is equal to* 3 *if* T *is dense-in-itself and not compact, and equal to* 4 *otherwise.*

14. The inner diamater.

We again assume that K is a given compact space and σ a given involutory automorphism, and that the notations $\Sigma, \partial\Sigma, \delta$ refer to the space $X = C_\sigma(K)$. Our aim in this section is to show that, except for the cases covered by Theorem 12.3, the inner diameter of $\partial\Sigma$ is always 3 and is very strongly attained. It is the proof of the latter assertion that turns out to be surprisingly intricate. We preface the main argument with results for two special cases.

14.1. Lemma. *Assume that* $f,g \in \partial\Sigma$ *and* $f(K) = [-1,1]$. *Then there exists a curve of length* ≤ 3 *from* f *to* g *in* $\partial\Sigma$.

Proof. By Theorem 5.3, f is a flat spot, i.e., $\delta(-f,f) = 2$. Lemma 11.3 then yields $\delta(f,g) \leq \frac{1}{2}(\delta(-f,f)+4) = 3$; and if equality holds, there exists a plane curve of length 3 from f to g in $\partial\Sigma$. If $\delta(f,g) < 3$, the existence of a curve as required in the statement follows from the definition of δ.

14.2. Lemma. *Assume that* $f,g \in \partial\Sigma$ *and that the following conditions are both satisfied.*

(a): *there exist* $t_1, t_2 \in K^\sigma$ *such that* $t_2 \neq \sigma t_1$ *and* $f(t_1) = g(t_2) = 1$;

(b): $f(t)g(t) \geq 0$ for some $t \in K^\sigma$.

Then there exists a curve of length $\leq 3-\rho$ *from* f *to* g *in* $\partial\Sigma$, *where* $\rho = \max\{\min\{f(t),g(t)\} : t \in K,\ f(t) \geq 0,\ g(t) \geq 0\} \geq 0$.

Proof. 1. On account of (b), ρ is well defined, and there is a point $t_0 \in K^\sigma$ such that

(14.1) $$0 \leqq \rho = \min\{f(t_0), g(t_0)\} \leqq 1.$$

2. Assume first that $g(t_1) = \rho$. By assumption (a) there exists a function $z \in C_\sigma(K)$ such that $\|z\| = z(t_1) = z(t_2) = 1 - \rho \geqq 0$ (use a skew Tietze extension; $t_1 = t_2$ is not excluded: it would imply $\rho=1, z=0$). Set $h = \mathrm{mid}(g+z, 1, -1) \in C_\sigma(K)$ (Lemma 3.2). Then $h(t_1) = h(t_2) = \|h\| = 1$. Since $f(t_1) = h(t_1) = h(t_2) = g(t_2) = 1$, the polygon p with consecutive vertices f, h, g is a curve from f to g in $\partial\Sigma$. Now $g - h = \mathrm{mid}(-z, -1+g, 1+g)$; therefore $\ell(p) = \|h-f\| + \|g-h\| \leqq \|h\| + \|f\| + \|z\| = 1 + 1 + (1-\rho) = 3 - \rho$, as required.

The same proof, with f, g interchanged, is applicable if $f(t_2) = \rho$.

3. We may assume in the rest of the proof that $f(t_2), g(t_1) < \rho \leqq 1$; this implies that t_0, t_1, t_2 are pairwise distinct. Since $f(\sigma t_1) = g(\sigma t_2) = -1$, (14.1) and assumption (a) then imply that $t_0, t_1, t_2, \sigma t_0, \sigma t_1, \sigma t_2$ are pairwise distinct.

We define $f_\rho = \mathrm{mid}(f, \rho, -\rho)$ and $g_\rho = \mathrm{mid}(g, \rho, -\rho)$; by Lemma 3.2, $f_\rho, g_\rho \in C_\sigma(K)$. (14.1) implies

(14.2) $$\begin{aligned} &f_\rho(t_0) = f_\rho(t_1) = \|f_\rho\| = g_\rho(t_0) = g_\rho(t_2) = \|g_\rho\| = \rho \\ &(f-f_\rho)(t_1) = \|f-f_\rho\| = (g-g_\rho)(t_2) = \|g-g_\rho\| = 1 - \rho. \end{aligned}$$

Let U_1, U_2, U_3 be open neighborhoods of t_0, t_1, t_2, respectively, such that the six sets $U_i, \sigma U_i, i = 0,1,2$ are pairwise disjoint and such that

$$(14.3) \qquad f(U_1), g(U_2) \subset [0,1].$$

There exist $z_0, z_1, z_2 \in C_\sigma(K)$ such that

$$(14.4) \quad z_i(t_i) = \|z_i\| = 1-\rho, \qquad z_i(U_i) \subset [0,1-\rho],$$
$$z_i(K \setminus (U_i \cup \sigma U_i)) = \{0\}, \ i = 0,1,2.$$

