# ON THE GEOMETRY OF SPHERES
# IN L-SPACES

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Abstract

It is shown that any two points on the surface of the unit ball of $L^1$(fi), where the measure \i is non-atomic, may be joined in the surface by a curve whose length is equal to the straight-line distance between its endpoints. This property is contrasted with the metric properties of the unit sphere in other L-spaces.

## 1. Introduction

In [3], Harrell and Karlovitz call a Banach space X *flat* if there exists on the surface of the unit ball a curve of length 2 with antipodal endpoints. They observe that $L^1$(ji), where \i is Lebesgue measure on the unit interval, is flat, and that in fact such a curve exists for *every* prescribed pair of antipodal endpoints. Our first aim is to show, in Theorem *2,* that $L^1$*(fi)* for this and every other non-atomic measure \x is even "flatter" than this observation alone indicates.

This question belongs to an area of investigation developed in [5], dealing with certain metric parameters of the unit spheres of norraed spaces. In Section 3 we discuss the values of these parameters for all L-spaces.

If X is a normed space, S(X) denotes its unit ball, and $d\Sigma(X)$ the boundary of 2(X). A *curve* in X is a "rectifiable geometric curve" as defined in [1; pp. 23-26]; for terminological details see [5; p. 61]. The length of a curve c is *l(c)* .

We shall consider in what follows a measure space (S,S,fi) and the corresponding Banach space $L^1 = L^1$(fi) of (equivalence classes of) real-valued functions. The argument \i is omitted when confusion is unlikely. We assume once and for all that dim $L^1$ > 1; i.e., that there exist E,E'eS with 0 < **μ(E)** < **μ(E')** < oo.

## 2. Non-atomic measures

1• Lemma. If \i JLS^ non-atomic and pe^S($L^1$), there exists a. curve of length || |p|-p|| from p jto |p| ±n d£($L^1$).

Proof. Consider the measure v on S defined by

v(E) = $\int_E$ |p |d|Lt, EeS. Then (S,£>,v) is a non-atomic measure space with v(S) = 1. There exists, therefore (see, e.g., [2]), a function G : [0,1]-*S such that G(0) = 0, G(l) = S, G(S) c G(t) whenever s <; t, and such that v(G(t)) = t, 0 £ t ^ 1.

We now set f(t) = $|p|x_{G(t)}$ + $P*_{s}\backslash_{G(t)}$' ° ^ t ^ 1- Then |f(t) I = |p|, whence ||f(t)|| = 1, for all t. If s ^ t we have f(t)-f(s) = ( |p|-P)$x_{G(t)} \backslash_{G(s)}$ > whence

(1) $$||f(t)-f(s)|| = \int_{G(t)\backslash G(s)} (|p|-p)d\mu .$$

From (1), ||f(t)-f(s)|| ^ 2v(G(t)\G(S)) = 2(t-s), so that f : [0,13-^$L^1$ is Lipschitzian. Since f(0) = p, f(l) = [p|. f is a parametrization of a curve from p to |p| in ∂Σ($L_1^-$).

Again from (1), ||f(t)-f(s)|| = <p(t)-<p(s), where <p(t) = $\int_{G(t)}$ (|p|-p)dfi. Therefore the length of the curve is

$$\varphi(1)-\varphi(0) = \int_S (|p|-p)d\mu = \||p|-p\|.$$

2. Theorem, jjf Jt is non-atomic and p,qeBE($L^1$), there exists a. curve of length ||q-p|| from p Jto q ±n S£($L^1$) .

Proof. The preimage of (-00 ,0) under the function p+q is an element of S; changing the values of each and every function in $L^1$ to their opposites on this set constitutes a congruence of $L^1$ onto itself. Modulo this congruence we may assume without loss that

(2) p+q ^ 0.

Now

(3) q-p = (q-|q|) + (|q|-|p|) + (|p|-p).

It follows from (2) that any two of the three summands in the second member of (3) are almost nowhere of strictly opposite signs. Therefore |q-p| == ( |q|-q) + | |q| - |P I I + ( IP |-P), and

(4) ||q-p|| = |||q|-q|| + |||q|-|p|||l + IIIP|-PI|.

