APPLICATIONS OP FORCING
TO DEFINABILITY PROBLEMS
IN THE ARITHMETICAL HIERARCHY

by

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Abstract

Forcing arguments are used to obtain generalizations of some well-known theorems about the degrees of unsolvability with the jump operator.

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<u>Introduction</u>. It has been remarked by Addison [1] and Hinman [6] that applications of forcing techniques (Cohen [2]; Feferman [3]) often allow results about recursiveness to be extended to higher levels of the arithmetical (and hyperarithmetical) hierarchy. In this paper we present a description of the forcing method, and then use this technique to obtain generalizations of some well-known theorems about the degrees of unsolvability. We prove:

(1) a generalized Spector's theorem [15],

$$\forall \mathbf{a}\exists \mathbf{b}\exists \mathbf{c}[\mathbf{a}^{(n)} = \mathbf{b}^{(n)} = \mathbf{c}^{(n)} = \mathbf{b} \vee \mathbf{c}],\ n < \omega,$$

and

$$\exists \mathbf{a}\exists \mathbf{b}[\mathbf{a}^{(\omega)} = \mathbf{b}^{(\omega)} = \mathbf{a} \vee \mathbf{b} = \mathbf{0}^{(\omega)}];$$

(2) a generalized Friedberg's theorem [4],

$$\forall \mathbf{a}\forall \mathbf{b}\exists \mathbf{c}[\mathbf{c}^{(n)} = \mathbf{c} \vee \mathbf{a}^{(n)} = \mathbf{b} \vee \mathbf{a}^{(n)}],\ n < \omega; \text{ and}$$

(3) a generalized Kleene-Post theorem [11],

$$\exists A\exists B[A \notin \Delta_n^B \text{ and } B \notin \Delta_n^A \text{ and } \mathbf{d}(A) \leq \mathbf{0}^{(n)} \text{ and } \mathbf{d}(B) \leq \mathbf{0}^{(n)}],\ n < \omega.$$

Each of these theorems concern the existence of sets (characteristic functions) of natural numbers. Our proofs will involve the construction of a total function as the union of a chain of initial segments. This general approach to degree problems was initiated by Kleene and Post in [11]. In the original Kleene-Post construction one is presented with a sequence of recursive conditions, and then defines a function (or functions) to satisfy these conditions by successively choosing greater initial segments in order to meet each condition one by one. In substance, we do the same. Forcing however, allows us to handle sequences of prescribed arithmetical conditions that are not necessarily recursive.

1. Preliminaries.

The purpose of this section is to present notation and set forth some definitions. Much of the contents are standard and refer mainly to [9] and [13].

Prime number factorization. Let the prime numbers in order of magnitude be $p_0, p_1, \ldots, p_i, \ldots$ $(p_0 = 2)$. Let a be an arbitrary natural number. By the fundamental theorem of arithmetic there is a unique representation of a, if $a > 0$, of the form

(1) $$a = p_0^{a_0} \cdot p_1^{a_1} \cdot \ldots \cdot p_i^{a_i} \cdot \ldots .$$

As shown by Kleene in [9], the following functions are all primitive recursive:

p_i = the $i+1$-th prime number;

$$(a)_i = \begin{cases} \text{the exponent } a_i \text{ of } p_i \text{ in (1), if } a \; j\& \; 0; \\ 0, \text{ if } a = 0; \end{cases}$$

$$lh(a) = \begin{cases} \text{the number of non-zero exponents in (1), if } a \wedge 0? \\ 0, \text{ if } a = 0. \end{cases}$$

We can represent any finite sequence $a_{\cap}, ..\#, a_s$ of natural numbers by the number $a = p_{\cap}^{a_0+1} \cdot \; ... \; * p_s^{a_s+1}$; then $lh(a)$ is the length $s + 1$ of the sequence represented by a.

A <u>sequence number</u> is a number $a = p_{\cap}^{a_0} \cdot \; \cdot . \cdot \; {}^{\#}p_s^{a_s}$ so that for all $i <\pounds s$, $a. > 0$. For any two sequence numbers a and p, define $a > p$ if and only if $lh(a) \wedge lh(P)$ and $(p)_i = (a)_i$, for all $i < lh(p)$.

Let f be any partial function whose domain includes the set $(0,1,2,\cdot \cdot \cdot,n\}$. Define $\bar{f}(n+l) = \prod_{i \leq n} p_i^{f(i)+1}$. $\bar{f}(n+l)$ is a sequence number. Moreover, if a is any sequence number, and if a partial function f is defined by $f(i) = (a)_i - 1$, for all $i < lh(cx)$, then $a = "\bar{f}(lh(a))$.

<u>Arithmetical properties</u>. Let 7 be a one-place function variable ranging over number theoretic functions.

<u>Definition</u> 1. A predicate $-A(r, x_1, \ldots, X_k)$, $k \geq 0$, is an <u>arithmetical property</u> if and only if it is expressible in the form

$$Q_1 y_1, \ldots, Q_j y_j \; R(\bar{\tau}(y_j), x_1, \ldots, x_k, y_1, \ldots, y_{j-1}),$$

where j ^ 1, for each i £ j, Q^ is 3yi or Vy^ and R(w,x1,...,*Lk,y1,...,yj-^) is a recursive predicate.

Observe that by Definition 1, the negation of an arithmetical property is an arithmetical property.

Lemma 1. Suppose R(w,x1,...,xk) is any recursive predicate. Define a new predicate R*(w,x1,...,xk) by R*(w,x1,...,xk) = 3v[lh(v) £ lh(w) & Vi < lh(v) ((v)i = (w)i) & R(v,xx,...,x^)].

Then,

(i) R*(w,x1,...,xk) is a recursive predicate;

(ii) 3y R*(T(y),x1,...,xk) = 3y R(r"(y)^x^•..,x^); and

(iii) if a and p are two sequence numbers and a < p, then R*(a,*^...,x^) implies R^-(P,x1,...,xk).

Lemma 2. Suppose R(w,x1,...,xk) is any recursive predicate. Define a new predicate R'(w,x1,...,xk) by R1(w,x1,...,aik) = Vv[lh(v) i lh(w) & Vi < ih(v) ((v)i = (w)i) -> R(v,x1,...,xk)].

Then,

(i) R1(w,x1,...,xk) is a recursive predicate;

(ii) Vy Ri(T(y),xx,...,x^) = Vy R(T"(y),x;L,...,xj); and

(iii) if a and p are two sequence numbers and a > (3, then R'(a,x][,...,3^) implies R1(p,x]L,-...,x^).

The proofs are immediate. A recursive predicate that satisfies clause (iii) of Lemma 1 will be called <u>monotonic increasing</u>, and a recursive predicate that satisfies clause (iii) of Lemma 2 will be called <u>monotonic decreasing</u>. We will assume, without loss of generality, that every arithmetical property is expressed in the form

$$Q_1 y_1 \cdots Q_j y_j \; R(\bar{f}(y_j), x_1, \ldots, x_k, y_1, \ldots, y_{j-1})$$

with $R(w, x_1, \ldots, x_k, y_1, \ldots, y_{j-1})$ monotonic — monotonic increasing if $Q_j y_j$ is $\exists y_j$ and monotonic decreasing if $Q_j y_j$ is $\forall y_j$.

In particular we will use the starred and primed versions respectively of T^{*}_{1n} and L^{*}_{1n} as defined in $[D]_1$. T_{1n} as defined in [D] enables the normal form and enumeration theorems to be written using $\bar{f}$ instead of $\tilde{f}$.

2. <u>Forcing</u>.

As briefly explained in the introduction of this paper, the forcing method will be applied to construct functions f as unions of chains of initial segments $f_0, f_1, \ldots, f_n, \ldots$. We desire relativized forms of such constructions. Therefore, at the n-th stage of a construction, f^{n+1} is chosen from some <u>admissible</u> subset of the set of all initial segments. Accordingly, forcing is defined relative to a notion of admissibility.

<u>Definition 2</u>. A <u>characteristic sequence number</u> is a sequence number $a = \prod_{i<n+1} p_i^{a_i}$ so that $a_i \in \{1,2\}$, for all $i < n+1$.