We set $z_0' = \mathrm{mid}(\mathrm{mid}(z_0, 1-\rho+f-f_\rho, -(1-\rho)+f-f_\rho), 1-\rho+g-g_\rho, -(1-\rho)+g-g_\rho) \in C_\sigma(K)$. By (14.1),(14.2),(14.4) we have

$$(14.5) \quad \begin{aligned} &z_0'(t_0) = \|z_0'\| = 1-\rho, \qquad z_0'(K \setminus (U_0 \cup \sigma U_0)) = \{0\}. \\ &(z_0'+f_\rho-f)(U_0), (z_0'+g_\rho-g)(U_0) \subset [-(1-\rho), 1-\rho]. \end{aligned}$$

We set $h_1 = z_1 + z_0' + f_\rho$, $h_2 = z_2 + z_0' + g_\rho$. Since z_0', z_1, z_2 have pairwise disjoint supports, (14.4), (14.2),(14.5) imply $1 = \rho + (1-\rho) = h_1(t_0) \leqq \|h_1\| \leqq \|f_\rho\| + \max\{\|z_0'\|, \|z_1\|\} = \rho + (1-\rho) = 1$, so that equality holds; and likewise $h_2(t_0) = \|h_2\| = 1$. Thus $h_1, h_2 \in \partial\Sigma$, and the same formulas imply $f(t_1) = h_1(t_1) = h_1(t_0) = h_2(t_2) = g(t_2) = 1$. Therefore the polygon p with consecutive vertices f, h_1, h_2, g is a curve from f to g in $\partial\Sigma$. We proceed to estimate its length.

Now $h_1 - f$ coincides with $z_0' + f_\rho - f$ on $U_0 \cup \sigma U_0$, and with $f_\rho - f$ elsewhere. By (14.2),(14.3),(14.4) we find $(z_1+f_\rho)(U_1) \subset z_1(U_1) - (f_\rho - f)(U_1) \subset [0,1-\rho] - [0,1-\rho] = [-(1-\rho), 1-\rho]$. Combining this with (14.2) and (14.5) we find $\|h_1 - f\| \leqq 1 - \rho$.

In a similar fashion we find $\|g-h_2\| \leqq 1 - \rho$. Finally, $h_2 - h_1 = z_2 - z_1 + g_\rho - f_\rho$; since z_1, z_2 have disjoint supports, $\|h_2-h_1\| \leqq \|f_\rho\| + \|g_\rho\| + \max\{\|z_1\|, \|z_2\|\} = 2\rho + (1-\rho) = 1 + \rho$. Thus

$$\ell(p) = \|h_1-f\| + \|h_2-h_1\| + \|g-h_z\| \leqq (1-\rho) + (1+\rho) + (1-\rho) = 3 - \rho,$$

as required.

14.3. Theorem. *Assume that* K^σ *is dense-in-itself and not split. If* $f, g \in \partial\Sigma$, *there exists a curve of length* ≤ 3 *from* f *to* g *in* $\partial\Sigma$.

Remark. Under the assumptions of the theorem, it is easy to show that, for given $\epsilon > 0$, there exists $f' \in \partial\Sigma$ such that $\delta(f,f') \leqq \varepsilon$ and such that f', g satisfy the assumptions of Lemma 14.2. It follows at once that $\delta(f,g) \leqq 3$. The difficulty of the following proof therefore lies in the actual construction of a curve of length not exceeding 3.

Proof. We consider the non-empty compact sets $T(f) = \{t \in K : f(t) = 1\}$, $T(-g) = \{t \in K : -g(t) = 1\}$ of K^σ, and the following exhaustive set of mutually exclusive possibilities:

(A): $T(f) \cap T(-g) = \emptyset$, i.e., $\|f-g\| < 2$;

(B): $T(f) \cap T(-g) \neq \emptyset$, but not both sets are singletons;

(C): $T(f) = T(-g) = \{t_1\}$ for some $t_1 \in K^\sigma$.

On the other hand, consider the following exhaustive set of mutually exclusive possibilities:

(X): $f(t)g(t) > 0$ for some $t \in K^\sigma$;

(Y): $f(t)g(t) \leqq 0$ for all $t \in K$, with equality for some $t \in K^\sigma$;

(Z): $f(t)g(t) < 0$ for all $t \in K^\sigma$.

