ON A LEMMA OF KY FAN

by

Om Parkash Kapoor

Report 71-21

April 1971

University Libraries
Carnegie Mellon University
Pittsburg, PA 15213-3890

HUNT LIBRARY
CARNEGIE-MELLON UNIVERSITY

ON A LEMMA OF KY FAN

by

Om Parkash Kapoor

Abstract

The object of this note is to give two applications of an intersection lemma of Ky Fan. First it is used to obtain a variational property of a strongly continuous function on a weakly compact convex subset of a normed space. In the second half we apply the lemma to obtain a direct proof of a result on the extension of monotone sets in topological linear spaces. It was established separately by Debrunner and Flor, Fan, and Browder.

ON A LEMMA OF KY FAN

by

Om Parkash Kapoor

The object of this note is to give two applications of a lemma of Ky Fan *[7,* Lemma 1]. First it is used to prove a variant of a result of Ky Fan restated as theorem 1 below. In the second half of this note we apply the lemma to obtain a direct proof of a theorem about the extension of monotone sets in topological vector spaces. It was established separately by Debrunner and Flor [5], Fan [8] and Browder [4].

The lemma is

Lemma jL. Let K be a nonempty subset of a Hausdorff linear topological space E. Let for each X€K, F(x) be a closed subset of K such that

(1) The convex hull of any finite subset $\{x_1,x_2,...,x_n\}$ of K is contained in $\overset{n}{\underset{i=l}{U}} F(x_x)$.

(2) F(x) is compact for some xeK.

Then $\underset{XGK}{D} F(x)$ ^ *(p.*

We shall use the following notation: the topological vector spaces will be Hausdorff and over the reals as scalars.____> and —*. denote convergence in the given topology and the weak topology respectively. A function f will be called strongly continuous, if $x_\alpha \rightharpoonup x \Rightarrow f(x_\alpha) \to f(x)$; weakly continuous, if $x_\alpha \rightharpoonup x \Rightarrow f(x_\alpha) \rightharpoonup f(x)$; weakly closed, if $x_\alpha \rightharpoonup x$, $f(x_\alpha) \rightharpoonup y \Rightarrow f(x) = y$; compact if it takes bounded sets into precompact sets; and completely continuous, if it is continuous and compact. The convex hull of a set X will be denoted by <u>co</u> X.

Ky Fan [6] has recently obtained the following theorem.

<u>Theorem 1</u>. Let K be a nonempty compact convex set in a normed vector space E. For any continuous mapping $f : K \to E$, there exists a point $y_0 \in K$ such that

$$\|y_0 - f(y_0)\| = \min_{x \in K} \|x - f(y_0)\|.$$

This result reduces to the Schauder fixed point theorem, if $f(K) \subset K$. It is of interest to obtain more results of this kind. We have the following result.

<u>Theorem 2</u>. Let E be a normed linear space and K a nonempty, weakly compact convex subset of E. Let f be a strongly continuous mapping of K into E, then there exists a point $y_0 \in K$ such that,

$$\|y_0 - f(y_0)\| = \min_{x \in K} \|x - f(y_0)\|. \tag{3}$$

<u>Proof</u>. For each $x \in K$ define

$$F(x) = \{y \in K : \|y - f(y)\| \le \|x - f(y)\|\}.$$

For each x, F(x) is weakly closed. For, let $y_\alpha \to y$, $\{y_\alpha\}$ a net in F(x). By strong continuity of f, $f(y_\alpha) \to f(y)$. So that $y_\alpha - f(y_\alpha) \to y - f(y)$ and $x - f(y_\alpha) \to x - f(y)$. Now because norm is weakly lower semi-continuous, we have the following.

$$\|y_\alpha - f(y_\alpha)\| \le \|x - f(y_\alpha)\|$$

$$\therefore \quad \liminf \|y_\alpha - f(y_\alpha)\| \le \liminf \|x - f(y_\alpha)\|$$

$$\|y - f(y)\| \le \liminf \|y_\alpha - f(y_\alpha)\|$$

$$\le \liminf \|x - f(y_\alpha)\| = \|x - f(y)\|$$

and therefore $y \in F(x)$.

