COLLECTIONWISE NORMALITY AND

THE EXTENSION OF FUNCTIONS

ON PRODUCT SPACES

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The concept of extending a continuous pseudometric is char-
acterized in terms of extending functions on product sets (satis-
fying no pseudometric axioms). For example a subspace S is
P-embedded in X (every continuous pseudometric on S extends
continuously to X) if and only if for all locally compact hemi-
compact Hausdorff spaces A, the product set S x A is C*-embedded
in the product set S x βS is C*-embedded in the product space
X x PS (where PS is the Stone-Cech compactification of S) if
and only if the product set S x A is C*-embedded in X x A for
all compact Hausdorff spaces A. For Tichonov spaces S this
says that S is P-embedded in X if and only if $u(x x pX) = ux x PX$.
If X is of non-measurable cardinal then $u(X x A) = ux x A$ for all
compact Hausdorff spaces A. Also our results show that if S is
C-embedded in X then S x M is C-embedded in X x M   for all
compact metric spaces M. Using our results and results of Morita
it follows that if A is a compact Hausdorff space with a base for

its natural uniformity of cardinality at most y then $X \times A$ is

Y-paracompact and normal if X is. Finally if A is any compact

Hausdorff space in which X is C*-embedded, then X is collection-

wise normal if and only if $F \times A$ is C*-embedded in $X \times A$ for

all closed subsets F.

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1. <u>Introduction</u> <u>and</u> <u>Preliminary</u> <u>Results</u>.

Let S be a nonempty subset of a topological space X.
The subset S is said to be P-<u>embedded</u> in X if every continuous
pseudometric on S extends to a continuous pseudometric on X.
The subset S is C-<u>embedded</u> (respectively C*-<u>embedded</u>) in X if
every continuous (respectively bounded continuous) real valued func-
tion on S extends to a continuous (respectively bounded contin-
uous) real valued function on X.   It is clear that every C-embedded
subset is a C*-embedded subset; moreover every P-embedded subset is
a C-embedded one (see Theorem 2.4 of [7]).   The concept of P-embed-
ding characterizes collectionwise normal spaces in the same way as
C-embedding (and also C*-embedding) characterize normal spaces. Spe-
cifically a topological space X is collectionwise normal if and
only if every closed subset of X is P-embedded in X   (see [12]).

Since a pseudometric on a space x is a function on the pro-
duct set X x X, it is of interest to relate the extension of pseudo-
metrics to the extension of functions on X x X   (without the tri-
angle inequality).   In [1] we showed that a

subset S is P-embedded in X if and only if every continuous
function from S into a bounded, closed convex subset of a Banach
space extends to a continuous function on X with values in the
convex subset. Using results developed in [1] to demonstrate this
result, we also showed in [1] that if L is any Fréchet space,
then every uniformly continuous function from S into L can be
extended to a continuous function on X. Also uniform continuity
of the extended function is shown not to be attainable.

We now turn our attention to relating P-embedding to the exten-
sion of functions from product sets. By utilizing results from [1]
we show that a subspace S is P-embedded in the Tichonov space X
if and only if for all locally compact hemicompact Hausdorff spa-
ces A, the product set S x A is C*-embedded in the product space
X x A if and only if the product set S x PS is C*-embedded in
the product space X x f3S, where |3S is the Stone-Čech compactifi-
cation of S. As a corollary we have that the subspace S is P-em-
bedded in the Tichonov space X if and only if for all compact
Hausdorff spaces A, S x A is C*-embedded in X x A. For Tichonov
spaces X this implies that X is P-embedded in ux if and only
if u(XxPX) = ux x 0X. Moreover for spaces X of non-measurable
cardinality, this also implies that u(XxA) = ux x A for all com-
pact Hausdorff spaces A. This is similar to a result in [4].

In section 3 the results of section 2 are generalized to give results concerning P'-embedding and y-collectionwise normality. See [1] for definitions of these terms. From these results we are able to show that if S is a c-embedded subset of a topological space X then S x M is C-embedded in X x M for all compact metric spaces M. In [10] Morita considers y-paracompact normal spaces and such spaces are always Y-collectionwise normal. We obtain characterizations of Y-collectionwise normal spaces. As a corollary to results of Morita in [10], we show that if X is a Y-paracompact normal space and if A is a compact Hausdorff space with a base for its natural uniformity of cardinality at most y, then X x A is a y-paracompact normal space. This theorem is then related to our results on P^Y-embedding and Y-collectionwise normal spaces.

