A SPECTRAL SEQUENCE THAT RELATES
THE HOMOLOGY OF A POLYHEDRON
TO THE HOMOLOGY PRESHEAF

by

Richard N. Cain

Report 71-20

April 1971

/ps -- 4/15/71

A SPECTRAL SEQUENCE THAT RELATES THE HOMOLOGY OF A POLYHEDRON TO THE HOMOLOGY PRESHEAF

by

Richard N. Cain

ABSTRACT

In section 1 we describe first the notion of the (generalized) homology presheaf of a polyhedron, and secondly, the outline of a spectral sequence from cochains in this presheaf to homology of the polyhedron. Sections 2 and 3 provide, respectively, a certain space pair (AlxX,K) that strongly suggests existence of the spectral sequence, and a series of lemmas that prove this existence.

A SPECTRAL SEQUENCE THAT RELATES THE HOMOLOGY OF A POLYHEDRON TO THE HOMOLOGY PRESHEAF

by

Richard N. Cain

J-.^O Let X be a polyhedron, i.e., a space for which there can be found a CW complex structure. Let h = © hq be a homology theory in the sense of [2]. For open sets U for which X-U is a subcomplex of X under some CW structure, consider the groups

$$P(U) = h(X,X-U) .$$

They form the homology presheaf P over X.

.1..1 Let X be given a CW complex structure. Let U = {u ^{i}| iel} be an open covering of X such that the index set I is finite and each U^{i} is the complement of a corresponding subcomplex of X. By C^{n}(U;P) (for n = 0,1,...) one means the group of all families § = {?^{i}*|i = (i,...,i)el^{n+1}} with the properties (a) each ?^GP(U^{lo}n-••nu^{ln}) , (b) §^{x}* = 0 if $i_0,...,i_n$ are not distinct, (c) 5^{1}* = (sgn7r) ?^{X}* if i' = (i ,_,....,i , for some permutation TT:{0,1,...,n}-»[0,1,...,n}. These groups c^{n}(U;P) form a sequence

$$2f(U):h(X)\text{—>} C^{-}(U;P) \text{~^->} C (U;P) \text{—4} ...,$$

in which 6 and the 5^{n}s (n = 0,1,...) have the definitions

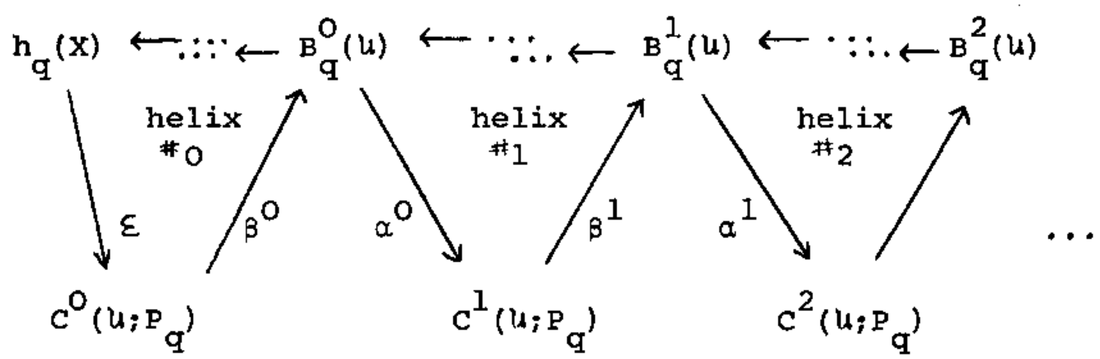
(d) for each $\theta \in h(X)$, $t(\theta) = (\theta|_{A \cap U})|_{A, A-U}$; **(e)** for each $\xi \in C^n(U;P)$ and $i_* = (i_0, \ldots, i_{n+1}) \in I^{n+2}$,

$$(\delta^n \xi)_{i_*} = \sum_{t=0}^{n+1} (-1)^t \, \xi_{i_*^t}|_{X - U^{i_*}} \quad (i_*^t = (i_0, \ldots, i_{t-1}, i_{t+1}, \ldots, i_{n+1})).$$

1.2 With h(X) and ε omitted from $\tilde{C}(U)$, call the remaining sequence C(U). We shall obtain a spectral sequence $E_1(U), E_2(U), \ldots$, and a filtration $F^0h(X) = h(X) \supset F^1h(X) \supset F^2h(X) \supset \ldots$ such that

$$\begin{cases} C(U) \cong E_1(U), \\ \bigoplus_n F^n h(X)/F^{n+1}h(X) \cong E_\infty(U). \end{cases}$$

1.3 More significantly, what we obtain is a diagram of exact helices



such that $\alpha^n \beta^n = \delta^n$. This will be recognized as a bigraded exact couple whose spectral sequence satisfies the description claimed in 1.2. Here, P_q stands for the presheaf of groups

$$P_q(U) = h_q(X, X-U),$$

while the definition β^i, α^i, $B_q^i(U)$ $(i \geq 0)$ will be apparent from the sequel.