<u>Definition</u> 3. Let $\mathrm{adm}(\alpha)$ be a unary relation defined on the set of all characteristic sequence numbers. For any two characteristic sequence numbers α and β, define $\alpha >_{\mathrm{adm}} \beta$ if and only if $\alpha > \beta$, $\mathrm{adm}(\alpha)$, and $\mathrm{adm}(\beta)$. Let $n^1, \ldots, n^k$ be arbitrary numerals. Then, the relation α adm-forces $A(T, n^1, \ldots, n^k)$, in symbols $\alpha \Vdash^{\mathrm{adm}} A(T, n^1, \ldots, n^k)$, is defined inductively for arbitrary sequence numbers α and arithmetical properties $A(T, x^1, \ldots, x^k)$, as follows:

(i) $\alpha \Vdash_{\mathrm{adm}} \exists y\, R(\bar{T}(y), n_1, \ldots, n_k)$, where $R(w, x_1, \ldots, x_k)$ is recursive, if $\mathrm{adm}(\alpha)$ and $R(\alpha, n_1, \ldots, n_k)$;

(ii) $\alpha \Vdash_{\mathrm{adm}} \forall y\, R(\bar{T}(y), n_1, \ldots, n_k)$, where $R(w, x_1, \ldots, x_k)$ is recursive, if $\mathrm{adm}(\alpha)$ and for each $\beta >_{\mathrm{adm}} \alpha$, $R(\beta, n_1, \ldots, n_k)$;

(iii) $\alpha \Vdash_{\mathrm{adm}} \exists y\, A(T, n_1, \ldots, n_k, y)$, if for some $n \in \omega$, $\alpha \Vdash_{\mathrm{adm}} A(\tau, n_1, \ldots, n_k, n)$;

(iv) $\alpha \Vdash_{\mathrm{adm}} \forall y\, A(T, n_1, \ldots, n_k, y)$, if $\mathrm{adm}(\alpha)$, and for each $\beta >_{\mathrm{adm}} \alpha$ and for each $n \in \omega$ there is some $\gamma >_{\mathrm{adm}} \beta$ so that $\gamma \Vdash_{\mathrm{adm}} A(T, n_1, \ldots, n_k, n)$.

<u>Lemma 3. If</u> $\alpha \Vdash_{\mathrm{adm}} A$ and $\beta >_{\mathrm{adm}} \alpha$, then $\beta \Vdash_{\mathrm{adm}} A$.

<u>Proof</u>; if $\alpha \Vdash_{\mathrm{adm}} \exists y\, R(\bar{T}(y), n_1, \ldots, n_k)$, and $R(w, x_1, \ldots, x_k)$ is recursive, then $R(\alpha, n_1, \ldots, n_k)$. Thus $\beta >_{\mathrm{adm}} \alpha$ implies $R(\beta, n_1, \ldots, n_k)$, because $R(w, x_1, \ldots, x_k)$ is monotonic increasing.

If a ||-$_{adm}$ ^{V}Y *R(T(y) ,n$_x$, . . . ^* and Rfw^, . . . ,*x^)* is recursive, then p >$_{adm}$ a implies R(P,x$_1$, . . .,3^) by definition. Since $^>$*$_{a6m}$* is transitive, p II'$_{adm}$ ^{Vv} ^{R}(.^{T}(y) J^* • • • J^1^) * $^{for each}$ p >$_{adm}$ a.

Suppose a ll-$_{adm}$ 3y A(T,n$_1$, . . . ^n^y) and A (r, x$_1$, . . . ^x^y) is an arithmetical property. Then, for some ne o> , ex ||-$_{adm}$ A(T,n$_1$, . . . ,n$_k$,n). Assume as induction hypothesis that p ||-$_{adm}$ A(T,n$_1$, . . . ,n$_k$,n) , for every p >$_{adm}$ a. Then, by definition, for each 0 >$_{adm}$ a, p ||-$_{adm}$ 3y A(r,n$_r$ . . . ,1^,7) .

Suppose a ||-$_{adm}$ Vy A(r,n$_{ir}$. . . ^n^y) and *A(T,11^ . . . ,x^,y)* is an arithmetical property. For each *y* >$_{adm}$ a and n€U> there is 6 >$_{adm}$ Y so that 6 ||-$_{adm}$ A(r,n$_1$, . . . ,n$_k$,n) . Let p >$_{adm}$ a. >$_{adm}$ is a transitive relation, therefore for each *y* >$_{adm}$ p and n€ u) there is a 6 $^>$$_{adm}$ Y so that 6 ||-$_{adm}$ A(r,n ^$_1$ . . ,n^,n) . Thus, p ||-$_{adm}$ Yy A(T,n$_1$, . . . ^^y) .

<u>Lemma</u> 4. For each a so that adm(a), numerals n$_1$,...,n$_k$ and arithmetical property A(T,x$_1$ . . . , x$_k$), there is some P so that P>$_{adm}$a and either P H-$_{adm}$A(T,n$_1$, ...,n$_k$) or P !h$_{adn}$ A(r,n$_r$. . . ,n$_k$).

<u>Proof</u>; The proof is by induction on the number of quantifiers, j, under which A(r,x$_1$ . . . ,X$_k$) is expressible in the form given in Definition 1.

<u>Case</u> j = 1. For some recursive predicate R(w,x$_1$. . . ,x^), A(r,x$_1$,...,x$_{fc}$) is expressible in the form 3y R(T(y) ,x$_1$,...,x$_k$)

or $\forall y\ \bar{R}(\bar{T}(y),x_1,\ldots,x_k)$. It follows from Definition 3 that either $\exists \beta \geq_{adm} \alpha\ \ \beta \Vdash_{adm} \exists y\ R(T(y),n_1,\ldots,n_k)$ or $\alpha \Vdash_{adm} \forall y\ \bar{R}(\bar{T}(y),n_1,\ldots,n_k)$. Thus, if $A(T,x_1,\ldots,x_k) \equiv \exists y\ R(T(y),x_1,\ldots,x_k)$, then there is some $\beta \geq_{adm} \alpha$ so that $\beta \Vdash_{adm} A(T,n_1,\ldots,n_k)$ or $\alpha \Vdash_{adm} \bar{A}(\bar{T},n_1,\ldots,n_k)$. And, if $\bar{A}(T,x_1,\ldots,x_k) \equiv \forall y\ \bar{R}(T(y),x_1,\ldots,x_k)$, then $\alpha \Vdash_{adm} A(T,x_1,\ldots,x_k)$ or there is some $\beta \geq_{adm} \alpha$ so that $\beta \Vdash_{adm} \bar{A}(T,n_1,\ldots,n_k)$.

Case $j > 1$. Assume as induction hypothesis that Lemma 4 is true for each arithmetical property expressible in the form given in Definition 1 with fewer than j quantifiers. Let $A(T,x_1,\ldots,x_k)$ be expressible in the form with j quantifiers. Then, there is an arithmetical property $B(T,x_1,\ldots,x_k)$ so that $A(T,x_1,\ldots,x_k) \equiv \exists y\ B(T,x_1,\ldots,x_k,y)$ or so that $A(T,x_1,\ldots,x_k) \equiv \forall y\ \bar{B}(T,x_1,\ldots,x_k,y)$ and so that $B(T,x_1,\ldots,x_k,y)$ is expressible in the form with fewer than j quantifiers. Suppose $\alpha \not\Vdash_{adm} \forall y\ \bar{B}(T,n_1,\ldots,n_k,y)$. Then, there exists $\beta \geq_{adm} \alpha$ and $n \in \omega$ so that for each $\gamma \geq_{adm} \beta$, $\gamma \not\Vdash_{adm} \bar{B}(T,n_1,\ldots,n_k,n)$. Therefore, by induction hypothesis $\exists \gamma \geq_{adm} \beta\ \ \gamma \Vdash_{adm} B(T,n_1,\ldots,n_k,n)$. Since $\gamma \geq_{adm} \beta$, and $\beta \geq_{adm} \alpha$, $\gamma \geq_{adm} \alpha$. Thus, for some $\gamma \geq_{adm} \alpha$ $\gamma \Vdash_{adm} B(T,n_1,\ldots,n_k,n)$. It follows that for some $\gamma \geq_{adm} \alpha$ $\gamma \Vdash_{adm} \exists y\ B(T,n_1,\ldots,n_k,y)$. Thus, if $A(T,x_1,\ldots,x_k) \equiv \exists y\ B(T,x_1,\ldots,x_k,y)$, then there is some $\gamma \geq_{adm} \alpha$ so that $\gamma \Vdash_{adm} A(T,n_1,\ldots,n_k)$ or $\alpha \Vdash_{adm} \bar{A}(T,n_1,\ldots,n_k)$.