Now (A) $\vee$ (B) is condition (a) of Lemma 14.2, and (X)$\vee$(Y) is condition (b) of the same lemma; in cases (AX),(AY),(BX),(BY) the conclusion of the theorem then follows from Lemma 14.2

<u>Case</u> (CX). With ρ defined as in Lemma 14.2, condition (X) and (14.1) imply $\rho > 0$. Choose $\varepsilon, 0 < \varepsilon < \rho$. The open set $g^{-1}((1-\varepsilon,1])$ contains σt_1; but this point is not isolated; therefore there exists $t_2 \in K^\sigma$ such that $\sigma t_1 \neq t_2$ and $1 - \varepsilon < g(t_2) < 1$. Set $g' = \mathrm{mid}((1+\varepsilon)g,1,-1) \in C_\sigma(K)$; then $g'(\sigma t_1) = g'(t_2) = \|g'\| = 1$. If $t_0 \in K^\sigma$ satisfies (14.1), we have $\min\{f(t_0),g'(t_0)\} \geqq \rho$. By Lemma 14.2, $\delta(f,g') \leqq 3 - \rho$. But $g'(\sigma t_1) = g(\sigma t_1) = 1$; therefore the straight-line segment with endpoints g',g lies in $\partial\Sigma$, and $\delta(g',g) = \|g-g'\| \leqq \varepsilon$, so that $\delta(f,g) \leqq \delta(f,g') + \delta(g',g) \leqq 3 - \rho + \varepsilon < 3$. The conclusion follows from the definition of δ.

<u>Case</u> (AZ). By condition (A) there exists $\varepsilon, 0 < \varepsilon < \frac{1}{2}$ such that the open sets $U_1 = f^{-1}((1-\varepsilon,1])$ and $U_2 = g^{-1}((1-\varepsilon,1])$ satisfy $U_1 \cap \sigma U_2 = \emptyset$. By the choice of ε and by condition (Z), the four sets $U_1, U_2, \sigma U_1, \sigma U_2$ are pairwise disjoint. We choose $t_1 \in T(f) \subset U_1$, $t_2 \in \sigma T(-g) \subset U_2$.

Since f,g vanish nowhere on K^σ, we have

$\eta = \min\{\varepsilon, \min\{|f(t)| : g(t) \geqq 1-\varepsilon\}, \min\{|g(t)| : f(t) \geqq 1-\varepsilon\}\} > 0$.

As shown at the beginning of the proof of Theorem 13.2, the set $g^{-1}((-\eta,\eta)) \cap K^{\sigma}$ is not empty; let t_0 be a point in it. By assumption (Z) we have $f(t_0)g(t_0) < 0$; if $|f(t_0)| < |g(t_0)|$, we interchange f and g; further replacing t_0 by σt_0, if necessary, we may assume

$$(14.6) \qquad -\varepsilon \leqq -\eta < g(t_0) < 0 < |g(t_0)| \leqq f(t_0).$$

By the definition of η, t_0 does not lie in $U_1, U_2, \sigma U_1$, or σU_2. We may choose an open neighborhood U_0 of t_0 so small that $U_0 \cap \sigma U_0 = \emptyset$ and, on account of (14.6),

$$(14.7) \qquad g(U_0) \subset (-\eta, 0), \quad f(U_0) \subset (0, 1-\varepsilon).$$

It follows from these conditions that the six sets $U_i, \sigma U_i, i=0,1,2$, are pairwise disjoint. There exists $z \in C_\sigma(K)$ satisfying

$$(14.8) \quad z(t_0) = \|z\| = 1, \quad z(U_0) \subset [0,1], \quad z(K\setminus(U_0 \cup \sigma_0)) = \{0\}.$$

We set $z' = \mathrm{mid}(\mathrm{mid}(z, 1-f(t_0)+f, -(1-f(t_0))+f), 1-g(t_0)+g, -(1-g(t_0))+g) \in C_\sigma(K)$ (Lemma 3.2), and find, using (14.6), (14.8),

$$z'(t_0) = \|z'\| = 1,$$

$$(14.9) \quad (z'-f)(U_0) \subset [-(1-f(t_0)), 1-f(t_0)]$$

$$(z'-g)(U_0) \subset [-(1-g(t_0)), 1-g(t_0)] = [-(1+|g(t_0)|), 1+|g(t_0)|]$$

Define $\varphi \in C_\pi([-1,1])$ by

$$\varphi(t) = -\varphi(-t) = \begin{cases} 0 & 0 \leqq t \leqq 1-\varepsilon \\ 1-\varepsilon^{-1}(1-t) & 1-\varepsilon \leqq t \leqq 1, \end{cases}$$

and set $h_1 = z' + (\varphi \circ f), h_2 = z' + (\varphi \circ g)$, both elements of $C_\sigma(K)$. Now

$$(14.10) \quad \begin{aligned} \|\varphi \circ f\| &= (\varphi \circ f)(t_1) = 1, \quad (\varphi \circ f)(U_1) \subset [0,1], \quad (\varphi \circ f)(K\setminus(U_1 \cup \sigma U_1)) = \{0\} \\ \|\varphi \circ g\| &= (\varphi \circ g)(t_2) = 1, \quad (\varphi \circ g)(U_2) \subset [0,1], \quad (\varphi \circ g)(K\setminus(U_2 \cup \sigma U_2)) = \{0\}. \end{aligned}$$

It follows that $h_1(t_1) = h_1(t_0) = \|h_1\| = h_2(t_2) = h_2(t_0) = \|h_2\| = 1$, so that $h_1, h_2 \in \partial\Sigma$. Since also $f(t_1) = g(t_2) = 1$, the polygon p with consecutive vertices f, h_1, h_2, g is a curve from f to g in $\partial\Sigma$. We estimate its length.