In Theorem 4.1 we improve a characterization on collectionwise normal spaces given by H. Tamano in [13] and close with some open questions concerning P-embedding.

The major tool of this paper will be Theorem 1.2 below (which appeared as Theorem 2.3 of [1]). We state the relevant portion of this result here for convenience. However first we need the following definitions.

Definition 1.1: Let X be a topological space and let y be an infinite cardinal number. A function f on X is said to be a

(L,M) ~~valued~~ ~~function~~ on X if f $_{maps}$ x into a complete, con-
vex, metrizable subset M of a locally convex topological vector
space L. A function f on x is said to be a $(L,_Y,M)$-valued
function on x if ifc $\wedge$ a (L,M)-valued function and if the image
of X under f is a Y-separable subset of *M* (that is, there is
a dense subset A of f(x) and the cardinality of A is not great-
er than Y)• A _Fréchet space_ is a complete, metrizable locally
convex topological vector space.

Let us recall that the set of all bounded, real valued contin-
uous functions on X is a Banach space under the sup norm, that is
$[\|f\| = \sup_{x \in X} |f(x)|$. This Banach space will be denoted by C*(X) .

Theorem 1.2: Let S be j^* nonempty subspace of X and let y
be an infinite cardinal number. The following statements are equiva-
lent:

!) The subspace S $i^{\wedge} P^Y$-embedded in X,

2) Every continuous (L,Y,M)-valued function on S extends
to £ continuous (L,M)-valued function on X.

3) Every continuous (L,Y.M)-valued function on S extends
to ja continuous function from X to L.

4) Every continuous function from S to a Fréchet space,
such that the image of S .is Y-separable, extends to a
continuous function on X.

5) Every continuous function from S into C*(S), such
that the image of S is y-separable, extends to a
continuous function on X.

Furthermore, the above conditions are also equivalent to the
conditions obtained from (2) through (5) by requiring the image
of S to be a bounded subset of the locally convex space in ques-
tion.

Since a subspace S of a topological space X is P-embedded
in X if and only if it is P^γ-embedded in X for all infinite
cardinal numbers y (Theorem 2.8 of [12]), it is clear that we ob-
tain characterizations of P-embedding from Theorem 1.2 by removing
all mention of cardinality.

2. P-embedding and the Extension of Functions on Product Spaces.
Now we develop the material needed to characterize P-embedding
in terms of product spaces. If A and B are topological spaces,
C(A,B) will denote the set of all continuous functions from A to B
equipped with the compact-open topology. A subbase for this topology
is the collection of all sets (K,W) = (feC(A,B): f(K) c w} where K
is a compact subset of A and W is an open subset of B. On
page 80 of [11], the following proposition is shown.

Proposition 2.1: If A is a Hausdorff topological space and
if B is ci locally convex topological vector space, then C(A,B)
is also a locally convex topological vector space.

In addition we need to know when $C(A,B)$ is a Fré'chet space. Consequently a _hemicompact_ space is defined as one that is a countable union of compact subsets $(K_i)_{i \in \mathbb{N}}$ such that every compact subset of the space is contained in some K_i. Compact spaces, Euclidean spaces R^n, and countable direct sums of compact spaces are examples of hemicompact spaces. In [3] Richard Arens showed that the concept of hemicompact spaces is useful in showing when $C(A,B)$ is a Fréchet space.

Proposition 2.2: _J^f A is a hemicompact Hausdorff space and if B is .§. Fréchet space, then $C(A,B)$ JIs a Fréchet space._

We note that if A is a compact Hausdorff space and if B is a Banach space, then by the above $C(A,B)$ is a Fréchet space. More than this, it is a Banach space under the norm $\|f\| = \sup_{a \in A} \|f(a)\|_B$ (see [3]). It is easily verified that in this case the topology defined by this norm is equivalent to the compact-open topology.

The following lemma is due to Ralph Fox (see [6]).

Lemma 2.3: _Let X,A, and B be topological spaces and let f be a continuous function from X x A into B. The function <p defined from X into C(A,B) by (<p(x))(a) = f(x,a) for all x in X and a jIn A JIS continuous. Conversely, if A JIS regular and locally compact, and if <p jIs ^a continuous map from X into C(A,B), then the map f from X x A into B defined by f(x,a) = (<p(x))(a) for all (x,a) in X x A is continuous._

Using Theorem 1.2 as stated in section 1, we can now prove the following product space characterizations of P-embedding.