And, if $A(\tau,x_1,\ldots,x_k) \equiv \forall y\, \bar{B}(\tau,x_1,\ldots,x_k,y)$, then there is some $\gamma >_{adm} \alpha$ so that γ H-adm^'V-'-'V or $\alpha \Vdash_{adm} A(\tau,n_1,\ldots,n_k)$.

Definition 4. If f is a number theoretic function, define adm(f) if for every natural number n, adm($\bar{f}(n)$). If $A(\tau,x_1,\ldots,x_k)$ is an arithmetical property and $n_1,\ldots,n_k$ are numerals, the relation $f \Vdash_{adm} A(\tau,n_1,\ldots,n_k)$ is defined by $f \Vdash_{adm}$ A f r ^ , . . . , n^) if and only if adm(f) and there is some n so that

$$\bar{f}(n) \Vdash_{adm} A(\tau,n_1,\ldots,n_k).$$

Definition 5. A set G of arithmetical properties is closed if:

(i) each arithmetical property in G is expressible without free number variables (if A is an arithmetical property, then τ is free in A);

(ii) for arbitrary numerals $n_1,\ldots,n_k$ and recursive predicate $R(w,x_1,\ldots,x_k)$, if $\exists y\, R(\tau(y),n_1,\ldots,n_k)$ belongs to G, then $\forall y\, \bar{R}(\tau(y),n_1,\ldots,n_k)$ belongs to G, and conversely;

(iii) for arbitrary numerals $n_1,\ldots,n_k$ and arithmetical property $\exists y\, A(\tau,x_1,\ldots,x_k,y)$, if $\exists y\, A(\tau,n_1,\ldots,n_k,y)$ belongs to G, where $A(\tau,x_1,\ldots,x_k,y)$ is also an arithmetical property, then $\forall y\, \bar{A}(\tau,x_1,\ldots,x_k,y)$ belongs to G, and for each new, $A(\tau,n_1,\ldots,n_k,n)$ belongs to G;

(iv) if $\forall y\, A(T, n_1, \ldots, n_k, y)$ belongs to G, where $A(T, x_1, \ldots, x_k, y)$ is also an arithmetical property, then $\exists y\, \bar{A}(T, n_1, \ldots, n_k, y)$ belongs to G, and for each $n \in \omega$, $A(T, n_1, \ldots, n_k, n)$ belongs to G.

Lemma 5. Let G be a closed set of arithmetical properties and let f be a number theoretic function so that adm(f). If for each $A \in G$, $f \Vdash_{adm} A$ or $f \Vdash_{adm} \bar{A}$, then for each $A \in G$, $A(f)$ if and only if $f \Vdash_{adm} A$.

Proof: Suppose first that $R(w, x_1, \ldots, x_k)$ is recursive and $\exists y\, R(\bar{T}(y), n_1, \ldots, n_k) \in G$. $f \Vdash_{adm} \exists y\, R(\bar{T}(y), n_1, \ldots, n_k) \leftrightarrow$ for some n, $\bar{f}(n) \Vdash_{adm} \exists y\, R(\bar{T}(y), n_1, \ldots, n_k) \leftrightarrow$ for some n, $R(\bar{f}(n), n_1, \ldots, n_k) \leftrightarrow \exists y\, R(\bar{f}(y), n_1, \ldots, n_k)$.

Suppose $\forall y\, R(\bar{T}(y), n_1, \ldots, n_k) \in G$. By Lemma 3, not both $f \Vdash_{adm} \forall y\, R(\bar{T}(y), n_1, \ldots, n_k)$ and $f \Vdash_{adm} \exists y\, \bar{R}(\bar{T}(y), n_1, \ldots, n_k)$. Therefore, $f \Vdash_{adm} \forall y\, R(\bar{T}(y), n_1, \ldots, n_k) \leftrightarrow f \not\Vdash_{adm} \exists y\, \bar{R}(\bar{T}(y), n_1, \ldots, n_k) \leftrightarrow$ there is no n so that $\bar{R}(\bar{f}(n), n_1, \ldots, n_k) \leftrightarrow$ for each n, $R(\bar{f}(n), n_1, \ldots, n_k) \leftrightarrow \forall y\, R(\bar{f}(y), n_1, \ldots, n_k)$.

Suppose $\exists y\, A(T, n_1, \ldots, n_k, y) \in G$. $f \Vdash_{adm} \exists y\, A(T, n_1, \ldots, n_k, y) \leftrightarrow$ for some n, $\bar{f}(n) \Vdash_{adm} \exists y\, A(T, n_1, \ldots, n_k, y) \leftrightarrow$ given n, there is m so that $\bar{f}(n) \Vdash_{adm} A(T, n_1, \ldots, n_k, m) \leftrightarrow$ there is m so that $f \Vdash_{adm} A(T, n_1, \ldots, n_k, m)$. By induction hypothesis this is equivalent to: there is m so that $A(f, n_1, \ldots, n_k, m)$. Thus, $f \Vdash_{adm} \exists y\, A(T, n_1, \ldots, n_k, y) \leftrightarrow \exists y\, A(f, n_1, \ldots, n_k, y)$.

Suppose $\forall y\, A(r,n_1,\ldots,n_k,y) \in G$. $f \Vdash_{adm} \forall y\, A(f,n_1,\ldots,n_k,y) \leftrightarrow f \Vdash_{adm} \exists y\, \overline{A}(r,n_1,\ldots,n_k,y) \leftrightarrow$ for all n, $f \Vdash_{adm} \overline{A}(r,n_1,\ldots,n_k,n)$ $\leftrightarrow$ for all n, $f \Vdash_{adm} A(r,n_1,\ldots,n_k,n)$, by hypothesis, $\leftrightarrow$ for all n, $A(f,n_1,\ldots,n_k,n)$, by induction hypothesis, $\leftrightarrow \forall y\, A(f,n_1,\ldots,n_k,y)$.

3. Theorems.

For each $k > 0$, let p^k be a recursive one-one mapping of ω^k onto ω with recursive inverse functions $\pi_1^k,\ldots,\pi_k^k$. That is, for all z, $p^k(\pi_1^k(z),\ldots,\pi_k^k(z)) = z$. (Explicit examples are given in [13, p. 64].) We abbreviate $p(x_1,\ldots,x_k)$ as $\langle x_1,\ldots,x_k\rangle$.

For two degrees of unsolvability $\underset{\sim}{a}$ and $\underset{\sim}{b}$, $\underset{\sim}{a} \in \Sigma_n^{\underset{\sim}{b}}$ will mean that there is an A in $\underset{\sim}{a}$ and a B in $\underset{\sim}{b}$ so that $A \in \Sigma_n^B$. References [13] and [14] are cited as standard references to the fundamental concepts in the study of degrees, $\leq_T$ will denote relative recursiveness. The following Theorem 1 for the case $n = 1$ without the additional properties $\underset{\sim}{b} \notin \Pi_1^{\underset{\sim}{a}}$ and $\underset{\sim}{a} \notin \Pi_1^{\underset{\sim}{b}}$ is due to Spector [16]. The technique used to prove $\underset{\sim}{b} \notin \Pi_n^{\underset{\sim}{a}}$ and $\underset{\sim}{a} \notin \Pi_n^{\underset{\sim}{b}}$ is due to Shoenfield [15].