Now $h_1 - f$ coincides with $z' - f$ in $U_0 \cup \sigma U_0$, and with $\varphi \circ f - f = (\varphi - u) \circ f$ elsewhere (here $u \in C_\pi([-1,1])$ is given, as in Section 5, by $u(t) = t$, $t \in [-1,1]$). Using (14.9), we find $\|h_1 - f\| \leqq \max\{\|z' - f\|, \|(\varphi - u) \circ f\|\} \leqq \leqq \max\{1 - f(t_0), \|\varphi - u\|\} = 1 - \min\{f(t_0), \varepsilon\}$. Similarly, $\|g - h_2\| \leqq \max\{\|z' - g\|, \|\varphi - u\|\} \leqq 1 + |g(t_0)|$. Finally, $h_2 - h_1$ coincides with $-(\varphi \circ f)$ on $U_1 \cup \sigma U_1$ and with $\varphi \circ g$ on $U_2 \cup \sigma U_2$ and is zero elsewhere. Therefore (14.10) yields $\|h_2 - h_1\| = 1$. Then we compute, using (14.6),

$$\ell(p) = \|h_1 - f\| + \|h_2 - h_1\| + \|g - h_2\| \leqq$$

$$\leqq (1 - \min\{f(t_0), \varepsilon\}) + 1 + (1 + |g(t_0)|) \leqq 3,$$

as required.

<u>Cases</u> (BZ), (CY), (CZ). 1. If either $f(K)$ or $g(K)$ is $[-1,1]$, Lemma 14.1 gives the desired conclusion. We shall therefore assume in the remainder of the proof that

$$(14.11) \qquad f(K) \neq [-1,1] \neq g(K),$$

and under this assumption we shall prove $\delta(f,g) < 3$; the conclusion then follows from the definition of δ.

Under assumption (B) or (C) there exists $t_1 \in K^\sigma$ such that $f(t_1) = -g(t_1) = 1$. Under assumption (Y) or (Z), the functions $f, -g$ are nowhere of strictly opposite signs; therefore "$f(t) > 0$ or $-g(t) > 0$" is equivalent to "$(f-g)(t) > 0$".

We consider the class $\underset{\sim}{B} = \{B \subset K : B$ closed-and-open, $t_0 \in B$, $(f-g)(B) \subset (0,1]\}$. We shall see in a moment that $\underset{\sim}{B}$ is not empty. Meanwhile, we observe that $B \in \underset{\sim}{B}$ implies that B is a σ-block; since K^σ is not split, $K^\sigma \setminus (B \cup \sigma B) \neq \emptyset$. Further, $\underset{\sim}{B}$ is closed under non-empty finite unions and intersections. By these remarks, the numbers

$$\alpha_1(B) = \max\{f(t) : t \in K \setminus (B \cup \sigma B)\}, \qquad \omega_1(B) = \min\{f(t) : t \in B\},$$
$$\alpha_2(B) = \max\{-g(t) : t \in K \setminus (B \cup \sigma B)\}. \qquad \omega_2(B) = \min\{-g(t) : t \in B\}$$

are well defined for each $B \in \underset{\sim}{B}$, and lie in $[0,1]$.

By (14.11) and the fact that $f(K) = -f(K)$ and $g(K) = -g(K)$, there exist $s_1, s_2 \in (0,1)$ such that $s_1 \notin f(K)$, $s_2 \notin -g(K)$. Therefore the sets $B_1 = f^{-1}((s_1,1])$, $B_2 = (-g)^{-1}((s_2,1])$ belong to $\underset{\sim}{B}$, which is thus non-empty; and we also find

$$\alpha_i(B_i) < s_i < \omega_i(B_i), \quad i = 1,2. \tag{14.12}$$

We set $\alpha_{0i} = \inf\{\alpha_i(B) : B \in \underset{\sim}{B}\}$, $\omega_{0i} = \sup\{\omega_i(B) : B \in \underset{\sim}{B}\}$, $i = 1,2$. It follows from (14.12) that $\alpha_{0i} < \omega_{0i}$, $i = 1,2$.