Theorem 2.4: Let S be ja subspace of a^ completely regular T_1 space X. The following are equivalent;

1) The subspace S is P-embedded in X.

2) For all locally compact, hemicompact Hausdorff spaces A, the product set S x A is P-embedded in the product space X X A.

3) For all locally compact, hemicompact Hausdorff spaces A, the product set S x A is C-embedded in the product space X x A.

4) For all locally compact, hemicompact Hausdorff spaces A, the product set S x A is C*-embedded in the product space X x A.

5) The product set S x PS is. P-embedded in the product space X X PS.

6) The product set S x |3S jjs C-embedded in the product space X x pS.

7) The product set S x 3S JIS C*-embedded in the product space X x pS.

Proof: To show that (1) implies (2), let A be a locally compact, hemicompact Hausdorff space. By the equivalence of (1) and (4) in Theorem 1.2 and the remark after Theorem 1.2 relating to

P-embedding, it is sufficient to prove that if f is a continuous

function from the product set S x A into a Fréchet space B,

then f extends to a continuous function on X x A. Let f be

a continuous function from S x A into a Fréchet space B, and

define a map φ from S into C(A,B) by ($\varphi(x)$) (a) = f(x,a) for all

a in A and all x in S. Then φ is continuous by Lemma 2. 3

and C(A,B) is a Fréchet space by Proposition 2.1. By assumption S

is P-embedded in X. Hence by Theorem 1.2 the map *(f>* extends to a

continuous function φ^* from X into C(A,B) . Define a map f*

from X x A into B by f*(x,a) = ($\varphi^*(x)$)(a) for all (x,a) in

X x A. The function f* is an extension of f and is continuous

by Lemma 2.3.

The implications (2) implies (3) implies (4) implies (7),

and (2) implies (5) implies (6) implies (7) are clear. It remains

to show that (7) implies (1). By Theorem 1.2 it is sufficient to

prove that every bounded continuous function from S into C*(S)

extends to a continuous function on X. Let φ be such a function.

The Banach space C*(S) is isomorphic to C*(PS), by the mapping

$f \rightarrow \dot{f}^{Q}$ which assigns to every bounded real valued continuous function

f on S its unique extension f^{Q} to PS. Hence we may think of φ

as mapping S into C*(pS).

Define f from S x PS into R by f(x,z) = (($\varphi(x)$)(z) for all

(x,z) in S x PS. The map f is continuous by Lemma 2.3. Since φ

is bounded, there is a constant K such that $\|\varphi(x)\| \leq K$ for all

x in S. Hence $\sup_{z \in 3S} |\varphi(x)(z)| = \sup_{z \in gS} |f(x,z)| \leq K$ for all x in S.

Therefore, f is a bounded continuous function on S x PS. By hypothesis f extends to a continuous real valued function f* on X x PS. Defining a function φ^* from X into C*(PS) by $((p^*(x))(z) = f^*(x,z)$ for all x in X and z in gS, we see that φ^* is continuous by Lemma 2.3 and is the desired extension of the map φ.

As a result of this theorem extending pseudometrics from a subspace S to the space X is the same as extending bounded continuous real valued functions from S x A to X x A where A is a compact Hausdorff space.

Corollary 2.5: Let S bjs a subspace of a completely regular T_1 space X. Then S is P-embedded in X if and only if S x A is C*-embedded in X x A for all compact Hausdorff spaces A.

As mentioned in the introduction, Theorem 2.4 gives new characterizations of Tichonov spaces which are collectionwise normal. Recall that a space is collectionwise normal if and only if every closed subset is P-embedded. We state here two of the characterizations for these spaces which result from 2.4.

Corollary 2.6: Let X be a completely regular T_d space. The following are equivalent;

1) The space X is collectionwise normal.

2) For all locally compact, hemicompact Hausdorff spaces

A and for all closed subsets F of X, the product

set F x A is C*-embedded in X x A.

3) For all closed subsets F pf X, the product set

F x pF is C*-embedded in X x PF.