Theorem 1. $\forall \underset{\sim}{a} \exists \underset{\sim}{b} \exists \underset{\sim}{c} [\underset{\sim}{a}^{(n)} = \underset{\sim}{b}^{(n)} = \underset{\sim}{c}^{(n)} = \underset{\sim}{b} \vee \underset{\sim}{c}\ \&\ \underset{\sim}{b} \notin \Pi_n^{\underset{\sim}{a}}\ \&\ \underset{\sim}{c} \notin \Pi_n^{\underset{\sim}{a}}]$,

~~Proof;~~ Let h be a function with degree $\underset{\sim}{a}$. Two functions f and g will be defined so that:

(i) $f \leq_T h^{(n)}$ & $g \leq_T h^{(n)}$;

(ii) $f^{(n)} \leq_T f \vee g$ & $g^{(n)} \leq_T f \vee g$;

(iii) h ^ f & h ^ g; and

(iv) d(f) differs from the degree of every set which is Σ_n in h, d(g) differs from the degree of every set which is Σ_n in h.

Define adm(a) ⟺ ∀x[2x < lh(a) → $(a)_{2x} \dot{-} x$ = Mx)]. (iii) will be satisfied if f and g are defined so that adm(f) and adm(g) — for in that case, for each x, h(x) = f(2x) = g(2x).

Let $C_{n,e}$ denote the characteristic function of the e-th set Σ_n in h. Let φ_z^f, for any function f, denote the z-th function recursive in f. (iv) will be satisfied if f and g are defined so that:

(iv)' ∀x[∀z f ≠ $\varphi_z^{C_{n,x}}$ ∨ ∀y $C_{n,x}$ ≠ φ_y^f] and

∀x[∀z g ≠ $\varphi_z^{C_{n,x}}$ ∨ ∀y $C_{n,x}$ ≠ φ_y^g].

Let $B_n(\tau^{-1}(x_n),e,e,x_1,\ldots,x_{n-1})$ denote $T_n(\tau^{-1}(x_n),e,e,x_1,\ldots,x_{n-1})$ if n is odd, and denote $\overline{T}_n(\tau^{-1}(x_n),e,e,x_1,\ldots,x_{n-1})$, if n is even. Let Qx_i denote $\exists x_i$, if i is odd, and denote $\forall x_i$, if i is even. Let n be fixed. Let $e,x_1,\ldots,x_{n-k}$ be constants, where 1 ≤ k ≤ n, and let m = <e,x_1,...,x_{n-k}>. To the arithmetical property $Qx_{n-(k-1)}\ldots Qx_n\, B_n(\tau(x_n),e,e,x_1,\ldots,x_{n-k},x_{n-(k-1)},\ldots,x_n)$ we associate the index number n·m + k. Define [n·m + k] to be the arithmetical property with index number n·m + k. Clearly, to each integer I, I ≥ 1, there exist unique m and k = 1,...,n so that I = n·m + k. Thus with n fixed this indexing is unambiguous.

For ease of notation, we will write $\neg[\ell]$ for the negation of the arithmetical property $[\ell]$, rather than $\overline{[\ell]}$. The set of all arithmetical properties of the form $[\ell]$ and $\neg[\ell]$, $\ell \geq 1$ is a <u>closed</u> set of arithmetical properties.

Construction of f and g:

<u>Stage</u> 0. Define $f_0 = g_0 = 1$. Since $lh(f_0) = lh(g_0) = 0$, $adm(f_0)$ and $adm(g_0)$.

<u>Stage</u> $4\ell + 1$. By induction hypothesis $f_{4\ell}$ and $g_{4\ell}$ are defined; $adm(f_{4\ell})$, $adm(g_{4\ell})$, and $lh(f_{4\ell}) = lh(g_{4\ell})$.

<u>Case</u> 1. $\exists m, k[\ell + 1 = n \cdot m + k \;\&\; 0 < k < n]$. By Lemma 4, there is an α so that $\alpha \geq_{adm} f_{4\ell}$ and either $\alpha \Vdash_{adm} [\ell + 1]$ or $\alpha \Vdash_{adm} \neg[\ell + 1]$. Define $f_{4\ell+1}$ to be the least such α. Define

$$g_{4\ell+1} = g_{4\ell} \cdot \prod_{lh(f_{4\ell}) \leq i < lh(f_{4\ell+1})} p_i^{(f_{4\ell+1})_i}.$$

<u>Case</u> 2. $\exists m[\ell + 1 = n \cdot m + n]$.

If $\exists \alpha \geq_{adm} f_{4\ell}\; \alpha \Vdash_{adm} [\ell + 1]$, then let $\beta = \mu\alpha(\alpha \geq_{adm} f_{4\ell} \;\&$ $lh(\alpha)$ is odd $\&\; \alpha \Vdash_{adm} [\ell + 1])$. Define $f_{4\ell+1} = \beta \cdot p_{lh(\beta)}$, and define

$$g_{4\ell+1} = g_{4\ell} \cdot \prod_{lh(f_{4\ell}) \leq i < lh(\beta)} p_i^{(\beta)_i} \cdot p^1_{lh(\beta)}.$$

Otherwise, let $\beta = \mu\alpha(\alpha \geq_{adm} f_{4\ell} \;\&\; lh(\alpha)$ is odd $\&$ $\alpha \Vdash_{adm} \neg[\ell + 1])$.

In this case, define $f_{4\ell+1} = f_{4\ell} \cdot p^{1}_{lh(f_{4\ell})}$ and define

$$g_{4\ell+1} = g_{4\ell} \cdot \prod_{lh(f_{4\ell}) \leq i < lh(\beta)} p_i^{(\beta)_i} \cdot p^{2}_{lh(\beta)} .$$

<u>Stage</u> $4\ell + 2$. $f_{4\ell+2}$ and $g_{4\ell+2}$ are to be defined as in stage $4\ell + 1$, but with f and g interchanged.

<u>Stage</u> $4\ell + 3$. By induction hypothesis $f_{4\ell+2}$ and $g_{4\ell+2}$ are already defined, $adm(f_{4\ell+2})$, $adm(g_{4\ell+2})$, and $lh(f_{4\ell+2}) = lh(g_{4\ell+2})$. Let $I = <x,y,z>$. $f_{4\ell+3}$ shall be constructed at this stage so that for all admissible extensions F of $f_{4\ell+3}$ either, $\varphi_y^{n,x} \neq C^{n,x}$ or $F \neq \varphi_z^{n,x}$. Let $f^0 = f_{4\ell+2} \cdot p^{h(x)+1}_{lh(f_{4\ell+2})} \cdot p^{1}_{lh(f_{4\ell+2})+1}$ and

$f^1 = f_{4\ell+2} \cdot p^{h(x)+1}_{lh(f_{4\ell+2})} \cdot p^{2}_{lh(f_{4\ell+2})+1}$, if $lh(f_{4\ell+2})$ is even. If $lh(f_{4\ell+2})$ is odd, then let $f^0 = f_{4\ell+2} \cdot p^{1}_{lh(f_{4\ell+2})}$ and

$f^1 = f_{4\ell+2} \cdot p^{2}_{lh(f_{4\ell+2})}$.

<u>Case</u> 1. There do not exist characteristic sequence numbers α and β so that $adm(\alpha)$, $f_{4\ell+2} <\cdot\, \alpha$, $\beta < \varphi_y^{\alpha}$, and $f^0 < \varphi_z^{\beta}$. In this case define $f_{4\ell+3} = f^0$.

<u>Case</u> 2. There do exist characteristic sequence numbers α and β satisfying the hypotheses of case 1, but there do not exist characteristic sequence numbers α and β so that $f_{4\ell+2} <\cdot\, \alpha$, adm $\beta < \varphi_y^{\alpha}$ and $f^1 < \varphi_z^{\beta}$. In this case define $f_{4\ell+3} = f^1$.

<u>Case</u> 3. There exist characteristic sequence numbers a^0, p^0, a^1, and p^1 so that $f_{4\ell+2} \Vdash_{adm} \alpha^0$, $f_{4\ell+2} \Vdash_{adm} \alpha^1$, $p^0 \ll p$, $p^1 \ll p$, $f^0 < \beta^0_z$, and $f^1 < \beta^1_z$. Choose such a^0, p^0, a^1, and p^1. Since f^0 and f^1 differ for some argument, p^0 and p^1 must disagree for some argument. Hence, either p^0 or p^1, disagrees with $C_{n,x}$ for that argument. If p^0 disagrees, define $f_{4\ell+3} = a^0$; otherwise define $f_{4\ell+3} = a^1$.