Let $\{x_1, x_2, \ldots, x_n\}$ be a finite subset of K. The $co\{x_1, x_2, \ldots, x_n\} \subset \bigcup_{i=1}^{n} F(x_i)$. If not, suppose $z \in co\{x_1, x_2, \ldots, x_n\}$ and $z \notin \bigcup_{i=1}^{n} F(x_i)$. There exist $a_1, a_2, \ldots, a_n$ such that $a_i \ge 0$, $\sum_{i=1}^{n} a_i = 1$, and $z = \sum_{i=1}^{n} a_i x_i$. $z \notin F(x_i)$ for $i = 1, \ldots, n$ means that $\|z - f(z)\| > \|x_i - f(z)\|$ for $i = 1, 2, \ldots, n$. Hence

$$\|z - f(z)\| = \left\|\sum_{i=1}^{n} a_i x_i - f(z)\right\| = \left\|\sum a_i (x_i - f(z))\right\| < \|z - f(z)\|,$$

which is a contradiction and the conditions of lemma 1 are satisfied and there exists $y_0 \in K$ such that $y_0 \in \bigcap_{x \in K} F(x)$. Hence we have a point $y_0 \in K$ with the property $\|y_0 - f(y_0)\| = \min_{x \in K} \|x - f(y_0)\|$.

In particular we have a special case of Altman's result [1].

Corollary 1. Let B be a Banach space which is reflexive. U_r = fx : ||x|| £ r}. Let f : $U_r \longrightarrow B$ be a strongly continuous mapping satisfying

(4) ||f(x) - x||^2 ^ ||f(x)||^2 - ||x||^2 for every x with ||x|| = r.

Then f has a fixed point in U_r.

Proof. By theorem 2, there exists a point y_neU_r such that

(5) ||y_0 - f(y_Q)ll = Min f||x - f(y_Q)|| : x∊U_r}.

We shall show that y_o is a fixed point. If not, we must have l|f(y_o)ll > r. Moreover ||y_o|| = r. If ||y_o|| < r, then there is a point x on the open line segment (y_o,f(y_o)) which is in U_r i.e., x = Ay_o + (l-A)f(y_o) for some A such that 0 < A < 1 and xeU_r.

By (5)

lly$_0$ - ^{f}(y$_Q$)H £ 11% $^+$ d-^f^) - ^{f}(y$_0$$^{}i|}$

= M\Y$_0$ - f(y$_o$)ll < lly$_0$ - f(y$_o$)ll,

which is a contradiction. Therefore lly$_n$ll = *''. By (4) we have

(6) ||f(y$_0$) - y$_o$l|^2 ^ ||f(y$_o$)||^2 - ||y$_o$l|^2 = ||f(y$_Q$)||^2 - r^2

and by (5)

f((y_o))

lly$_0$ - f((y_o))ll<1H Hrjifj^iī - f(y$_o$)ll = ||f(y_0)|| - r

(7)

$\|y_0 - f(y_0)\|^2 \leq (\|f(y_0) - r)^2$.

Combining (6) and (7) we get a contradiction and therefore $\|f(y_0)\| \le r$ and y_0 is a fixed point of f.

Remark 1. Another interesting consequence of theorem 2 is the well-known fact that any weakly compact convex set K in a normed space E is an existence set, i.e., for each point x in E, there exists at least one point $z \in K$ such that $\|x - z\| = \min_{y \in K} \|x - y\|$. We apply theorem 2 to the constant map $f(y) = x$ for each y in K.

Remark 2.- In a Banach space, the condition that f be strongly continuous can be replaced by the equivalent condition that f be weakly closed and completely continuous; that continuity is not enough can be seen from the

Example. Take the Hilbert space l_2, K the closed unit ball in it, and the function f defined by $x = (x_1, x_2, \ldots, x_n, \ldots) \to f(x) = (\sqrt{1-\|x\|^2}, x_1, x_2, \ldots)$. Because $\|f(x)\| = 1$ therefore $f(K) \subset K$. If there were a point $y_0 \in K$ satisfying $\|y_0 - f(y_0)\| = \min_{x \in K} \|x - f(y_0)\|$, it must be a fixed point of f. But it is easily seen that f has no fixed point in K.

We now turn to another use of lemma 1. The following theorem 3 in its present form was proved by Browder [3]. His approach was based on

(i) The Brouwer's fixed point theorem,

(ii) The existence for a finite covering of a compact space of a partition of unity subordinated by the covering.