The other characterizations can be obtained in a likewise fash-
ion. We also obtain a result on the Hewitt realcompactification of
a product that is similar to a result of W. W. Comfort (see Theor-
em 1.2 of [4]) .

Corollary 2.7: J^f x is. ja completely regular T_1 space,
then X is P-embedded in ux if and only Lf u(xxPX) = ux x PX.
Moreover, if X has noimmeasurable cardinality, then u(xxA) = ux x A
for all compact Hausdorff spaces A.

Proof; If X is P-embedded in ux, then by Theorem 2.4 the
space X x pX is C-embedded in ux x PX. The product of a compact
Hausdorff space and a realcompact space is realcompact. Therefore,
ux x pX is a realcompact space in which X x PX is dense and
C-embedded. By the uniqueness of the Hewitt realcompactification,
it follows that u(xxPX) = ux x PX. Conversely, if u(XxPX) =
ux x PX, then X x PX is c-embedded in ux x PX. Therefore,
by Theorem 2.4 it follows that X is P-embedded in ux.

To prove the second statement we note that in [12] it was shown that if S is a dense C-embedded subset of a completely regular T_1 space X and if the cardinality of S is nonmeasurable, then S is P-embedded in X.   Consequently if X has nonmeasurable cardinality then X is P-rmbedded in uX.   Thus the statement follows from (1) implies (2) of Theorem 2.4. This completes the proof.

A nonempty subset S of a topological space X is Z-<u>embedded</u> in X if for every zero set Z of S there is a zero set Z^1 of X such that Z' fl S = Z.   Every C*-embedded subset is Z-embedded but the converse is not true. This concept was studied in [2]. In Theorem 2.4, statement (7) cannot be improved by stating "the product set S x PS is Z-embedded in the product space X x PS." In fact an equivalence to (1) cannot be obtained even if S is required to be closed.   The following examples are instructive.

Let R be the real line and let S be the open interval (0,1).   Since S is a cozero set of R and since S x PS is a cozero set of R x PS, then S is Z-embedded in R and S x PS is Z-embedded in R x PS (see [2]).   However S is not P-embedded in R since it is not C-embedded.

Now let X be the Tichonov plank, that is X = *[0,d]* x [0, cu]\(n, ɑɹ) and let F be the closed subset { (O,a) : a<u>}.   Since F is a Lindeldf subset of X and since F x PF is a Lindelöf subset of X x pF,

it follows that F x pF is Z-embedded in X x pF.   However, F is

not P-embedded in X since it is not C*-embedded in X.

3. P^Y ~~embedding and Product Spaces.~~

This section attempts to generalize the results of the pre-

vious section to the case of P^Y-embedding.   It will be seen that a

generalization of Corollary 2.5 is possible, with suitable restric-

tions on the spaces A referred to in the corollary.

Lemma 3.1:   Let A be any topological space and let Y be

an infinite cardinal number.   Let S be a subspace of ji topological

space X.   If S x A JLS_ P^Y-embedded in X x A, then S .is. P^V-em-

bedded in X.   JEf S x A JLS_ P-embedded in X x A, then S .is_ P-em-

bedded in X.

Proof;   The second statement follows from the first by choosing

Y to be the cardinality of S.   For if S x A is P-embedded in

X x A, by Theorem 2.8 of [12], it is P^Y-embedded in X x A.   There-

fore by the first statement, S is P^Y-embedded in X.   But since

every continuous pseudometric on S is y-separable, this means that

S is P-embedded in X.

To prove the first statement let d be a y-separable continu-

ous pseudometric on S.   By Theorem 2.1 of [12] it is sufficient to

show that d extends to a continuous pseudometric on X.   Define

a pseudometric e on S x A by

$$e((x,a),(x',a')) = d(x,x«)$$

for all (x,a) and (x^1,a') in $S \times A$. Then e is a continuous pseudometric on $S \times A$ since $S_e((x,a),£) = S_d(x,t) \times A$ for all $£. > 0$. From this relationship it is also clear that the Y-separability of d implies the y-separability of e. Therefore by assumption e extends to a continuous pseudometric e^* on $X \times A$. Fix b in A and define a function d^* on $X \times X$ by $d^*(x,x') = e^*((x,b),(x^1,b))$, for all x,x' in X. Then d^* is a pseudometric on X which extends d. To see that d^* is continuous, let $y \in S_{d^*}(x,£)$. Then $(y,b) \in S_{e^*}((x,b),£)$. Hence there are U and V, neighborhoods of y_o and b respectively such that $U \times V <= s_{e^*}((x,b),£)$. Therefore $y \in U \subset S_{a^*}(x,£)$. This completes the proof of the lemma.