Define

$$g_{4\ell+3} = \Pi\, p_i^{(f_{4\ell+3})_i}, \quad lh(f_{4\ell+2}) \leq i < lh(f_{4\ell+3}).$$

<u>Stage</u> $4\ell + 4$. $f_{4\ell+4}$ and $g_{4\ell+4}$ are to be defined as in stage $4\ell + 3$, but with f and g interchanged.

Define $f(y) = (f_{\mu m(y<lh(f_m))})_y \dot{-} 1$, and define $g(y) = (g_{\mu m(y<lh(f_m))})_y \dot{-} 1$.

<u>Claim</u> i. $f \leq_T h^{(n)}$ and $g \leq_T h^{(n)}$.

<u>Proof</u>; It is only necessary to see that the conditions used to define f and g are at most Σ_n in h. First, consider cases 1 and 2 of stage $4\ell + 1$. if $\ell + 1 = n \cdot m + k$, $0 < k < n$, then $[\ell + 1]$ has fewer than n alternating quantifiers. Thus, by Definition 3 of the forcing relation, $a \Vdash_{adm} [\ell + 1]$ is Σ_i in h, for some $i < n$. Thus, $\exists a \supseteq f_{4\ell}(a \Vdash_{adm} [\ell + 1] \vee a \Vdash_{adm} \neg [\ell + 1])$ is at most Σ_n in h. If $\ell + 1 = n \cdot m + n$, then by Definition 3

of the forcing relation, a Ih$_{adm}$ I^1 + 1] is $\%_n$ in h. Thus
3a >$_{adm}$ $f_{A\ell}$4v(a ||-$_{aam}$ [I + 1]) is $£_n$ in h. Similarly, the hypotheses in stage 41 + 2 are $£_n$ in h. It is easy to observe that the hypotheses of cases 1, 2, and 3 of stages 41 -f 3 and 41+4 are $£_n$ in h. Therefore, both f and g are recursive in predicates which are $£_n$ in h. That is, f $\leq_r$ $h^{(n)}$ and
$g \leq_r h^{(n)}$.

<u>Claim</u> ii. $f^{(n)}$ <^ f V g and $g^{(n)}$ ^ f V g.
<u>Proof</u>; We prove $f^{(n)}$ $\leq_1$ f V g# $g^{(n)}$ $\leq_r$ f V g is proved <u>mutatis mutandis</u>» For each m and k = l,2,...,n, f ||-$_{adm}$ [n«m + k] or
f ||-^{adm} —j[n*m -f k]. Therefore, by Lemma 5, f ||-^{adm} [n*m + k] if and only if [n*m + k] (f) • In particular, since
3x,...Qx$_n$ B_n(T(x_n),e,e,x_1,...x_{n-1}) is [n.e + n],
3x,...Qx$_n$ B^2_n(f(x_n),e,e,x_1,.•#,x_{n-1}) if and only if f ||-$_{aam}$ [n*e + n].
We show that for each e,

f l^adm [n*e + n]« 3a >ad» f4((n-e+n)-1) a »" adm ^e + n],

in fact, suppose 3a ^ f$_{4((n \cdot e+n)-1)}$ a ||- ^ [n-e + n]. Then, f4(fn*e+n)-l) -fl is an admissit)le extension of such an a. Therefore, by Lemma 3, f$_{4((n \cdot e+n)-1)+1}$ ||-$_{adm}$ fn-e + n]. Thus, for some m, f(m) ||-$_{adm}$ fn*e + n], that is, f Ih$_{adm}$ [n*e + n]. Now, suppose there exists an m so that T(m) ||-$_{aam}$ [n-e + n]. For such an m, if I(m) ^dm f4((n-e+n)-l)' then b^ Lemma 3,

Ea >ad» f4((n.e+n)-l)+l a "" adm [n•e + n].

If f<m) >adn f4((n.e+n)-l)' then of course

$\exists\alpha >_{adm} f4((n\text{-}e_+n)\text{-}l) * lh adm [n \cdot e + n].$

Therefore,

f ⊩ adm [n- e + n] f-> 3x > adm ,4((n*e+n)-l) a ⊩ adm [n* e + n] .

Define a function K by

K(o) = /ix[f(x) ^ g(x)],

K(x + 1) = Atyfy > K(x) & f(y) ^ g(y)j.

f and g have been constructed so that

aa > * adm f-Att 4((n*e+n)-l) ix a ⊩ * adm [n-e + n]

if and only if f(K(2e)) = 1. Hence,

$\exists x_1 \ldots Qx_n \; B^1_n(\overline{f}(x_n), e, e, x_1, \ldots, x_{n-1})$

if and only if f(K(2e)) = 1. Thus $f^{(n)} \leq_r f \vee g$.

<u>Claim</u> iii. $h \leq_r f$ and $h \leq_r g$, since adm(f) and adm(g).

Claim ivf . Vx[Vz f ^ <P n,x V Vy c ^ Vf] and
z J n,x y

C

Vx[Vz g * <Pzn>* V Vy c ^ x ^ ^] .

<u>Proof</u>: It will be shown for every x,y, and z, that either

C

Cn,x / <pry or f ^ if) n>x.
z

For any x,y, and z, let I = <x,y,z>. If $f_{4\iota+3}$ is constructed according to case 1 or case 2 of stage 41 + 3, and if

$C_{n,x} = \varphi_y^{f}$, then $f_{4\ell+2} \neq \varphi_z^{C_{n,x}}$. Hence, $f \neq \varphi_z^{C_{n,x}}$. If $f_{4\ell+3}$ is constructed according to case 3 of stage $4\ell+3$, then $C_{n,x} \neq \varphi_y^{f_{4\ell+2}}$. Hence, $C_{n,x} \neq \varphi_y^{f}$.

To complete the proof of Theorem 1, let $b = d(f)$ and let $c = d(g)$. By (i), (ii), and (iii), $a^{(n)} \leq b^{(n)}_a \leq b \vee c \leq a^{(n)}_a$ and $a^{(n)} \leq c^{(n)} \leq b \vee c \leq a^{(n)}$. By (iv), $b \notin \mathcal{E}$ and $c \notin \mathcal{E}$.

In Theorem 1 an arbitrary number n is given, and then remains fixed throughout the entire proof. The idea of the following theorem is to force the set of all arithmetical properties and negations of arithmetical properties of the form $[n*m + k]$, for all n,m, and $k = 1,2,\ldots,n$. (Of course, our indexing must be altered since it is ambiguous if n is not fixed.) Also, the theorem will not be presented in a relativized form, so every sequence number a is admissible, and we will write $\Vdash$, rather than $\Vdash_{adm}$.

Theorem 2. $\exists a \exists b[a^{(\omega)} = b^{(\omega)} = 0^{(\omega)} = a \vee b]$.

Proof; Two functions f and g will be defined so that:

(i) $d(f) \leq 0^{(\omega)}$, $d(g) \leq 0^{(\omega)}$ and

(ii) $f^{(\omega)} \leq_r f \vee g$ and $g^{(\omega)} \leq_r f \vee g$.

As before, let $B_n(T^n(x_n), e, e, x_1, \ldots, x_{n-1})$ denote $T_n^1(T^n(x_n), e, e, x_1, \ldots, x_{n-1})$, if n is odd, and denote $T_n^1(T^n(x_n), e, e, x_1, \ldots, x_{n-1})$, if n is even. Let Qx_i denote $\exists x_i$ if i is odd, and denote $\forall x_i$, if i is even. For each natural number $I = \langle n,m\rangle$, n]≥ 1, and m ^≥ 1, define [I] to be the arithmetical property

n-(k-l) n n^v v n ' 9 V ' n-k' n-(k-l)* ' n-1

where m = n-q + k, 1 ^ k ^ n, and q = <e,x -, ..#,x ^>. The set 1 n-* yc

of all arithmetical properties [I] and T[£], for -t = <n,m>, n ^ 1, and m ^ 1, is a closed set of arithmetical properties.

Construction of f and g:

Stage 0. Define $f_0 = g_0 = 1$.

Stage 2b + 1. By induction hypothesis f-f and g -, are defined and have the same length.

Case 0. There do not exist n ^≥ 1 and m ^ 1 so that $I = \langle n,m\rangle$.