2^.0, In the case of standard homo logy there is a bicomplex whose spectral sequence is the desired one: to get the bicomplex, simply replace h by the cellular chain functor throughout the construction of C(U). However, this approach cannot be extended to non-standard h. An alternative approach is outlined next. (See 2.4 for the essential conjecture.)

^.jL Let A be some functor from subsets of I (and their inclusion maps) to (affine) simplexes (and their simplicial inclusion maps), with the property that, for each s c x_9 the number of vertices on AS is the same as the number of members of s. Without loss of generality, assume $(AS)_O$ = s.

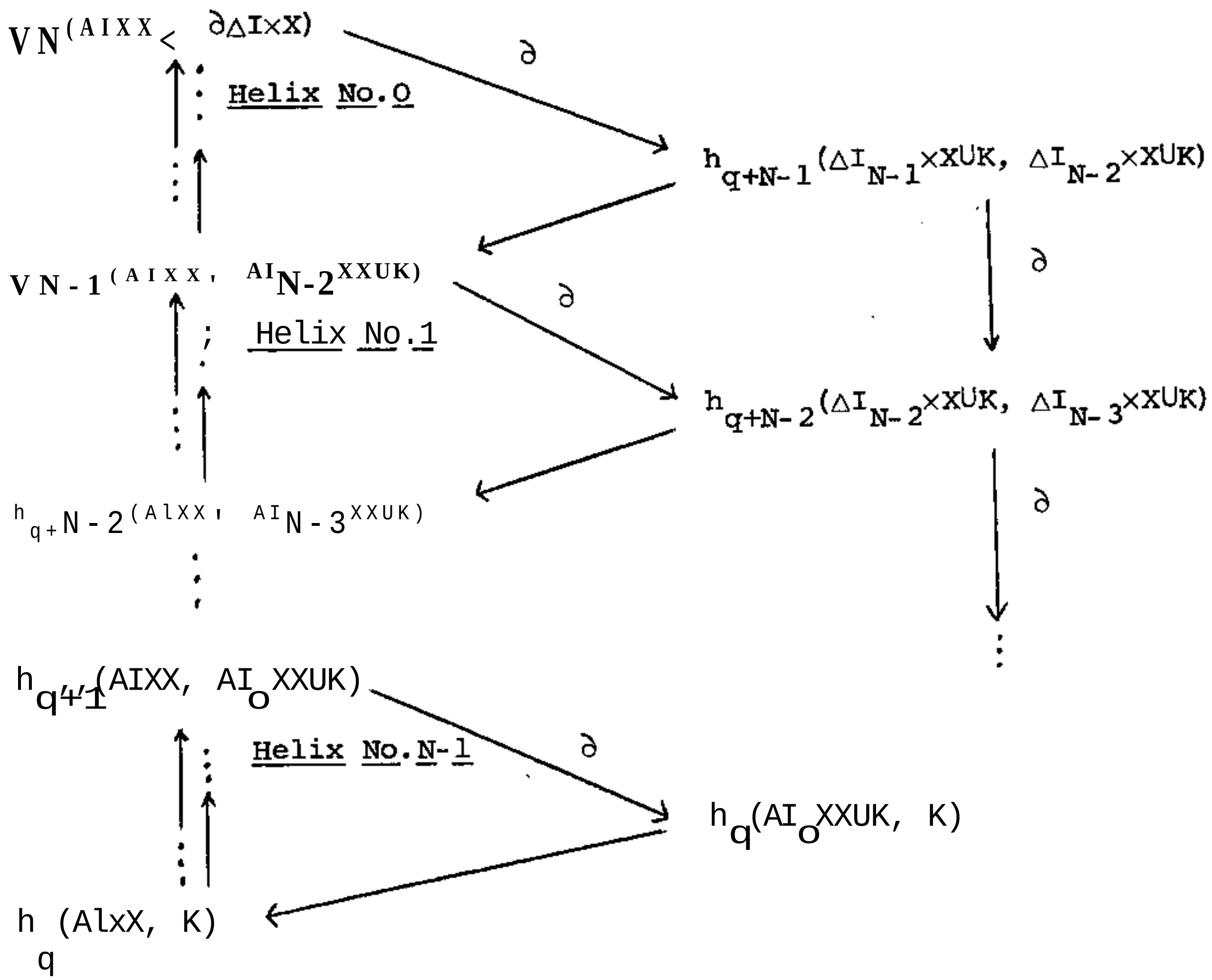
^.^2 A certain subspace K of AlxX will next be defined: for each s c i write X_s for X - fl u[1], and set K = U A(I-{i})xX_i.
ies ie l