The following result is probably known, but we cannot find a proof of it in the literature. The proof will be included here for completeness. Recall that a compact Hausdorff space possesses a unique admissible uniformity, that is generated by all continuous pseudometrics on the space. In the case of a compact Hausdorff space, this uniformity will be referred to as its natural uniformity.

Lemma 3.2: Let X be a compact Hausdorff space and let y be an infinite cardinal number. Then X has a^ base for its natural uniformity of cardinality at most y JJ[and only if X has a^ base for its topology of cardinality at most y.

<u>Proof</u>; Suppose that JB forms a base of cardinality at

most y for the natural uniformity of X. For each d in jS

and each n in N, consider the cover of X by d-spheres about

the points of X with radii 1/n. Let $(S., (x., 1/n))$. , , ,.
                                     ^{v} d^v 1   i=l,...,m(d)

be a finite subcover. The collection of the spheres in these fi-

nite subcovers, for all d in JB and n in N, has cardinality

at most Y and forms a base for the topology of X. To see this

let $x^{\circ}eU$, an open subset of X. The natural uniformity is admis-

sible. Therefore there exists d in $ and n in N such that
$x_o eS_{\overset{.}{a}}(x_o,1/n)$ c u. Let x_o be in $S_{\overset{.}{a}}(x_{\cdot i},1/2n)$, an element of the

finite subcover associated with d and the natural number 2n.

It follows that $S_n(x.,1/2n)$ is contained in U.
                   d   l

Conversely, suppose that U is an open base for the topology

of X of cardinality at most *y.* We may assume that a finite

union of elements in U is also in U, since these unions will

not increase the cardinality of U. Suppose that F c: G, where F

is a closed set and G is an open set in X. By an easy argument

involving the compactness and normality of X, it can be shown

that there are U and V in U such that F c u c clU c v c G.

Consider a pair (U,V), where clU c v and U,V are in U. Since

every compact Hausdorff space is normal, there exists a continuous

function f on X with values in [0,1] such that f(x) =0 for

all x in clU, and f(y) =1 for all y in X\V. Pick a func-

tion f with these properties for each pair (U,V) such that

clu c v and U,V are in U.   The collection of finite supremum's

of the $\$_f$ is a base for the natural uniformity on X.   Note that

this collection has the proper cardinality.

To show this we will show that for every continuous pseudo-

metric d on X and   £ > 0, there exists $f,,...,f$ in this
                                                1 n

collection and 6 > 0 such that $\$_{f_1}$ v ... v $\$_{f_n}$ (x,y) £ 6   implies

that d(x,y) £ £.   for all x,y in X.

Let d be a continuous pseudometric on X and let 0 < £ < 1.

Let $(S_d(x_I, /2))_{i=1,.\,.\,.,n}$ be a finite subcover of the covering

of X by d-spheres of radii £/2.   Since this cover is normal,

there is an open cover $(W_I)_{i=1,.\,.\,.,n}$ of X such that

clW_i c $S_d(x_i,£/2)$ for i = 1,...,n. By the remark above, for each

$1 < i < n$ we may choose U_i and V_1 in U such that

U_i c clu_i <= v_i. c $S_{d'}(x_{I^1},£/2)$ and such that $_v(U_I)_{i=1,...,n}$ covers X.

Then for 1 £ i £ n, (U_1,V_1) are in the pairs mentioned above, and

therefore we have the corresponding functions f_1 for i = 1,...,n.

Assume that $\$_{f_i}$ (x,y) £ £/2 for $1 \le i \le n$.   This means that

$| f_i (x) - f_i (y) |$ £ *e/2* for 1 £ i £ n. Since $(U_i)_i$,   is

an open cover, x is in some U_j. If y is not in $S_d(x_j, /2)$,

then y is in $X \backslash V_j$. Hence $f_j(x) = 0$ and $f_j(y) = 1$, which can-

not be. Therefore $y \in S_d(x_j,£/2)$, which implies that d(x,y) £ £.   .

We can now state the generalization of Corollary 2.5 for

P^Y-embedding.