Define $f_{2l+1} = f_{2l}$ and $g_{2\ell+1} = g_{2\ell}$.

Case 1. There exist integers n,m,q, and k so that n ^ l, m ;≥ 1, $I = \langle n,m\rangle$, m = n^q + k, and 0 < k < n.

Define

$$f_{2l+l} = V^* > f_{2l} \ [a \Vdash [I] \text{ or } a \Vdash \neg[\ell]].$$

Define

$$g_{2\ell+1} = g_{2\ell} \cdot \prod_{lh(f_{2\ell}) \leq i < lh(f_{2\ell+1})} p_i^{(f_{2\ell+1})_i}.$$

Case 2. There exist integers n,m, and q so that n J≥ 1, m ^ 1, -t = <n,m>, and m = n*q + n.

If 3a > £$_{2t}$ a ||- [I], then let 0 = jua > f^fa ||- [*]). Define $f_{2\ell+1} = \beta \cdot p^2_{lh(\beta)}$, and define

$$g_{2\ell+1} = g_{2\ell} \cdot \prod_{lh(f_{2\ell}) \leq i < lh(\beta)} p_i^{(\beta)_i}.$$

Otherwise, let p = Jtc > £, (a ||—1[^]). In this case, define $f_{2\ell+1} = \cdot p^1_{lh(\beta)}$, and define

$$g_{2\ell+1} = g_{2\ell} \cdot \prod_{lh(f_{2\ell}) \leq i < lh(\beta)} p_i^{(\beta)_i} \cdot p^2.$$

<u>Stage</u> 2^+2, f.. . and g .. . are to be defined as in stage 24+1, but with f and g interchanged.

Define f(y) = (f$_{\mu m}$ ^ ^ ^ ,,) $_y$ -̇ 1, and define g(y) = $(g_{\mu m}(y<lh(g_m)))_y$ " 1'

Claim i. d(f) < 0$^{(U))}$ and d(g)<0$^{(t0)}$.

~~Proof;~~ The definitions of f-. - from f- and of f-. o from f-, . are recursive in predicates which are recursive in £ , where the n can be found effectively from *I.* Therefore f is recursive in 0· • Similarly, g is recursive in 0· .

<u>Claim</u> ii. f^{v} } $\leq_r$ f V g and g^{v} ; $\leq_r$ f V g.

<u>Proof:</u> We prove f$^{\cdot x}$ $\leq_r$ f V g. 9 · $\leq_r$ f V g is proved <u>mutatis mutandis</u>.

For each *I, f* \\- *[I]* or f ||--?[£]. Therefore, by Lemma 5, f ||- *[I]* if and only if *[I]* (f). In particular, since for each n,

$\exists x_1 \ldots Q x_n\ B_n^1(\overline{T}(x_n), e, e, x_1, \ldots, x_{n-1})$ is [<n,n«e + n>],

$\exists x_1 \ldots Q x_n\ B_n^1(\overline{f}(x_n), e, e, x_1, \ldots, x_{n-1})$ if and only if f ||- [<n,n-e + n>].

For each n and e, f ⊩ [<n,n-e + n>] if and only if

$$\exists a > f_{2^*<n,n^*e+n>} \wedge a \Vdash [<n,n\text{-}e + n>].$$

The proof of this statement is identical to the proof presented for the similar statement in claim (ii) of Theorem 1.

Define a function K by

K(l) = /ix 3n ;> 1 3m [x = 2*<n,n*m + n> + 1 or x = 2*<n,n*m + n>+2],

K(y+1) = μx 3n ^ 1 3m[x > K(y) & (x = 2*<n,n-m + 1> + 1

or x = 2*<n^n«m + n>+2)].

The y^{th} argument x for which f(x) ^ g(x) is introduced at stage K(y) of the construction of f and g.

Define p(n,m) = My[K(y) = 2«<n,n»m + n> + 1]. At stage 2«<n,n»m + n> + 1, the p(n^m)-th argument x for which f(x) ≠ g(x) is introduced.

Define h(l) = μx[f(x) ≠ g(x)], h(y+l) = μx[x > h(y) & f(x) ≠ g(x)].

f and g have been constructed so that

$$\exists a > f_{2\cdot<n,n\cdot e+n>} \wedge a \Vdash [<n,n*e + n>]$$

if and only if f(h(p(n,e))) = 1. Thus, for each n and each e, $\exists x_1 \ldots Q x_n\ B_n^1(\overline{f}(x_n), e, e, x_1, \ldots, x_{n-1})$ if and only if f(h(p(n,e))) = 1.

Therefore $f^{(n)}$ is uniformly recursive in $f \vee g$. By definition $\langle x,y\rangle \in f^{(\omega)} \leftrightarrow x \in f^{(y)}$. But, $x \in f^{(y)} \leftrightarrow f(h(p(y,x))) = 1$. Therefore, $f^{(\omega)} \leq_r f \vee g$.

To complete the proof of Theorem 2, let $\underset{\sim}{a} = \underset{\sim}{d}(f)$ and let $\underset{\sim}{b} = \underset{\sim}{d}(g)$. By (i) and (ii), $\underset{\sim}{0}^{(\omega)} \leq \underset{\sim}{a}^{(\omega)} \leq \underset{\sim}{a} \vee \underset{\sim}{b} \leq \underset{\sim}{0}^{(\omega)}$, and $\underset{\sim}{0}^{(\omega)} \leq \underset{\sim}{b}^{(\omega)} \leq \underset{\sim}{a} \vee \underset{\sim}{b} \leq \underset{\sim}{0}^{(\omega)}$.

The following Theorem 3 for the case $n = 1$ is a relativized version of Friedberg's characterization of the complete degrees [4].

<u>Theorem</u> 3. $\forall a \forall b \exists c[c^{(n)} = c \vee a^{(n)} = b \vee a^{(n)}]$.

<u>Proof;</u> Let h be a characteristic function with degree $\underset{\sim}{a}$. Let g be a characteristic function with degree $\underset{\sim}{b}$. A function f will be defined so that:

(i) $f^{(n)} \leq_r h^{(n)} \vee g$;

(ii) $g \leq_r f \vee h^{(n)}$; and

(iii) $h \leq_r f$.

As in the proof of Theorem 1, define

$adm(a) \leftrightarrow \forall x[2x < lh(a) \rightarrow (a)_{2x} \dot{-} 1 = h(x)]$. (iii) will be satisfied if f is defined so that $adm(f)$.

Also let n be fixed, and let the arithmetical properties $Qx_{n-(k-1)} \ldots Qx_n B_n(\bar{T}^1_n(\bar{x}_n), e, e, x_1, \ldots, x_{n-1})$ be defined and indexed as in the proof of Theorem 1. Then, $[n*m + k]$, $k = 1,2,\ldots,n$, is the arithmetical property $Qx_{n-(k-1)} \ldots Qx_n B_n(\bar{T}^1_n(x_n), e, e, x_1, \ldots, x_{n-1})$, where $m = \langle e, x_1, \ldots, x_{n-k}\rangle$.

Construction of f:

Stage 0. Define $f_0 = 2^{h(0)+1}$. adm(f_0) and lh(f_0) is odd.

Stage $\ell + 1$. By induction hypothesis f_ℓ is defined, adm(f_ℓ), and lh(f_ℓ) is odd.

Case 1. $\exists m, k[\ell + 1 = n \cdot m + k \ \& \ 0 < k < n]$. Define

$$f_{\ell+1} = \mu\alpha\left[\alpha >_{adm} f_\ell \cdot p_{lh(f_\ell)}^{g(\ell)+1} \ \& \ lh(\alpha) \text{ is odd } \& \ (\alpha \Vdash_{adm} \ldots[\ell + 1] \text{ or } \alpha \Vdash_{adm} \neg[\ell + 1])\right].$$

By Lemmas 3 and 4, such an α exists.