2. Choose ε, $0 < \varepsilon < \frac{1}{3}\min\{\omega_{01}-\alpha_{01}, \omega_{02}-\alpha_{02}\}$. Then there exist $P_i, Q_i \in \underset{\sim}{B}$, $i = 1,2$, such that $\alpha_{0i} \leqq \alpha_i(P_i) \leqq \alpha_{0i} + \varepsilon$, $\omega_{0i} - \varepsilon \leqq \omega_i(Q_i) \leqq \omega_{0i}$, $i = 1,2$. Set $P = P_1 \cup P_2 \cup Q_1 \cup Q_2$, $Q = Q_1 \cap Q_2$. Then $P, Q \in \underset{\sim}{B}$, $Q \subset P$, and

$$\alpha_{0i} \leqq \alpha_i(P) \leqq \alpha_{0i} + \varepsilon, \quad \omega_{0i} - \varepsilon \leqq \omega_i(Q) \leqq \omega_{0i}, \quad i = 1,2. \tag{14.13}$$

Replacing f,g by $-g,-f$, if necessary, we may assume that

$$(14.14)\qquad \omega_1(Q) - \alpha_1(P) \leqq \omega_2(Q) - \alpha_2(P).$$

Now $A = P\setminus Q$ is empty or a σ-block. Set $K' = K\setminus(A\cup\sigma A) = K\setminus(P\cup\sigma P) \cup Q \cup \sigma Q$, and let σ' be the restriction of σ to K'. Let $\Sigma', \partial\Sigma', \delta'$ refer to $C_{\sigma'}(K')$. Let f', g' be the restrictions of f,g, respectively, to K'; then $f',g' \in C_{\sigma'}(K')$. Now $t_1 \in Q \subset K'$, and $f'(t_1) = -g'(t_1) = 1$. Therefore $f',g' \in \partial\Sigma'$. By Lemmas 12.1 and 11.1 if $A \neq \emptyset$, and trivially otherwise, we have $\delta(f,g) \leqq \max\{\delta'(f',g'),2\}$. <u>It will therefore be sufficient to prove</u> $\delta'(f',g') < 3$.

Since $K'^{\sigma'} = K' \cap K^{\sigma}$ is open in K^{σ}, it is dense-in-itself; since it contains t_1, it is not empty. We claim that it is not split: if B' were a σ'-block that splits $K'^{\sigma'}$, it is immediately verified that $A \cup B'$ would be a σ-block in K that splits K^{σ}, a contradiction.

3. We now work in $C_{\sigma'}(K')$. We claim that

$$(14.15)\qquad \Lambda(f') = \omega_1(Q) - \alpha_1(P).$$

By the construction of K' we indeed have $f'(K'^{\sigma'}) \cap$ $\cap\ (\alpha_1(P), \omega_1(Q)) = \emptyset$, so that

$$(14.16)\qquad \Lambda(f') \geqq \omega_1(Q) - \alpha_1(P).$$

On the other hand, since $K'^{\sigma'}$ is not split, the argument at the beginning of the proof of Theorem 13.2 shows that $\Lambda(f') = b - a$ for some interval $(a,b) \subset (0,1)$

with $f'(K'^{\sigma'}) \cap (a,b) = \emptyset$. Since f' does take the value $\alpha_1(P)$ on $K^{\sigma}\setminus(P\cup\sigma P) \subset K'^{\sigma'}$ and the value $\omega_1(Q)$ on $Q \subset K'^{\sigma'}$, the interval (a,b) must either coincide with $(\alpha_1(P),\omega_1(Q))$ or be disjoint from it; the latter alternative will now be disproved, and this will establish the claimed validity of (14.15).

Set $B' = f'^{-1}((a,1])$; then B' is closed-and-open. If $0 \leqq a < b \leqq \alpha_1(P)$, we find that $P \cup B' \in \underset{\sim}{B}$, since $f(P\cup B') \subset [\alpha_1(P),1] \cup [b,1] \subset (0,1]$, the other conditions being trivially verified. The definition of B' also implies $\alpha_1(P\cup B') = a$. Then (14.13) implies

$$\Lambda(f') = b-a \leqq \alpha_1(P)-\alpha_1(P\cup B') \leqq \alpha_1(P)-\alpha_{01} \leqq \varepsilon < \tfrac{1}{3}(\omega_{01}-\alpha_{01}) \leqq$$

$$\leqq \omega_1(Q) - \alpha_1(P),$$

contradicting (14.16). If, on the other hand, $\omega_1(Q) \leqq a < b \leqq 1$, we find that $B'\subset Q$, $B'\in\underset{\sim}{B}$, $\omega_1(B') = b$. Then

$$\Lambda(f') = b-a \leqq \omega_1(B') - \omega_1(Q) \leqq \omega_{01} - \omega_1(Q) \leqq \varepsilon < \tfrac{1}{3}(\omega_{01}-\alpha_{01}) \leqq$$

$$\leqq \omega_1(Q) - \alpha_1(P),$$

again contradicting (14.16). Thus (14.15) is proved.