Theorem 3.3: Let S be a subspace of a topological space X, and let A be a compact Hausdorff space with a base for its natural uniformity of cardinality at most y. If S is P^Y-embedded in X, then S x A is P^Y-embedded in X x A.

Proof: Let S be P^Y-embedded in X, and let A be a compact Hausdorff space with a base for its natural uniformity of cardinality at most y. Let f be a continuous function from S x A into a Banach space B such that f(SxA) is a y-separable subset of B. By the equivalence of (1), (4), and (5) in Theorem 1.2, it is sufficient to show that f extends to a continuous function on X x A.

Let IB be a dense subset of f(SxA) of cardinality at most y, and let & be a base of cardinality at most Y of continuous pseudometries for the natural uniformity of A. Define a function g from S into C(A,B) by (g(x))(a) = f(x,a) for all x in S and a in A. As was mentioned before Lemma 2.3, C(A,B) with the compact-open topology is a Banach space with norm $\|f\| = \sup_{a \in A} \|f(a)\|^B$. Hence Lemma 2.3 shows that g is continuous. Now if g(S) is a y-separable subset of C(A,B), then by the equivalence of (1) and (4) in Theorem 1.2, the mapping g will extend to a continuous map g* from X into C(A,B). Define then a function f* from X x A into B by f*(x,a) = g*(x)(a) for all (x,a) in X x A. The mapping f* is an extension of f and Lemma 2.3 shows that it is

continuous.   The remainder of the proof then will be devoted to

showing that g(S) is a y-separable subset of C(A,B).

For each d in \$ and each n in N let $(S_d(a_i,1/n))_{i=1j\ldots,m}$

be a finite subcover of the cover of A by d-spheres of radii 1/n.

Every open cover of a compact Hausdorff space is a normal cover

(see [12]).   Therefore, this cover has a partition of unity

$(h_i)_{i=1,\ldots,m}^n$ subordinate to it such that $A\backslash Z(h_i) \subset S_d(a_i,1/n)$

for 1 £ i £ m.

For each subset $p = (b_i)_{i=1,\ldots,m}$ of m elements of &
consider the function which maps A into B   and is defined by

$$(\sum_{i=1}^{m} Sh_i b_i)(a) = \sum_{i=1}^{m} Eh_i(a)b_i \quad (a \in A).$$

Now for each d in \$ and each n in N form the function

$$f_{d,n,p} = \sum_{i=1}^{m} h_i b_i.$$

Each function $h_i b_i$ is continuous.   To see this, let 0_r de-

note the continuous function which maps r in R to rb in 3,

where b is a fixed element of p.   Then $h_i b = 0 \ll h_i$, and consequent-

ly it is continuous.   Therefore, each $f_{d,n,P}$ is continuous.

The totality of these functions is a collection whose cardinality

does not exceed $y.$ We now show that every function in g(S) is

uniformly approximated by functions in this collection.   Since any

Y-separable metric space is hereditarily y-separable, this will

prove that g(S) is itself y-separable.

Let k be an element of g(S) and let   £ > 0.   The function
k maps A into B, and is uniformly continuous.   Therefore there
is a d in JS and an n in N such that d(a,a') £ 1/n implies
that ||k(a) - k(a>) II £ 6/2.   Let   (h..)_.   be the partition

$$1 1~~J.f \bullet \bullet \bullet y HI$$

of unity subordinate to the cover (S,(a.,1/n)). ,   .   For each
a   l   i=l,...,m

1 £. i⁻<L m, the image k(a_i.) is an element of f (SxA) .   Hence there
is a b_i in © such that ||k(a_i) - b_i|| £ £/2.   We will show that

$$\left\| k - \sum_{i\,l}^{m} h^i b^i \right\| £ £ .$$

$$\left\| k - E h_{i} b_{j} JI \right| = \sup_{aeA} \left| \left| k(a) - \sum_{i} h_{i}(a) b_{i} \right\| .$$

Let a be a fixed element of A.   Then

$$\left\| k(a) - \sum_{i} h_{i}(a) b_{i} \right\| \leq \left\| k(a) - \sum_{i} h_{i}(a) k(a_{i}) \right\| + \left\| \sum_{i} h_{i}(a) k(a_{i}) - \sum_{i} h_{i}(a) b_{l} \right\| .$$