Now by Lemma 1 there exist curves from p to |p| and from |q I to q in S2($L^1$ ) with lengths || |p |-p|| and |||q|-q||, respectively; and since |p|,[q| ^ 0, the straight-line segment from |p| to |q | (which has length |||q|-|p|||) lies entirely in ^($L^1$). Putting curve, segment, and curve together end-to-end, we obtain a curve from p to q in SE($L^1$ ), and its length is ||q-p|| on account of (4).

## 3. Spheres in L-spaces

We recall some further terminology and notation from [5]. Let X be a normed space with dim X > 1. The inner metric $6 = 6_{\mathbf{v}}$ of SS(X) is defined by 6(p,q) = ±*nf[l(c)* : c a curve from p to q in d£(X)} for all p,qed2(X). Using this inner metric, we define the parameters

$$m(X) = \inf\{6(-p,p) : p\in\partial\Sigma(X)\}$$
$$M(X) = \sup\{6(-p,p) : p\in\partial\Sigma(X)\}$$
$$D(X) = \sup\{6(p,q) : p,q\in\partial\Sigma(X)\}.$$

2m(X) and 2M(X) are, respectively, the girth and the perimeter of S(x); D(X) is the inner diameter of SS(X). We note in passing that it is a conjecture unresolved in general (though verified for, e.g., dim X ^ 3, or D(X) = 4) that M(X) = D(X) for all X.

We note an immediate consequence of the work in Section 2.

3. Corollary. If JI is non-atomic, $6_L$ i(p,q) = |lq-p|| for all p^edEfL[1]), and DCL[1]) = MfL[1]) = m(L[1]) = 2.

Proof. Immediate from the definitions and Theorem 2.

What if fx is not non-atomic? We refer to [9] and recall that, in a normed space X, a point u is a pole of X if 6(-u,u) = 4. Existence of a pole is sufficient but not necessary (unless dim X < oo ) for D(X) = M(X) = 4.

4. Theorem. If A€S is an atom for jit, then u = $(jx(A))^{-1}\chi_A$ i£ a pole £f $L^1$, and $D(L^1) = M(L^1) = 4$.

Proof. If $a(x) = \int_A x\,d\mu$, $x \in L^1$, we have $x = a(x)u + x\chi_{S\setminus A}$ for all $x \in L^1$, and $\|x\| = |a(x)| + \|x\chi_{S\setminus A}\|$.

Now $a : L^1 \to R$ is continuous, and $a(\pm u) = \pm 1$. On any given curve $c$ from $-u$ to $u$ in $BS(L^1)$ there must therefore be a point $v$ with $a(v) = 0$, whence $v = v\chi_{S\setminus A}$. Since $u\chi_{S\setminus A} = 0$, we find

$$\ell(c) \geq \|u-v\| + \|v+u\| = |a(u)| + \|v\chi_{S\setminus A}\| + |a(u)| + \|v\chi_{S\setminus A}\|$$
$$= 2 + 2\|v\| = 4.$$

Therefore $\delta(-u,u) \geq 4$, and the reverse inequality holds by [5; Theorem 3.5]. Thus $u$ is a pole. The proof could have been rephrased so as to use part of [9; Theorem 4.1 with Remark].

On the other hand, we record the following observation.

5. Theorem. If $\mu$ is not purely atomic, $L^1$ is flat, and $m(L^1) = 2$.

Proof. If $\mu'$ is the (non-null) restriction of $\mu$ to the non-atomic part of the measure space, $L^1(\mu)$ contains a subspace congruent to $L^1(\mu')$. By Theorem 2, this subspace is flat, hence so is $L^1(\mu)$ itself.

It remains to consider purely atomic measures. Now if $\mu$ is purely atomic, $L^1(\mu)$ is congruent $\ell^1(\kappa)$, where $\kappa$ is the cardinal of the set of (equivalence classes of) atoms; and for every cardinal $\kappa$, $\ell^1(\kappa)$ is of course itself $L^1(\mu_\kappa)$, where $\mu_\kappa$ is the (purely atomic) counting measure on $\kappa$. We may thus restrict our attention to the spaces $\ell^1(\kappa)$, $\kappa \geq 1$.

