# SOME GEOMETRIC PROPERTIES OF THE UNIT CELL IN SPACES C(X;B)

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If $X$ is a compact Hausdorff space and $B$ is a Banach space then the Banach space of $B$-valued continuous functions on $X$ with the usual supremum norm is denoted by $C(X;B)$. The purpose of the present note is to discuss certain smoothness properties of the norm in $C(X;B)$ in terms of the corresponding properties of the norm in $B$. Specifically we are concerned here with $G$-differentiability, $F$-differentiability (more generally $n$ times $F$-differentiability) of the norm. The norm in $C(X;B)$ in general does not have any of these properties globally and in this note we characterize the functions $f$ in $C(X;B)$ where the norm has one of the preceding properties.

We quickly recall a few definitions and notations useful in the subsequent discussion. If $(E, \|\ \|)$ is a Banach space then the norm is $G$-differentiable at $x \in E$ if $\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t}$ exists. It is $F$-differentiable at $x$ if there exists a linear functional $l_x$ on $E$ such that $\frac{\|x+h\| - \|x\| - \ell_x(h)}{\|h\|} \to 0$ as $\|h\| \to 0$. If such a linear functional $\ell_x$ exists then it is verified that $\ell_x$ is continuous on E. More generally the norm in E is said to be a K-times F-differentiable at x if there exist for each integer i, $1 \leq i \leq K$, a symmetric i-multilinear form $T^i_x$ continuous on the Cartesian product of i copies of E (denoted by $E^i$) such that

$$\frac{\|x+h\| - \|x\| - \sum_{i=1}^{K} \frac{1}{i!} T^i_x(h^i)}{\|h\|^K} \to 0$$

as $\|h\| \to 0$ where $h^i$ is the i-tuple (h,h, . . . ,h).

For the convenience of presentation if T is a i-multilinear form on B and $p \in X$ then $e(T;p)$ is the multilinear form on C(X;B) defined by $e(T;p)(f_1,\ldots f_i) = T(f_1(p),\ldots,f_i(p))$ where $f_j \in C(X;B)$ for $1 \leq j \leq i$. It is at once verified that if T is continuous (symmetric) i-form on B then so is e(T;p) on the space C(X;B).

In the rest of the paper we denote the unit cells of B and of its dual B* by $U_B$ and $U^*_B$ respectively. The unit cells of C(X;B) and of its dual by $U_X$ and $U^*_X$ respectively. In what follows the norms of various Banach spaces involved in the discussion are all denoted by the same symbol || || as there is no occasion for confusion.

Before proceeding to state the main results of the paper we restate few known useful results as lemmas.

<u>Lemma 1</u>. A linear functional L e U | is an extreme point of $U^*_X$ if and only if L = *e{b,p)* where *I* is an extreme point of $U^*_B$ and p is a point in X.

<u>Lemma 2</u>. The norm in E is G-differentiable at x, ||xj| = 1 , if and only if there is a unique hyperplane of support for the unit cell $U_E$ at x .

<u>Lemma 3</u>. The norm in E is F-differentiable at x, |]x|| = 1 , if and only if the diameter {f |f (x) )≥ 1-6, f *e* $U^*_E$ } 0 as 6 0.

For a proof of Lemma 1 we refer to Lemma 1 in Sundaresan [4]. For the proofs of Lemmas 2 and 3, see Mazur [2] and Smulyan [3] respectively. See also pages 111-114, Day [1].

In theorem 1 we provide a characterization of the G-differentiability of the norm in C(X;B) at a function f. This result is known, Sundaresan [4] and restated here with proof for the sake of completeness.

Theorem 1. The norm in C(X;B) is G-differentiable at f if and only if (1) there exists a point q e X such that ||f|! = Hf(q)|| > ||f(q')|! for all qV ^ q and (2) the norm in B is G-differentiable at f(q).