Since k(a) = £h_{i}(a)k(a), the first term of this sum becomes:

$$\left\| \sum_{i} h_{i}(a) k(a) - \sum_{i} h_{i}(a) k(a_{i}) \right\| = \left\| \sum_{i} h_{i}(a) (k(a) - k(a_{\pm})) \right\|$$

$$\leq \sum \| \hat{} t a) (k(a) - k(_{ai})) | | = \hat{} (a) \| k(a) - k(a_{\pm}) \| .$$

If $h_{\pm}$(a) *ji* 0, then aeA\Z(h_{i}).   Hence d(a,a_{i}) < 1/n, so
||k(a) - k(a_i)||_<_ /2.   Therefore S$h_{\pm}$(a) ||k(a) - k(a_{i}) || ^ Eh_{i}(a)£/2 = £/2.
Similarly, the second term of the sum can be shown to be £ £_/2.
This completes the proof.

The following corollary combines Theorem 3.3 and Lemma 3.1.

Corollary 3.4: Let A be a compact Hausdorff space with ja base for its natural uniformity of cardinality at most y. A subspace S is P^Y-embedded in X if and only if S x A is P^Y-embedded in X x A.

Since P^O-embedding is equivalent to C-embedding, we obtain the following from Lemma 3.2 and Theorem 3.3.

Corollary 3.5: Xf S is C-embedded in X, then S x M is C-embedded in X x M for all compact metric spaces M.

From Corollary 3.4, the following characterization of Y-collectionwise normality is obtained.

Corollary 3.6: Let X be a topological space, let Y be an infinite cardinal number, and let A be ^ compact Hausdorff space with a base for its natural uniformity of cardinality at most Y« The following statements are equivalent:

1) The space X is Y-collectionwise normal.

2) For all closed subsets F of X, the product F x A is P^Y-embedded in X x A.

It is known that every Y-paracompact normal space is Y-collectionwise normal. K. Morita proved the following theorems (see Theorems 2. 1 and 2. 2 of [10]): (1) If X is ja Y-paracompact space and if A is a^ compact space, then X x A is Y-paracompact; and (2) If X is ≤i Y-paracompact normal space and if A is a compact normal space with an open base of power at most ^{Y}, then X x A is normal.

Combining these two theorems and recalling Lemma 3.2, we have the following result.

Theorem 3.7: If X is a γ-paracompact normal space and if A is a compact Hausdorff space with a base for its natural uniformity of power at most γ, then X x A is a γ-paracompact normal space.

This result is parallel to that of Corollary 3.6 in the following sense. If X is a γ-paracompact normal space and if A is a compact Hausdorff space with a base for its natural uniformity of power at most Y, then X x A is a γ-paracompact normal space by 3.7. Hence X x A is γ-collectionwise normal, therefore every closed subset of the form F x A, where F is closed in X, is P^γ-embedded in X x A. But the latter can also be obtained by first noting that X is γ-collectionwise normal, and then using Corollary 3.6 to argue that if F is closed in X, then F x A is P^γ-embedded in X x A.

The next section deals with some special results for collection-wise normal spaces.

4. Product Space Characterizations of Collectionwise Normality.

Recall that every paracompact normal space is collectionwise normal. H. Tamano proved that if BX is any Hausdorff compactification of a completely regular T_1 space X, then X is paracompact if and only if X x BX is normal (see [13]). Since X x BX is

paracompact if X is paracompact, this can be restated as: A

Tichonov space X is paracompact if and only if X x BX is nor-

mal if and only if X x BX is paracompact. The next theorem is

a parallel result for collectionwise normality. If A is a com-

pact Hausdorff space in which the space X is C*-embedded, this

result will show that the collectionwise normality of X is equiv-

alent to the following conditions: (1) For all closed subsets F

of X, the product F x A is C*-embedded in X x A, and (2) For

all closed subsets F of X, the product F x A is P-embedded in

X x A. We know that every closed subset of a normal space is

C*-embedded and every closed subset of a paracompact Hausdorff space

is P-embedded. Hence it is clear that these conditions are weaker

than the normality or paracompactness of X x A.

Theorem 4.1: Let A be ja compact Hausdorff space in which X

is C*-embedded. The following statements are equivalent:

  1) The space X ±s collectionwise normal.

  2) For all closed subsets F p_f X, the product set

      F x A i§ P-embedded in the product space X x A.