6. Theorem, For every positive integer $n > 1$ we have $m(\ell^1(n)) = 2n(n-1)^{-1}$.

Proof. 1. Consider the sequence $(p_i : i=1,\ldots,n)$ of points of $BS(\ell^1(n))$ given by

$$p_i(j) = (n-1)^{-1}\operatorname{sgn}(j-i) \qquad i,j = 1,\ldots,n,$$

and set $p_0 = -p_n$. A straightforward verification shows that the polygon $p$ with consecutive vertices $p_0, p_1, \ldots, p_n$ is a curve from $p_0 = -p_n$ to $p_n$ in $BS(\ell^1(n))$, and $\|p_i - p_{i-1}\| = 2(n-1)^{-1}$, $i = 1,\ldots,n$. Therefore $m(\ell^1(n)) \le \ell(p) = 2n(n-1)^{-1}$. This argument was pointed out by L. Danzer (private communication, 1967).

2. To prove the reverse inequality, we observe that $B(\ell^1(n))$ is a polytope, and apply [6; Lemma 2]; this states that there exists a simple polygon $p$ in $BS(\ell^1(n))$ with antipodal endpoints such that $\ell(p) = m(\ell^1(n))$. Let $p_0, p_1, \ldots, p_k = -p_0$ be the consecutive vertices of this polygon. Now

$$(5) \qquad \ell(p) = \sum_{i=1}^{k} \|p_i - p_{i-1}\| = \sum_{i=1}^{k}\sum_{j=1}^{n} |p_i(j) - p_{i-1}(j)|.$$

For every fixed $j_0$, $1 \le j_0 \le n$, we have $p_k(j_0) = -p_0(j_0)$, so that the $j_0$th co-ordinate must vanish at some point of $p$. Since no co-ordinate vanishes at an interior point of a face of the polytope $B(\ell^1(n))$, it follows at once that, in fact $p_{i_0}(j_0) = 0$ for some $i_0$, $1 \le i_0 \le k$. But then (5) yields

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On p.7 replace Theorem 7 by:

7. <u>Theorem</u>. <u>If</u> A JLs atn <u>infinite set, then</u> m(£ 1(A) ) = 2; <u>but</u> 6(-p,p) >2 <u>for every</u> pe5S(^1(A))J <u>and</u> ^1(A) <u>is a fortiori not flat</u>.

<u>Proof</u>, $I^1$(A) contains a subspace congruent to $t^1$(n) for n = 2,3,...; therefore, by Theorem 6 and [5; Lemma 5.2 and Theorem 5.3(a)], 2 £ m(^1(A)) £ m(^1(n)) = 2n(n-l)~1, n = 2,3,...; whence m(l 1(A))) = 2. (Since I 1(A) is not reflexive, this conclusion also follows from [7].)

On the other hand, let pedSC^1(A)) be given, and choose $a_Q$eA such that P($a_Q$) ¥" 0. If c is a curve in 5£(£1(A)) from -p to p, there is a point q on c with <3($a_0$) = 0. Then

**t(c) ≥ ||p-q||+||q+p|| = 2|p(a 0)\+l ( | p(a) -q(a) | + | q(a)+p(a)| ) ≥**
**a a0**

≥ 2|p(a 0)| + aZi |q(a)| = 2(|p(a 0) |+||q||) = 2(|p(a 0)|+1).
a^a Q

Since c was arbitrary, we find 6(-p,p) > 2(|p($a_0$)|+1) > 2.

<u>Remark</u>, This strengthens the result in [3] that ^1(^o) is not flat.

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$$(6)\quad \ell(p) - \sum_{i=1}^{k} |p_i(j_0) - p_{i-1}(j_0)| = \sum_{i=1}^{k} \sum_{j\neq j_0} |p_i(j) - p_{i-1}(j)|$$

$$\geq \sum_{j\neq j_0} \left( |p_{i_0}(j) - p_0(j)| + |p_{i_0}(j) - p_n(j)| \right)$$

$$= \sum_{j\neq j_0} \left( |p_{i_0}(j) + p_0(j)| + |p_{i_0}(j) - p_0(j)| \right) \geq 2 \sum_{j\neq j_0} |p_{i_0}(j)|$$

$$= 2 \sum_{j=1}^{n} |p_{i_0}(j)| = 2\|p_{i_0}\| = 2.$$

Since (6) holds for $j_0 = 1, \ldots, n$, we find, using (5),

$$n\ell(p) \geq 2n + \sum_{j_0=1}^{n} \sum_{i=1}^{k} |p_i(j_0) - p_{i-1}(j_0)| = 2n + \ell(p).$$

Thus $m(\ell^1(n)) = \ell(p) \geq 2n(n-1)^{-1}$, as was to be shown.