By Theorem 13.1 and (14.15) we know that $\delta'(-f',f') = 2 + \Lambda(f') = 2 + \omega_1(Q) - \alpha_1(P)$. To prove that $\delta'(f',g') = \delta(-f',-g') < 3$ it will suffice to prove that $\delta'(f',-g) < 1 - (\omega_1(Q) - \alpha_1(P))$. Since $f'(t_1) = -g'(t_1) = 1$, the straight-line segment with endpoints $f',-g'$ lies in $\partial\Sigma'$; thus $\delta'(f',-g') = \|f'+g'\|$, and <u>all we have to prove is</u>

(14.17) $$\|f'+g'\| < 1 - (\omega_1(Q) - \alpha_1(P)).$$

4. Now f and $-g$ are nowhere of strictly opposite signs, so

(14.18) $$\|f'+g'\| = \max\{\mu',\mu''\}$$

where

(14.19) $$\mu' = \max\{|f(t)-(-g(t))|:t\in K\setminus(P\cup\sigma P)\} \leqq \max\{\alpha_1(P),\alpha_2(P)\}$$

(14.20) $$\mu'' = \max\{|f(t)-(-g(t))|:t\in Q\} \leqq \max\{1-\omega_1(Q),1-\omega_2(Q)\}.$$

Suppose $\alpha_1(P),\alpha_2(P) > 0$; if equality holds in (14.19), there exists $t_0\in K\setminus(P\cup\sigma P)$ such that either $f(t_0) = \alpha_1(P) > 0$, $-g(t_0) = 0$, or $-g(t_0) = \alpha_2(P) > 0$, $f(t_0) = 0$; in either case, $t_0\in K^\sigma$, and condition (Z) is excluded in favour of (Y). The same conclusion is reached if $\alpha_1(P)$ or $\alpha_2(P)$ is 0, since then either f or $-g$ vanishes in all $K^\sigma\setminus(P\cup\sigma P) \neq \emptyset$. <u>Thus condition</u> (Z) <u>requires</u> $\alpha_1(P),\alpha_2(P) > 0$ <u>and strict inequality in</u> (14.19).

Likewise, suppose $\omega_1(Q),\omega_2(Q) < 1$; if equality holds in (14.20), there exists $t_0\in Q$ such that either $f(t_0) = \omega_1(Q) < 1$, $-g(t_0) = 1$, or $-g(t_0) = \omega_2(Q) < 1$ and $f(t_0) = 1$; in either case $t_0 \neq t_1$, and either $T(-g)$ or $T(f)$ is not a singleton; this excludes condition (C) in favor of (B). The same conclusion is reached if $\omega_1(Q)$ or $\omega_2(Q)$ is 1, since then Q, which is not a singleton (there are no isolated points in K^σ), is contained in either $T(f)$ or $T(-g)$. Thus <u>condition</u> (C) <u>requires</u> $\omega_1(Q),\omega_2(Q) < 1$ <u>and strict inequality in</u> (14.20).

We substitute (14.19) and (14.20) in (14.18); using (14.14), we find

$$(14.21)\quad \|f'+g'\| = \max\{\mu',\mu''\} \leqq \max\{\alpha_1(P), 1-\omega_1(Q), \alpha_2(P), 1-\omega_2(Q)\} \leqq$$
$$\leqq \max\{1-(\omega_1(Q)-\alpha_1(P)), 1-(\omega_2(Q)-\alpha_2(P))\} =$$
$$= 1-(\omega_1(Q)-\alpha_1(P)).$$

To prove (14.17), and thereby complete the proof for the cases under scrutiny, it remains to exclude equality between extreme members of (14.21).

Suppose equality does hold. Assume first that $\|f'+g'\| = = \mu' = \alpha_i(P) = 1-(\omega_1(Q)-\alpha_1(P))$ for $i = 1$ or $i = 2$. By (14.14) we then have $1 \geqq \omega_i(Q) = \omega_i(Q) - \alpha_i(P) + \alpha_i(P) = (\omega_i(Q) - \alpha_i(P)) + 1 - (\omega_1(Q) - \alpha_1(P)) \geqq 1$, so that equality holds, and $\omega_1(Q) = 1$; since equality also holds in (14.19), neither (C) nor (Z) holds, contrary to the assumption for the present cases. If, on the other hand, $\|f'+g'\| = \mu'' = 1 - \omega_i(Q) = 1 - (\omega_1(Q) - \alpha_1(P))$ for $i = 1$ or 2, we again conclude, from (14.14), that $0 \leqq \alpha_i(P) = \omega_i(Q) - (\omega_i(Q) - \alpha_i(P)) = (\omega_1(Q) - \alpha_1(P)) - - (\omega_i(Q) - \alpha_i(P)) \leqq 0$, so that $\alpha_i(P) = 0$ and equality holds in (14.20); thus (C) and (Z) are again both excluded, contrary to the assumption. Thus equality cannot hold between members in (14.21), and (14.17) is proved.