Case 2. $\exists m[\ell + 1 = n \cdot m + n]$. If

$$\exists\alpha >_{adm} f_\ell \cdot p_{lh(f_\ell)}^{g(\ell)+1} \left[\alpha \Vdash_{adm} [\ell + 1]\right],$$

then define

$$f_{\ell+1} = \mu\alpha\left[\alpha >_{adm} f_\ell \cdot p_{lh(f_\ell)}^{g(\ell)+1} \ \& \ lh(\alpha) \text{ is odd } \& \ \alpha \Vdash_{adm} [\ell + 1]\right].$$

Otherwise, define

$$f_{\ell+1} = \mu\alpha\left[\alpha >_{adm} f_\ell \cdot p_{lh(f_\ell)}^{g(\ell)+1} \ \& \ lh(\alpha) \text{ is odd } \& \ \alpha \Vdash_{adm} \neg[\ell + 1]\right].$$

Note that for all x, $x < lh(f_x)$. Define $f(x) = (f_x)_x \dot{-} 1$. Define a function K by $K(x) = f_x$. As in the proof of claim i of Theorem 1 and 2, it is easy to see that $K \leq_r h^{(n)} \vee g$. Thus, it is proved that $f \leq_r h^{(n)} \vee g$. We need to prove the stronger statement:

Claim i. $f^{(n)} \leq_r h^{(n)} \vee g$.

Proof; Again, as in the proof of Theorem 1,

$\exists x_1 \ldots Q x_n B_n^1(T(x_n),e,e,x_1,\ldots,x_{n-1})$ if and only if $f \Vdash_{adm} [n\cdot e + n]$.

Suppose $f_{n\cdot e+n} \Vdash_{adm} [n\cdot e + n]$. Then of course, for some m, $\bar{f}(m) \Vdash_{adm} [n\cdot e + n]$. Conversely, suppose that for some m, $\bar{f}(m) \Vdash_{adm} [n\cdot e + n]$. If $\bar{f}(m) \geq_{adm} f_{(n\cdot e+n)-1} * \lambda^{g((n\cdot e+n)-1)+1}(lh(f_{(n\cdot e+n)-1}))$, then by definition, $f_{n\cdot e+n} \Vdash_{adm} [n\cdot e + n]$. If $\bar{f}(m) \leq_{adm} f_{(n\cdot e+n)-1} * \lambda^{g((n\cdot e+n)-1)+1}(lh(f_{(n\cdot e+n)-1}))$, then still, by Lemma 3, $f_{n\cdot e+n} \Vdash_{adm} [n\cdot e + n]$. Therefore,

$$f \Vdash_{adm} [n\cdot e + n] \leftrightarrow f_{n\cdot e+n} \Vdash_{adm} [n\cdot e + n].$$

Thus,

$$\exists x_1 \ldots Q x_n B_n^1(T(x_n),e,e,x_1,\ldots,x_{n-1}) \leftrightarrow K(n\cdot e + n) \Vdash_{adm} [n\cdot e + n].$$

The right hand side is recursive in $h^{(n)}$ and g. Thus, $f^{(n)} \leq_r h^{(n)} \vee g$.

Claim ii. $g \leq_r f \vee h^{(n)}$.

Proof; $g(x) = f(lh(K(x)))$, for all x. Using the definition of K, substitute $f(lh(K(x)))$ for $g(x)$ in the definition of K, to obtain $K \leq_r f \vee h^{(n)}$. Then, use $g(x) = f(lh(K(x)))$, to obtain $g \leq_r f \vee h^{(n)}$.

Claim iii. $h \leq_r f$, since adm(f).

To complete the proof of Theorem 3, let $c = d(f)$. By (i), (ii), and (iii), $c^{(n)} \leq a^{(n)} \vee b \leq c \vee a^{(n)} \leq c^{(n)}$.

Corollary 1. $\forall a \forall b [a^{(n)} < b \rightarrow \exists c [c^{(n)} \equiv b \;\&\; c \mid a^{(n)}]]$.

The proof is immediate.

Corollary 2. $\forall a \forall b \exists c [c^{(n+1)} = c^{(n)} \vee a^{(n+1)} = b \vee a^{(n+1)}]$.

Proof;

$c^{(n+1)} \wedge c \vee a^{(n+1)} \leq c^{(n)} \vee a^{(n+1)} = (c \vee a^{(n+1)}) \vee c^{(n)} \leq c^{(n+1)}$.

Corresponding to the original Kleene-Post construction [11] of $\leq_1$-incomparable sets in Δ_2, we can now prove the existence of $\leq_n$-incomparable sets in Δ_{n+1}. In the following theorem we incorporate ideas from Theorem 3 to get a stronger result. Peter Hinman [7] has proved, corresponding to the Friedberg-Muchnik theorem ([5] and [12]), that there exist Δ_{n+1}-incomparable sets in Σ_{n+1}.

Theorem 4. $\exists A \exists B [A \not\leq_n B \;\&\; B \not\leq_n A \;\&\; d(A)^{(n)} = 0^{(n)} = d(B)^{(n)}]$.

Proof; Two functions f and g will be defined so that

(i) $d(f^{(n)}) \leq 0^{(n)}$ and $d(g^{(n)}) \leq 0^{(n)}$; and

(ii) $f \not\leq_n g$ and $g \not\leq_n f$.

Let $B^1_{nn}(\wedge(x\ ), e, u, x_1, \ldots, x_n)$ denote $T^1_{nn}(T^{\prime\prime}(x\ ), e, u, x_1, \ldots, x_n)$, if n is odd, and denote $\overline{T}^1_n(T^{\wedge}x_n), e, u, x_1, \ldots, x_n)$, if n is even. Let Qx_i denote $\exists x_i$, if i is odd, and denote $\forall x_i$, if i is even. Let $e, x_1, \ldots, x_{n-k}$ be constants, where $1 \leq k \leq n$, and let

$m = \langle e, x_1, \ldots, x_{n-k} \rangle$. To the arithmetical property $Qx_{n-(k-1)} \ldots Qx_n B^1_n(T(x_n), e, u, x_1, \ldots, x_{n-1})$ with one free number variable u, we associate the index number $n \cdot m + k$. Define $[n \cdot m + k](u)$ to be the arithmetical property with index number $n \cdot m + k$. With n fixed this indexing is unambiguous.

Observe that $[n \cdot e + n](e)$ is the arithmetical property $\exists x_1 \ldots Qx_n B^1_n(T(x_n), e, e, x_1, \ldots, x_{n-1})$. (It may be assumed that $\langle x \rangle = x$, for all x.)

Construction of f and g:

Stage 0. $f_0 = g_0 = 1$.

Stage $6\ell + 1$. By induction hypothesis $f_{6\ell}$ and $g_{6\ell}$ are defined.

Case 1. $\exists x,a,m,k[\ell = \langle x,a \rangle \;\&\; x = n \cdot m + k \;\&\; 0 < k < n]$.

In this case define $f_{6\ell+1} = \mu a > f_{6\ell}[a \Vdash [x](a) \text{ or } a \Vdash \neg[x](a)]$, and define $g_{6\ell+1} = g_{6\ell}$.

Case 2. $\forall x,a,m,k[(\ell = \langle x,a \rangle \;\&\; x = n \cdot m + k) \to (k = 0 \text{ or } k = n)]$.

Define $f_{6\ell+1} = f_{6\ell}$ and $g_{6\ell+1} = g_{6\ell}$.

Stage $6\ell + 2$. $f_{6\ell+2}$ and $g_{6\ell+2}$ are to be defined as in stage $6\ell+1$, but with f and g interchanged.

Stage $6\ell + 3$. By induction hypothesis $f_{6\ell+2}$ and $g_{6\ell+2}$ are defined.

Case 1. $\exists x\,[a > f_{6\ell+2} \;\&\; a \Vdash [n \cdot \ell + n](lh(g_{6\ell+2}))]$.

Define

$$f_{6\ell+3} = \mu\alpha > f_{6\ell+2}\ \alpha \Vdash [n\cdot\ell + n](lh(g_{6\ell+2})).$$

Define

$$g_{6\ell+3} = g_{6\ell+2}\cdot p^{1}_{lh(g_{6\ell+2})}.$$

Case 2. $\forall\alpha\,[\alpha > f_{6\ell+2} \rightarrow \alpha \nVdash [n\cdot\ell + n](lh(g_{6\ell+2}))]$. By Lemma 4,

$$\exists\alpha > f_{6\ell+2}\ \alpha \Vdash \neg[n\cdot\ell + n](lh(g_{6\ell+2})).$$

Define

$$f_{6\ell+3} = \mu\alpha > f_{6\ell+2}\ \alpha \Vdash \neg[n\cdot\ell + n](lh(g_{6\ell+2})).$$

Define

$$g_{6\ell+3} = g_{6\ell+2}\cdot p^{2}_{lh(g_{6\ell+2})}.$$

Stage $6\ell + 4$. $f_{6\ell+4}$ and $g_{6\ell+4}$ are to be defined as in stage $6\ell + 3$, but with f and g interchanged.