Here we use lemma 1 which is a generalization of Knaster, Kuratowski, and Mazurkiewicz's theorem which was used by them for their proof of Brouwer's theorem. It may be mentioned that theorem 3 generalizes earlier results of Minty [10] and Grünbaum [9] which have interesting applications to nonlinear boundary value problems.

Theorem 3. Let K be a nonempty compact convex subset of the topological vector space E, and F a topological vector space, with a bilinear pairing between E and F to the reals which we denote by (w,u) for w in F and u in E. We suppose that the mapping of $K \times F$ into reals which carries $[u,w]$ into (w,u) is continuous. Let T be a continuous mapping of K into F and let G be a monotone subset of $K \times F$ i.e., for each pair of elements $[u,w]$ and $[u_1,w_1]$ of G, we have

$$(w-w_1, u-u_1) \geq 0.$$

Then there exists an element u_0 of K such that for all $[u,w]$ in G

$$(Tu_0-w, u_0-u) \geq 0.$$

Proof. Let $A = \{x \in K : [x,w] \notin G$ for any $w \in F\}$ and let $B = K \sim A = \{x \in K : x \notin A\}$. Now define for each $x \in K$, $F(x)$ as follows:

$F(x) = K$, if $x \in A$, and

$F(x) = \{y \in K : (Ty-w, y-x) \geq 0$ for all $w \in F$ for which $[x,w] \in G\}$.

We shall prove that these $F(x)$'s satisfy the conditions of lemma 1. Clearly $F(x)$ for each x is a closed subset of K, the function T and the bilinear pairing being continuous on K and $K \times F$ respectively. To prove that the convex hull of any finite subset $\{x_1, x_2, \dots, x_n\}$ of K is contained in $\bigcup_{i=1}^{n} F(x_i)$, we consider two cases.

Case 1. At least one of the x_i's is in A_i. So $F(x_i) = K$ for at least one i and K being convex, we have the truth of the assertion.

Case 2. $x_i \in B$ for each $i = 1,2,\dots,n$.
Let us suppose the $co\{x_1, x_2, \dots, x_n\}$ is not contained in $\bigcup_{i=1}^{n} F(x_i)$.
Let $z = \sum_{i=1}^{n} a_i x_i$ where $a_i \geq 0$, $\sum_{i=1}^{n} a_i = 1$ and $z \notin F(x_i)$ for any $i = 1,2,\dots,n$. Therefore there exists $w_1, w_2, \dots, w_n \in F$ such that $[x_i, w_i] \in G$ for each $i = 1,\dots,n$ and $(Tz - w_i, z - x_i) < 0$, $i = 1,2,\dots,n$.
Now for any j and k from 1 to n we have

$$(Tz-w_j, z-x_k) + (Tz-w_k, z-x_j)$$

$$= (Tz-w_k, z-x_k) + (Tz-w_j, z-x_j) + (w_k-w_j, x_j-x_k).$$

The first two terms on the right hand side are negative and the third is non-positive, G being monotone; we have for all $j = 1,2,\dots,n$ and $k = 1,2,\dots,n$

(8) $$(Tz-w_j, z-x_k) + (Tz-w_k, z-x_j) < 0.$$

Multiply (8) by a_j and sum over j and using the fact that

$a_j \geq 0$ and $\Sigma x_j = 1$

$$(Tz-\Sigma x_j w_j, z-x_k) + (Ty-w_k, z-\Sigma x_j x_j) < 0,$$

(9)

$$(Tz-\Sigma x_j w_j, z-x_k) < 0, \quad (\forall z = \Sigma b_j x_j)$$

again multiplying (9) by a^k and summing over k we obtain

$$(Tz-\Sigma x_j w_j, z-\Sigma x_k x_k) < 0, \text{ or } 0 < 0,$$

which gives a contradiction. Hence by lemma 1, there exists a $y_0 \in K$ such that $y_0 \in \bigcap_{x \in K} F(x)$, which is equivalent to saying that there exists $y_0 \in K$ for which

$$(Ty_0-w, y_0-x) \geq 0 \text{ for all } [x,w] \in G,$$

which completes the proof of the theorem.

When E is a locally convex space: $F = E^*$, the dual with the topology of uniform convergence on bounded sets and the bilinear pairing is the natural one $[x,f] \to f(x)$, we have the following particular case. This corollary has been the basis of "monotonicity" methods for the solution of nonlinear equations in Banach spaces. For more references see Browder [4].