Proof. From the homogeneity of the norm it follows that the norm is G-differentiable at x if and only if it is G-differentiable at 7\ x , A ^ 0. Hence we can assume without loss of generality that ||f|| = 1. Suppose the norm in C(X;B) is G-differentiable at jlf|j =1 . If possible let there exist p, q *e* X such that ||f(p)|| = l|f(q)J! = 1. Let *l^*, $l_2$ be two linear functionals in E*, H^jj = 1 = |U2|| such that $\hat{}_1$(f(p)) = $*_2$(f(q)) = 1. The existence of such linear functionals is guaranteed by the Hahn-Banach theorem. From the choice of the functionals $l_\pm$ and $l_2$ it follows that e(-$t_1$,p)(f) = e($*_2$,q)(f) = 1. Further it is verified from the definition of e(£,p) that for a fixed p e X, ||e(-t,p)|l = |!-t|| for all *I e E**. Tlius we conclude that H e ^ p J H = ||e($*_2$,q)|| = 1 = e(-$t_1$,p)(f) = e($*_2$,q)(f) = ||f|| = 1. Hence e($\hat{}_1$,p)~$^1$(l) and e($\hat{}_2$,q)"$^1$(1) are supporting hyperplanes of $U_x$ at f . If p ^ q then consider a function g *e* C(X,B) such that g(p) = f($_p$) and g(q) = o . It is verified that *e($l_r$p)* (g) = 1 ^ o = e($*_2$,q)(g). Hence e($\hat{}_1$,p) / e(-$t_2$,q) and $U_x$ has two distinct hyperplanes of support at f *3* contradicting that the norm is G-differentiable at $f_9$ as seen from Lemma 2 . Thus f satisfies Condition (1) in the theorem. Next let q be the point in X such that ||f(q)!| = 1 . If the norm in B is not G-differentiable then there exist two distinct functionals ^,_ L e B^^ ||£, || = ||$*_2$|| = 1 such that ^(fCq)) = $4_2$(f(q)) = 1 . Thus e (£$_1$. ,q) (f) = e(^$_{2J}$,q)(f) = 1. Let x e B such that $l_{1X}$(x) ^ $l_{q\wedge}$(x) . Then if $c_X$ is the function in C(X;B) with range {x} then e f ^ q ) ($c_x$) ^ e ($*_2$,q) ($c_x$) . Thus once again it is verified that there are two distinct hyperplanes of support for the cell $U_v$ at $f_9$ once again contradicting the norm in C(X;B) is G-differentiable at f. Thus f is verified to satisfy Condition (2).

Conversely suppose that a function f in C(X;B) of unit norm satisfies Conditions (1) and (2). We proceed to show that there is only one hyperplane of support for $U_X$ at f . Consider the set

$$B_f = [L \mid L e U^*_X, \ ||L|| = L(f) = 1\} .$$

It is verified that $B_f$ is w* -closed convex subset of $U^*_X$ . Hence, byAlaoglu's theorem,, $B_f$ is w*-compact convex subset of $U^*_X$ . Further by the Hahn-Banach theorem

$B_f \neq \emptyset$ . It is also verified from the definition of an extreme point that Ext $B_f$ c Ext $U_X$ where Ext C is the set of extreme points of the set C . Since $B_f$ is w*-compact convex subset of B* it follows by the Krein-Milman theorem that $B_f$ = w*-closure of the convex hull of Ext $B_f$ . Thus if Card $B_f \geq 2$ Card Ext $B_f \geq 2$ •

Let $L_1, L_2 \in$ Ext $B_f$ , $L_1 \neq L_2$ . Since $L_1$, $L_2$ e Ext $U_X$ it follows from Lemma 1 that there are functionals $l_1$ , $l_2$ e Ext $U_B$ and points $p_1, p_2 \in X$ such that $L_i = e(l_i, p_i)$ , i = 1,2 . $L_i(f) = 1$ , it follows that $l_i(f(p_i)) = 1$ . Further noting that $||l_i|| = 1$, $||f|| = 1$, we conclude that $||f(p_i)|| = 1$ . Thus from condition (1) it follows that $p_1 = p_2 = q$ . From the condition (2), noting that $l_j(f(q)) = ||l_j|| = 1$ , i = 1,2, it is inferred that $l_1 = l_2$ . Thus $L_1 = L_2$ and Card $B_f = 1$ . Hence there is only one hyperplane of support for the cell $U_X$ at f i.e., the norm in C(X;B) is G-differentiable at f . This completes the proof of the theorem.

Remark 1. We note that if the norm in C(X;B) is G-differentiable at f , and if $q \in X$ such that $||f(q)|| = ||f||$ then q is a $G_{f}$-point, since $||f(\cdot)||$ is a continuous function on X . Let for completely regular $T_1$ spaces Y C(Y) be the Banach space of real valued bounded continuous function on $Y$ with usual supremum norm. Let $\beta Y$ be the Stone-Cech compactification of $Y$. From the definition of $\beta Y$ there is a linear isometry $T$ on $C(Y)$ onto $C(\beta Y)$ such that $Tf | Y = f$ for all $f \in C(Y)$. Hence the norm in $C(Y)$ is G-differentiable at function $f \in C(Y)$ if and only if there is a unique point $p \in \beta Y$ such that $Tf(p) = ||Tf||$. Since $\{p\}$ is a $G_\delta$ point of $\beta Y$ and since the cardinality of a nonempty closed $G_\delta$-subset of $\beta Y \sim Y$ is at least $2^{c}$ it follows that $p \in Y$. Thus we obtain a generalization of a theorem stated on page 170 in Banach [1] for compact metric spaces to completely regular $T_1$ spaces.