Stage $6\ell + 5$. By induction hypothesis $f_{6\ell+4}$ and $g_{6\ell+4}$ are defined.

If $\exists\alpha > f_{6\ell+4}\ \alpha \Vdash [n\cdot\ell + n](\ell)$, then define $f_{6\ell+5} = \mu\alpha > f_{6\ell+4}\ \alpha \Vdash [n\cdot\ell + n](\ell)$, and define $g_{6\ell+5} = g_{6\ell+4}$.

Otherwise, define $f_{6\ell+5} = \mu\alpha > f_{6\ell+4}\ \alpha \Vdash \neg[n\cdot\ell + n](\ell)$ and define $g_{6\ell+5} = g_{6\ell+4}$.

Stage $6\ell + 6$. $f_{6\ell+6}$ and $g_{6\ell+6}$ are to be defined as in stage $6\ell + 5$, but with f and g interchanged.

Define $f(x) = (f_{\mu m(x<lh(f_m))})_x \dot{-} 1$ and define $g(x) = (g_{(\mu m(x<lh(g_m))})_x \dot{-} 1$.

Define $A = \{x|f(x) = 1\}$, and $B = \{x|g(x) = 1\}$.

Claim i. $\underset{\sim}{d}(f^{(n)}) \leq \underset{\sim}{0}^{(n)}$ and $\underset{\sim}{d}(g^{(n)}) \leq \underset{\sim}{0}^{(n)}$.

Proof: For each e, the set G of all arithmetical properties and negations of arithmetical properties $[n \cdot m + k](e)$, $1 \leq k \leq n$, where $m = \langle e, x_1, \ldots, x_{n-k} \rangle$ is a closed set of arithmetical properties. Let e and $x_1, \ldots, x_{n-k}$, be arbitrary constants, and let $m = \langle e, x_1, \ldots, x_{n-k} \rangle$. At stage $6 \cdot \langle n \cdot m + k, e \rangle + 1$, $f_{6 \cdot \langle n \cdot m+k \rangle + 1}$ is chosen so that $f \Vdash [n \cdot m + k](e)$ or $f \Vdash \neg [n \cdot m + k](e)$. At stage $6e + 5$, f_{6e+5} is chosen so that $f \Vdash [n \cdot e + n](e)$ or $f \Vdash \neg [n \cdot e + n](e)$. Thus, given e, for each arithmetical property A in G, $f_1 \Vdash A$ or $f \Vdash A$. By Lemma 5,

$\exists x^1 \ldots Q x^n B^n(T(x^n), e, e, x^1, \ldots, x^{n-1})$ if and only if $f \Vdash [n \cdot e + n](e)$. $f \Vdash [n \cdot e + n](e)$ if and only if $f^{(n)}_{6e+5} \Vdash [n \cdot e + n](e)$. (This is easy to see, and has been argued previously.) Define a function K by $K(x) = f^{(n)}_x$ for all x. $d(K) \leq 0^{(n)}$. (The argument is similar to the proof of claim i of Theorems 1 and 2). It follows that $\exists x_1 \ldots Q x_n B_n(T(x_n), e, e, x_1, \ldots, x_{n-1})$ if and only if $K(6e + 5) \Vdash [n \cdot e + n](e)$. The right hand side is recursive in $0^{(n)}$.

Similarly it may be proved that $\underset{\sim}{d}(g^{(n)}) \leq \underset{\sim}{0}^{(n)}$.

Claim ii. $A \in \Pi_n^A$ & $B \in \Sigma_n^A$.

Proof: We will show that $B \, i \, th._n$. The proof that $A^{\wedge}J T^{R}_n$ is similar,

$B \, e \, I?''_n$ if and only if there is some e so that for all a,

g(a) = 1 if and only if $ax^{\wedge}-.Qx^{\wedge} B_n(f^1(xj, e, a, x_{;L}, \ldots \wedge x_{n-1})$ •

For each e, it will be shown that $9(^{lh}(g_{6e+2})) = 0$ if and only if $3x_r..Qx_n B^{\wedge}(T(x_n), e, lh(g_{62+2}), x_{l5}, \ldots, x_n{}^{\wedge}{}_1)$ -- from which it follows that $B^{\wedge}E^7{}^{\wedge}_n$.

For each e, the set Q of all arithmetical properties and negations of arithmetical properties $[n«m + k](lh(g_{6e+2}))$, $1 \pounds k \pounds n$, where $m = <e, x^1, \ldots, x^{n-} \wedge>$ is a closed set of arithmetical properties. Given numerals e and $x_J, \ldots, x_{n-is}$,, $0 < k < n$, let $m = <e, x^1, \ldots, x^{n-}{}_k>$ and let $I = <n-m + k, lh(g_{6e+2})><$ At stage 6-t + *1,* $f_g{}^{\wedge}{}_{+1}$ is chosen so that f ||- $[n-m + ^k](lh(g_{6e+2}))$ or f ||- $-i[n-e + k](lh(g_{6e+2}))$. At stage 6e + 3_5 f_{6e+3} is chosen so that f ||- $[n-e + n](lh(g_{6e+2}))$ or f $||-_n[n.e + n](lh(g_{6e+2}))$. Thus, for each arithmetical property A in Q, f ||- A or f ||-1A. $3x_1 \ldots Qx_n B^1_n(\overline{T}(x_n), e, lh(g_{6e+2}), \ldots)$ is $[n-e + n](lh(g_{6e+2}))$. • By Lemma 5, $3x_1 \ldots Qx_n B^1_n(T(x_n)_9 e_9 lYi\{q^{\wedge}_{be+2}), x_l, \ldots, x_{n-l})$ if and only if f |}- $[n.e + n](lh(g_{oe+2}))$. (It may be remarked that the necessity of stages 6£ + 1 is that for each e, $\wedge(g^{\wedge}_{e+2} J)$ is not known in advance,) Again f ||- $[n^*e + n](lh(g_{6e+2}))$ if and only if $3a[a > f_{6e+2}$ & a II- $[n^*e + n](lh(g_{6e+2}))]$. On the other hand, by definition of 9_{6e+3}, $3a[a > fg_{e+2}{}^{\&a} ll- [n-e + n](lh(g_{6e+2}))]$ if and only if $g(lh(g_{A_{6e+2}}^{\circ})) = 0$. This completes the proof of claim ii.

The proof of Theorem 4 is now complete: $A \in XT_n$, $B \in TT_n$ and by (i), $d(A)^{(n)} = Q^{(n)} = d(B)^{(n)}$.

Remark. Let fi be the set of all degrees. For each $n \geq 1$, let $«_n$ be the structure $<«, £^{(n)}>$. $\&_1$ is the structure $<«, \leq_1, ^f>$. It has been shown in this chapter that certain sentences which hold in $\&_1$ hold in $\$_n$ for all n. Is fi_n elementarily equivalent to $\&_m$, for $n,m \wedge 1$? This question has been answered in the negative by C. G. Jockusch, Jr., in private communication.

Let G be the set of all degrees of arithmetical sets. The proof given by Jockusch uses the fact that G can be simultaneously first-order defined in $\&_1$ and $\&_2$ (A corollary to this fact, is Jockusch's result, announced in [8], that the structures $<\$, <£., ^f>$ and $<G, \leq\wedge, ^f>$ are not elementarily equivalent.)

The method of proof leaves open two interesting questions. It is not known whether fi_n is elementarily equivalent to $\$_m$, for n and m both greater than one; and it is not known whether the structures $<G, £, ^{(n)}>$ and $<G, £, ^{(m)}>$ are elementarily equivalent, for $n,m \geq 1$.

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FOOTNOTES

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