<u>Proposition</u>. K = U AC x X_{I-c} .
cCI

<u>Proof</u>. K is certainly contained in the latter, which we shall denote temporarily as K*, for the <u>ith</u> term of K is the same as the <u>cth</u> term of K* if c = I - {i}. Conversely, for any c c i and s = I - c we have Ac x X_{x-c} = A(I-s) x X_s = U A(I-S) x X_1. c
ies

U A(I-{i}) x X_1 c K. Thus, R* c K, so they are equal. Q
ies

2.3 There is a diagram (N = dim ΔI)



in which each helix is exact and each circuit on the right is commutative. This diagram is a bigraded exact couple whose spectral sequence $E^1, E^2, \ldots$ has

$$E^1 = \bigoplus_{n,q} h_{q+N-1-n}(\Delta I_{N-1-n}\times X\cup K, \Delta I_{N-2-n}\times X\cup K)$$

$d_1 = \partial$ (vertical homomorphism on right of diagram)

$$\frac{F^n h_{q+N-n}(\Delta I\times X,\ \partial\Delta I\times X)}{F^{n+1} h_{q+N-n}(\Delta I\times X,\ \partial\Delta I\times X)}$$

$F^n h(\Delta I\times X,\ \partial\Delta I\times X)$ = Im[h(AlxX, A l ^ j^xXUK) -* h(AlxX, $\partial\Delta I\times X$)].

j2.4 Merely judging from the work of [1] and Proposition 2.2, we conjecture that the above spectral sequence has the properties announced in section 1.2. (see below for proof). Incidentally, it is regrettable that the steps in the construction so far are not functorial with respect to U (assuming X fixed) unless I is held fixed, but there probably exists a proof that at least the resulting spectral sequence is itself functorial in U.

Remark. For standard homology, the spectral sequence of 2.0 is functorial in U because C(U) is.

3.0 We now prove the conjecture of section 2.4, using methods of [1].

3.1 Lemma. For n = 0,1,...,N-l, there is an isomorphism

$$h_{q+N-1-n}(\Delta I^{N-1-n}\times X\cup K,\ \Delta I^{N-2-n}\times X\cup K)$$
$$\uparrow \cong$$
$$\bigoplus_{\dim \Delta s=n} h_{q+N-1-n}\Big(\big(\Delta(I-s),\ \partial\Delta(I-s)\big)\times\big(X,\ X_s\big)\Big),$$

which carries each direct summand via a map induced by inclusion of space pairs.

Proof. We note first that $\dim \Delta s = n$ iff $\dim \Delta(I-s) = N-1-n$ (by counting the number of vertices in Δs and $\Delta(I-s)$, respectively). The union $\bigcup_s \Delta(I-s)$ is therefore ΔI_{N-1-n}, and so there are isomorphisms

$$\bigoplus_{\dim \Delta s = n} h_{q+N-1-n}(\Delta(I-s)\times X,\ \Delta(I-s)\times X_s \cup \partial\Delta(I-s)\times X)$$

$$\Big\downarrow \cong \qquad \text{(excisions)}$$

$$\bigoplus_{\dim \Delta s = n} h_{q+N-1-n}(\Delta(I-s)\times X \cup \Delta I_{N-2-n}\times X \cup K,\ \Delta I_{N-2-n}\times X \cup K)$$

$$\Big\downarrow \cong \qquad \text{(direct sum)}$$

$$h_{q+N-1-n}(\Delta I_{N-1-n}\times X \cup K,\ \Delta I_{N-2-n}\times X \cup K)$$

(Note: In verifying the excisions, note that $K = \bigcup_s \operatorname{int} \Delta(I-s)\times X_s$, where int of a simplex means the simplex minus its bounding faces. Then, write $\Delta(I-s)\times X \cup_s \partial\Delta(I-s)\times X$ as $\operatorname{int} \Delta(I-s)\times X \cup_s \partial\Delta(I-s)\times X$. The rest is easy.) □

3.2 Lemma. There is a commutative diagram

$$\begin{array}{ccc}
h_{q+N-1-n}(\Delta I_{N-1-n}\times X\cup K,\ \Delta I_{N-2-n}\times X\cup K) & \xrightarrow{\ \partial\ } & h_{q+N-2-n}(\Delta I_{N-2-n}\times X\cup K,\ \Delta I_{N-3-n}\times X\cup K) \\
\cong \Big\downarrow F_n & & \cong \Big\downarrow F_{n+1} \\
C_{N-1-n}(\Delta I;\ h_q(X, X_{I-(\cdot)})) & \xrightarrow{\ d\ } & C_{N-2-n}(\Delta I;\ h_q(X, X_{I-(\cdot)}))
\end{array}$$

involving $n = 0,1,\dots,N-1$, where C_k stands for simplicial oriented chains (with indicated local coefficient system) and d is the simplicial boundary operator.