  3) For all closed subsets F of X, the product set

      F x A .is, C* -embedded in the product space X x A.

Proof: The implication (1) implies (2) follows from (1) im-

plies (2) of Theorem 2.4 and the fact that every closed subset of

a collectionwise normal space is P-embedded in it. The implication

(2) implies (3) is immediate. Hence it suffices to show that (3)

implies (1).

It is easy to show that (3) implies that every closed subset

of X is C*-embedded in X; hence X is normal. Let $(F_a)_{a \in \Delta}$ be

a discrete family of closed subsets of X. For each a in I, the

sets F_a and $H_a = U_{\beta \neq \alpha} F_e$ are disjoint closed subsets of X. For

each a in I, by the normality of X, there exists a continuous

function f_α on X with values in [0,1] such that $f_a(x) = 0$ for

all $x \in F_a$, and $f_a(y) = 1$ for all $y \in H_a$. Since X is C*-embedded

in A, let f_{ob} be a continuous mapping of A into [0,1] such that

f_a restricted to X is f_a. Let F be the union of F_a for a

in I. Then F is a closed subset of X.

Define a real-valued function f on F x A by $f(x,a) = f_a(a)$

for the unique a in I such that $f_\alpha(x) = 0$. The function f is

clearly bounded and is continuous. To see this, let (x_o, a_o) be in

F x A, let $f. > 0$, and suppose that x_o is in F_α. There is a

neighborhood U of a_o such that if a is in U, then

$|f_a^*(a) - f^*(a_o)| < a.$. If (x,a) is in F x U, then

$|f(x,a) - f(x_o, a_o)| < f.$. Therefore by assumption f extends to a

continuous real-valued function f* on X x A.

Define g from X into C(A) by $(g(x))(a) = f^*(x,a)$ for all

a in A and x in X. By Lemma 2.3 the function g is contin-

uous. Hence the pseudometric d* defined on X by

$d^*(x,y) = \big|\lg(x) - g(y)\big|\big|$ for all x,y in X is continuous. Let G$_\alpha$ = U S$_{\alpha}$(x,1/4) for all a in I. Then G$_\alpha$ is an open sub-set of X containing F$_{\alpha}$. It remains to show that (G$_\alpha$)$_{\alpha \in X}$ is a pairwise disjoint family.

Suppose t is in G$_a$ and G$_o$, where a $\wedge$ p. There exist x in F$_{ex}$ and y in F$_o$ such that d*(t,x) < 1/4 and d*(t,y) < 1/4. Hence $\|g(t) - g(x)\| < 1/4$ which means that sup$_{a \in A}$|f*(t,a) - f*(x,a)|< 1/4. Similarly, sup$_{a \in A}$|f*(t,a) - f*(y,a)| < 1/4. Let a = x. Then

$\big|f^*(t,x) - f^*(x,x)\big| = \big|f^*(t,x) - f(x,x)\big| = \big|f^*(t,x)\big| < 1/4$ and

$\big|f^*(t,x) - f»(y,x)\big| = \big|f»(t,x) - f(y,x)\big| = \big|f^*(t,x) - 1\big| < 1/4,$

which is a contradiction. This completes our proof.

Now as a corollary we have Tamano's original result. (See [14]).

Corollary 4.2: A completely regular T$_1$ space X ^s collec-tionwise normal if and only if F x |3X is C*-embedded in X x PX for all closed subsets F of X.

5. Concluding Remarks.

We have just obtained Tamano's original result on collection-wise normality as a corollary to Theorem 4.1. We have pointed out that our conditions are weaker than requiring the paracompactness or normality of X x A for A any compact Hausdorff space. It is an open question whether the above condition characterizes P-embed-ding. That is, is it true that if S is a subspace of X, then S is P-embedded in X if and only if S x PX is C*-embedded in X x βX?

In section 3 we discussed two results of Morita from [10]. In this paper his main result is the following: A topological space X is Y-P^a^acompact and normal if and only if $X \times I^Y$ is normal, where I denotes the unit interval. We do not know if there is a parallel result for Y-collectionwise normality. It might be conjectured that X is Y-collectionwise normal if and only if $F \times I^Y$ is C*-embedded in $X \times I^Y$ for all closed subsets F of X. This would be parallel to the result that X is collectionwise normal if and only if for all closed subsets F of X, the set $Px\ PF$ is C*-embedded in $X \times PF$.

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