Remark. If $n$ is odd, the inequality $m(\ell^1(n)) \geq 2n(n-1)^{-1}$ also follows from [8].

7. Theorem. If $N$ is an infinite cardinal, $m(\ell^1(N)) = 2$, but $\ell^1(N)$ is not flat.

Proof. $\ell^1(N)$ contains a subspace congruent to $\ell^1(n)$ for $n = 2,3,\ldots$; therefore, by Theorem 6 and [5; Lemma 5.2 and Theorem 5.3(a)], $2 \leq m(\ell^1(N)) \leq m(\ell^1(n)) = 2n(n-1)^{-1}$, $n = 2,3,\ldots$; whence $m(\ell^1(N)) = 2$. (Since $\ell^1(N)$ is not reflexive, this conclusion also follows from [7].) On the other hand, consider a curve in $\partial(\ell^1(N))$ with antipodal endpoints; it lies in a separable subspace of $\ell^1(N)$, and therefore in a subspace congruent to $\ell^1(N_0)$; but Harrell and Karlovitz have shown that $\ell^1(N_0)$ is not flat [3; Corollary to Theorem 5]; therefore the length of the curve is not 2, and $\ell^1(N)$ itself is not flat.

We can summarize some of our conclusions by restating them as a theorem on abstract L-spaces and relying on Kakutani's representation theorem [4], according to which the L-spaces are precisely the Banach lattices congruent and lattice-isomorphic to the spaces $L^1(\mu)$ for all measure spaces $(S, \Sigma, \mu)$.

8. Theorem. *Let X be an abstract L-space with* dim X > 1. *Then one and only one of the following four alternatives holds.*

| | | | | |
|---|---|---|---|---|
| dim X *is* | *infinite* | *infinite* | *infinite* | $n < \infty$ |
| D(X)=M(X)= | 2 | 4 | 4 | 4 |
| *does* X *have a pole?* | no | yes | yes | yes |
| m(X)= | 2 | 2 | 2 | $2n(n-1)^{-1}$ |
| *is* X *flat?* | yes | yes | no | no |
| X *is congruent to* $L^1(\mu)$, *where* $\mu$ *is* | non-atomic | neither non-atomic nor purely atomic | purely atomic; infinite set of atoms | purely atomic; n atoms. |

## 4. *Some problems*

The results of the preceding sections, and especially Theorem 2, suggest the following questions, given in order of increasing restrictiveness.

Problems. Characterize those Banach spaces X for which:

(a): M(X) = 2;

(b): D(X) = 2;

(c): 6(p,q) = l|q-p|| for all p,qe^S(X);

(d): there exists, for all P,qec)£(X), a curve of length ||q-p|| from p to q in SS(X).

The conjecture may be ventured that the only Banach spaces X that satisfy (d) are precisely those congruent to L 1(jit) for some non-atomic measure \i.

## References


[1] H. Busemann, The Geometry of Geodesies, New York, Academic Press, 1955.

[2] P.R. Halmos, "The range of a vector measure", Bull. Amer. Math. Soc. 54(1948), 416-421.

[3] R.E. Harrell and L.A. Karlovitz, "Girths and flat Banach spaces", Bull. Amer. Math. Soc. (to appear).

[4] S. Kakutani, "Concrete representation of abstract (L)-spaces and the mean ergodic theorem", Ann. of Math. 42(1941), 523-537.

[5] J.J. Schäffer, "Inner diameter, perimeter, and girth of spheres", Math. Ann. 173(1967), 59-79.

[6] J.J.SchUffer, "Addendum: Inner diameter, perimeter, and girth of spheres", Math. Ann. 173(1967), 79-82.

[7] J.J. Schäffer and K. Sundaresan, "Reflexivity and the girth of spheres", Math. Ann. 184(1970), 163-168.

[8] J.J. Schäffer, "Minimum girth of spheres", Math. Ann. 184(1970), 169-171.

[9] J.J. Schäffer, "Spheres with maximum inner diameter", Math. Ann. (to appear).



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