Corollary 2. (Browder [4], Proposition 1): Let K be a compact convex set in a locally convex linear space E, G a monotone subset of $K \times E^*$, T a continuous mapping of K into E^*. Then there exists an element u_0 of K such that for all elements $[u,w]$ of G

we have

$$(Tu_o\text{-}w,u_Q\text{-}u) \wedge 0.$$

In corollary *3,* we have a more special case of theorem 3. But we give a direct proof for it by using lemma 2 (below) which is a consequence of lemma 1 and was given in the same paper by Ky Fan [7], Browder [2] proved corollary 3 and used it for obtaining some fixed point theorems.

Lemma 2.. Let K be a nonempty compact convex subset of a Hausdorff linear topological space E and A is a closed subset of K x K having the properties

(10) (x,x)eA for every xeK,

(11) for each fixed yeK, the set {xeK : (x,y)/A) is convex (or empty).

Then there exists a point y_n^K such that K x *[y_n)* c A.

Corollary 3. Let E be a locally convex space, E* the dual of E. K is a nonempty compact convex subset of E. If T : K-*E* is a continuous mapping, then there exists a point Y_ne^K such that

$$(^T(Y_0)>y_0\text{-}x) \wedge 0, \text{ for all } xeK.$$

Proof. Let A = {(x,y)eKxK : (Ty,y-x) ;> 0}.

By continuity of T, A is closed c K x K. Let A_y = [x : (x,y)^A} and let x_1,x_2eA_y and 0 £ ^ £ 1, *z* = Ax_1 + (l-x)x_2 we have therefore (T(y),y-x_l) < O and (Ty,y-x_2) < O.

$$(Ty,y-z) = (Ty,y-\lambda x_1 - (1-\lambda)x_2)$$

$$= \lambda(Ty,y-x_1) + (1-\lambda)(Ty,y-x_2)$$

$$< 0$$

$\therefore$ (z,y)/A and $z \in A_y$.

A_y is convex for each $y \in K$.

By lemma 2, there exists $y^0 \in K$ such that $K \times \{y^0\} \subset A$ i.e., there exists y^0 in K such that $(Ty^0,y-x) \geq 0$ for all $x \in K$.

Remark 3. It must be mentioned that theorem 2 of this note is far from being satisfactory. We feel that the condition of strong continuity is too strong. The result should be true for completely continuous functions. Then Altman's result will follow in its full strength. We hope to improve upon the present form in the future.

Acknowledgments

This work was done while the author was on leave from the Indian Institute of Technology, Kanpur, India, as a USAID-Participant, at the Department of Mathematics, Carnegie-Mellon University, Pittsburgh, Pennsylvania. The author is grateful to Professor Kondagunta Sundaresan for his advice and encouragement.

References

[1] Altman, M., "A Fixed Point Theorem in Banach Space", Bull. Polish Acad. Sci. 1(1957), 19-22.

[2] Browder, F. E., "A New Generalization of the Schauder Fixed Point Theorem", Math. Ann. 174(1967), 285-290.

[3] ________________, "The Fixed Point and Minimax Theorems in Locally convex Linear Spaces", Math. Ann. 177(1968), 283-301.

[4] ________________, "Nonlinear Maximal Monotone Operators in Banach Spaces", Math. Ann. 175(1968), 89-113.

[5] Debrunner, H., u. P. Flor, "Ein Erweiterungssatz für Monotone Mengen", Arch. Math. JL5J1964), 445-447.

[6] Fan, K., "Extensions of Two Fixed Point Theorems of F. E. Browder", Math. Zeit. 112(1969), 234-240.

[7] _________, "A Generalization of Tychonoff s Fixed Point Theorem", Math. Ann. 142(1961), 305-310.

[8] _________, "Applications of A Theorem Concerning Set with Convex Sections", Math. Ann. 163 (1966), 189-203.

[9] Grünbaum, B., "A Generalization of Theorem by Kirsbraun and Minty", Proc. Amer. Math. Soc. JU(1962), 812-814.

[10] Minty, G. J., "On the Simultaneous Solution of A Certain System of Linear Inequalities", Proc. Amer. Math. Soc. 13(1962), 11-12.

HUNT LIBRARY
CARNEGIE-MELLON UNIVERSITY