14.4. Theorem. *If* K^σ *is dense-in-itself and not split, then* $D(C_\sigma(K)) = 3$; *otherwise* $D(C_\sigma(K)) = 4$. *In either case,* $D(C_\sigma(K))$ *is very strongly attained.*

Proof. We may assume that K^{σ} is dense-in-itself and not split, since the other cases are covered by Theorem 12.3. The conclusion under these assumptions is then evident from Theorem 14.3, provided we show that there exist $f,g \in \partial\Sigma$ with $\delta(f,g) \geq 3$.

Since K^{σ} is dense-in-itself, it is not scattered, and there exists, by Theorem 5.4, a function $h \in C_{\sigma}(K)$ with $h(K) = [-1,1]$. Define $v,w \in C_{\pi}([-1,1])$ by

$$v(t) = -v(-t) = t - \frac{1}{2} + |t - \frac{1}{2}| \quad 0 \leq t \leq 1,$$
$$w(t) = -w(-t) = -t - \frac{1}{2} + |t - \frac{1}{2}| \quad 0 \leq t \leq 1,$$

and set $f = v \circ h$, $g = w \circ h$. Then $f,g \in \partial\Sigma$, since $\|v\| = \|w\| = v(1) = -w(1) = 1$. We claim that $\delta(f,g) \geq 3$ (actually, equality must of course hold, by Theorem 14.3).

Let c be any curve from f to g in $\partial\Sigma$, and r a given number, $0 \leq r < 1$. Since $\|f-f\| = 0$, $\|g-f\| = \|w-v\| = 2$, there exists a point p on c such that $\|p-f\| = r$. Since $p \in \partial\Sigma$, there exists $t \in K^{\sigma}$ such that $p(t) = 1$. Now $v(h(t)) = f(t) \geq p(t) - \|p-f\| = 1 - r > 0$. From the definitions of v and w we must have $h(t) > \frac{1}{2}$, whence $g(t) = w(h(t)) = -1$. Then

$$\ell(c) \geq \|p-f\| + \|g-p\| \geq r + |g(t)-p(t)| = r + 2.$$

Since r is arbitrarily close to 1, and $\ell(c)$ is arbitrarily close to $\delta(f,g)$, we indeed conclude that $\delta(f,g) \geq 3$.

As in Section 13, Lemmas 3.1 and 12.2 yield a translation of the results into statements about $C_0(T)$.

14.5. _Theorem_. _Assume that_ T _is locally compact_. _If_ T _is dense-in-itself and not compact_, $D(C_0(T)) = 3$; _otherwise_ $D(C_0(T)) = 4$. _In either case_, $D(C_0(T))$ _is very strongly attained_.

15. Summary

In this final section we tabulate some of the results we have obtained for the spaces $C_\sigma(K)$, $C_0(T)$, and $C(T)$. We observe that the relevant properties of K^σ for the classification of the first of these types of function space can be put in the following sequence of increasing restrictiveness: (none); infinite; not scattered; dense-in-itself; without σ-blocks. Cutting across this hierarchy is the question of whether K^σ is split or not (of course, K^σ is always split if it is finite, and never split if it has no σ-blocks). On account of Lemma 12.1, the discussion of $C_\sigma(K)$ when K^σ is split is replaced, with some gain in clarity, by the discussion of $C(T)$ for compact T. To these case-distinctions correspond analogous ones for $C_0(T)$ with locally compact T.

15.1. *Theorem. Let* K *be a compact space and* σ *an involutory automorphism of* K, *and assume that* K^σ *is not split. Let* T *be a locally compact space that is not compact. Say that* $X = C_\sigma(K)$ *or* $C_0(T)$ *is in*

Case	*if* K^σ *is*	*or if* T *is*
I	*infinite, but scattered,*	*infinite, but scattered;*
II	*not scattered, but not dense-in-itself,*	*not scattered, but not dense-in-itself;*
III	*dense-itself, but with* σ*-blocks,*	*dense-in-itself, but with a compact component;*
IV	*without* σ*-blocks,*	*without compact components.*

Then the following holds:

If X is in Case	$m(X)=$	attained?	$m(X)=$	attained?	$D(X)=$	attained?
I	2	no	4	very strongly	4	very strongly
II	2	strongly*	4	very strongly	4	very strongly
III	2	strongly*	3	very strongly	3	very strongly
IV	2	strongly*	2	strongly**	3	very strongly

*sometimes very strongly, but not always; e.g., not for $C_0(T)$ when T is paracompact and first countable, in particular metrizable.