Proof. The dual of the proof given in [1], with $(\Delta I \times X, K)$ in place of (G, G^A), and using Proposition 3.1. •

(Note: The boundary operator d has the following formulas for simplicial oriented chains: (i) If C_{1^*} $(1^* = (i, \dots, 1_{UK}))$ is the unit chain for the oriented k-simplex whose ordering $(i_0, \dots, i_k)$ is positive, then $d\,C_{i^*} = \sum_{t=0}^{k} (-1)^t C_{i^*_t}$. (ii) Equivalently, if we work with alternating non-degenerate functions ? of $k+1$ variables in I, referring to these § as chains, then

'd«(ii) = $\sum_{i\in I}$ 5(i,i;) for 4-(ij,...-.^!).)

3.3 Lemma. There is a commutative diagram

$$\begin{array}{ccc}
h_{q+N}(\Delta I\times X,\ \partial\Delta I\times X) & \xrightarrow{\ \partial\ } & h_{q+N-1}(\Delta I_{N-1}\times X\cup K,\ \Delta I_{N-2}\times X\cup K) \\
\downarrow \cong & & \cong \downarrow F_0 \\
C_N(\Delta I; h_q(X)) & \xrightarrow{\ d\ } & C_{N-1}(\Delta I; h_q(X, X_{I-(\cdot)})).
\end{array}$$

Proof. Analogous to preceding. □

3.4 Let co be an orientation of the simplex Al.

Proposition. There is a diagram

$$\begin{array}{ccccc} C_N(\Delta I; h_q(X)) & \xrightarrow{d} & C_{N-1}(\Delta I; h_q(X, X_{I-(\cdot)})) & \xrightarrow{d} & \cdots \\ T_N \downarrow \cong & & T_{N-1} \downarrow \cong & & \\ h_q(X) & \xrightarrow{\xi} & C^0(U; P_q) & \xrightarrow{\delta} & \cdots \end{array}$$

that is commutative to within sign.

Proof. Let T_K send a k-chain ? (see preceding note) into the following n-cochain 17, where k = N-l-n: for each $i_* = (i_0, \ldots, i_n)$ with $i_0, \ldots, i_n$ distinct,

$$\eta^{i_*} = \xi_{j_*},$$

where $j_* = (j_0, \ldots, j_K)$ are chosen in I so that $(i_* j_*) \in \omega$. Then, for $i_\# = (i_0, \ldots, i_{n+1})$ and i^j^o), we have

$$(\delta\eta)^{i_*} = \sum_{t=0}^{n+1} (-1)^t \eta^{i_*^t}\Big|_{X, X_{i_*}} = \sum_{t=0}^{n+1} (-1)^t \xi_{j_*(t)}\Big|_{X, X_{i_*}},$$

where $i_*^t j_*(t) \in \omega$. Now,

$$\begin{aligned} & i_* j_* \in \omega \\ \Longleftrightarrow \quad & i_t i_*^t j_* \in (-1)^t \omega \\ \Longleftrightarrow \quad & i_*^t i_t j_* \in (-1)^{n+1} (-1)^t \omega. \end{aligned}$$

Therefore $?_{x\,\beta *} = (-1)^{n+1}(-1)^t\, \S_{\beta *(t)}$, and consequently,

$$t^{\wedge}\backslash^{x*}(6r)) = (-1)^{n+1} \; r\; e_{t=0}\, 5.^{i\,t\,J}{}_{*}\Big|_{X,X_{1*}} = (-1)^{n+1}(d\xi)_{1*},$$

i.e. we have coinmutatively up to sign, as required. ☐

3.5 Remark. The projection Al x X —> X induces an isomorphism h_q(AlxX,K)—> h_q(X,uX$_x$). (This is seen by methods of [ ].)

References

[1] Cain, R. N., "The Leray Spectral Sequence of A Mapping for Generalized Cohomology", Comm. Pure and Appl. Math, (to appear).

[2] Whitehead, G. W., "Generalized Homology Theories", Trans. Amer. Math. Soc. 102 (1962), 227-283.