**very strongly if and only if $m(X)$ is very strongly attained.

Proof. Except for footnote (*), by Theorems 5.4, 6.2, 13.3, 13.5, 14.4, 14.5. As concerns that footnote, examples of X that are very flat and examples of X that are not can be found for Cases II,III, and IV, as follows. Theorem 9.6 provides examples of $X = C_0(T)$ in Cases III or IV (depending on the connectivity of the factor spaces) that are very flat; an example for Case II is obtained by adding isolated points (Theorem 8.5). Theorem 9.5, on the other hand, shows that if T is paracompact and first countable (in particular, metrizable), then $X = C_0(T)$ is not very flat; such spaces exist in Cases II,III, and IV. Lemma 3.1 then yields instances for $X = C_\sigma(K)$ for all cases; further instances for Case IV can be found at the end of Section 7.

Remark: In Case IV, $M(X) = 2 < 3 = D(X)$; this refutes [3; Conjecture 9.1]; a special instance of this fact was noted in [7], and it was pointed out that here the inequality is as strong as the bounds recorded in Lemma 11.3 will permit.

15.2. *Theorem. Let* T *be a compact space with more than one point. Then* $M(C(T)) = D(C(T)) = 4$ *are both very strongly attained; further,*

if T *is*	$m(X) =$	*attained?*
finite (n *points*)	$2n(n-1)^{-1}$	*very strongly*
infinite, but scattered	2	*no*
not scattered	2	*strongly; sometimes very strongly, but not always.*

Proof. Theorems 12.3, 6.2, 9.3, 9.5, 9.6; for finite T, Lemma 2.1 and [4; Theorem 5].

15.3. *Theorem. Let* T *be a first countable paracompact space, in particular a metrizable space, with more than one point. Then* $M(C(T)) = D(C(T)) = 4$ *are both very strongly attained; further,*

if T *is*	$m(X) =$	*attained?*
finite (n *points*)	$2n(n-1)^{-1}$	*very strongly*
infinite, but compact and scattered	2	*no*
not compact, but locally compact and scattered	2	*very strongly*
not locally compact, or not scattered	2	*strongly, but not very strongly*

Proof. Theorems 12.3, 6.2, 9.5, and 15.2.

Results for other C(T) are contained in Theorems 6.2 and 12.3 and in Section 9. The results for L^∞-spaces are mainly covered by Theorems 10.2 and 10.5 (see also Conjecture 10.8); we merely add that, if $1 < \dim L^\infty = n < \infty$, then $m(L^\infty) = 2n(n-1)^{-1}$ (very strongly attained); and that, in any case, $M(L^\infty)=D(L^\infty)=4$ (both very strongly attained). This follows from Theorems 10.1 and 15.2.

For purposes of comparison, we record here the corresponding classification for L-spaces, according to [6].

15.4. *Theorem. Let* $(S,\underset{\sim}{S},\mu)$ *be a measure space, and set* $L^1 = L^1(\mu)$. *Then:*

if μ *is*	$m(L')=$	*attained*?	$M(L^1)=$	*attained*?	$D(L^1)=$	*attained*?
purely atomic: n atoms, n>1	$2n(n-1)^{-1}$	*very strongly*	4	*very strongly*	4	*very strongly*
purely atomic: infinitely many atoms	2	*no*	4	*very strongly*	4	*very strongly*
neither atomless nor purely atomic	2	*very strongly*	4	*very strongly*	4	*very strongly*
atomless	2	*very strongly*	2	*very strongly*	2	*very strongly*.

Proof. For atomless μ, [6; Theorem 2]. For $M(L^1)$ and $D(L^1)$ in all other cases, [6; Theorem 4] and Theorem 11.4. For $m(L^1)$ when μ is purely atomic, [6; Theorems 6 and 7] and Lemma 2.1. In the intermediate case, [6; Theorem 5] shows that $m(L^1) = 2$ is strongly attained; we have to show that it is very strongly attained. We denote by μ' the atomless part of μ; then $L^1(\mu')$ is a subspace of $L^1(\mu)$. The argument of the proof of [6; Theorem 7], adapted to this case, shows that if $f \in \partial\Sigma(L^1(\mu))$ is not in $L^1(\mu')$, then $\delta(-f,f) > 2$; therefore, every flat spot of $L^1(\mu)$ lies in $L^1(\mu')$; by [6; Theorem 2] it is then a strictly flat spot of $L^1(\mu')$, and a fortiori of $L^1(\mu)$.

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