Theorem 2. The norm in $C(X,B)$ is F-differentiable at $f$ if and only if (1) there exists an isolated point $p \in X$ such that $||f|| = ||f(p)|| > ||f(q)||$ for all $q \neq p$ and (2) the norm in $B$ is F-differentiable at $f(p)$. Further the norm in $C(X;B)$ is K-times F-differentiable, $K \geq 2$, at $f$ if and only if $f$ has property (1) stated above and the norm in $B$ is K-times F-differentiable at $f(p)$.

Proof. As in Theorem 1 without loss of generality it could be assumed that $||f|| = 1$. Let the norm in $C(X;B)$ be F-differentiable at $f$. Thus it is G-differentiable at f and there exists exactly one point $p \in X$ such that $\|f(p)\| = 1$. We verify that p is an isolated point of X. Since the norm is G-differentiable at f by the preceding there is only one functional $\ell \in B^*$ such that $\|\ell\| = \ell(f(p)) = 1$. If p is not isolated then there exists a net $\{p_\alpha \mid \alpha \in D\}$ in X such that $p_\alpha \to p$ with $p_\alpha \neq p$ for any $\alpha \in D$. For each $\alpha \in D$, $e(\ell, p_\alpha)$ is a functional in $U^*_X$ of unit norm. If $\delta > 0$ since $e(\ell,p_\alpha)(f) \to e(\ell,p)(f) = 1$ there exist $\alpha \in D$ such that $e(\ell,p_\alpha)(f) \geq 1-\delta$. Further for $\alpha \in D$ if $\|e(\ell,p_\alpha) - e(\ell,p)\| \geq 1$ for if g is a function in $C(X;B)$ such that $g(p) = f(p)$ $g(p_\alpha) = 0$ and $\|g\| = 1$ then $|e(\ell,p_\alpha)g - e(\ell,p)(g)| = 1 = \|g\|$. Thus

$$\text{diameter } \{L \mid L \in U^*_X,\ L(f) \geq 1-\delta\}$$

does not tend to 0 as $\delta \to 0$. Hence by Lemma 3 it follows that the norm is not F-differentiable at f, obtaining a contradiction. Thus f has property (1) of the theorem. Next we prove that the norm in B is F-differentiable at f(p) by showing (A) that $\text{diam}\{\ell \mid \ell \in U^*_E,\ \ell(f(p)) \geq 1-\delta\} \to 0$ as $\delta \to 0^+$. If $\ell \in U^*_E$ then $e(\ell,p) \in U^*_X$ since $\|e(\ell,p)\| = \|\ell\|$ for all $\ell \in E^*$ and for a fixed $p \in X$. Thus if (A) is false then

$$\text{diam}\,[L \mid L \in U_X^* ,\ L(f) \geq 1 - \delta] \to 0$$

as $\delta \to 0^+$ contradicting that the norm in $C(X;B)$ is F-differentiable at $f$. This completes the proof of the necessity of conditions (1) and (2).

Conversely suppose that $f$ has properties (1) and (2) in the theorem. Let $q$ be the unique point in $X$ such that $||f(q)|| = ||f|| = 1$. (*) Since $q$ is isolated there exists a positive number $\delta_1 > 0$ such that if $g \in C(X;B)$ and $||f - g|| < \delta_1$ then $||g(q)|| = ||g|| > ||g(q')||$ for all $q' \neq q$. Thus for $h \in C(X;B)$ of sufficiently small norm $||f+h|| = ||f(q)+h(q)||$. Since the norm in $B$ is F-differentiable at $f(q)$, if $\ell$ is the functional in $B^*$ such that $\ell(f(q)) = ||\ell|| = 1$ then $\ell$ is the F-differential of the norm in $B$ at $f(q)$ and

$$\frac{||f+h|| - ||f|| - \ell(h(q))}{||h||} \leq \frac{||f(q)+h(q)|| - ||f(q)|| - \ell(h(q))}{||h(q)||}$$

Hence from the F-differentiability of the norm in $B$ at $f(q)$ it follows that the norm in $C(X;B)$ is F-differentiable at $f$ with $\ell f(q)$ as the F-differential.

Next we proceed to the proof of the final part of the theorem. For convenience of notation if $P$ is a polynomial operator on $B$ defined by

$$P(h) = \sum_{i=1}^{n} T_i(h^i)$$

where $T_i$ is a continuous multilinear form on B $1 \le i \le n$, then if $p \in X$ we define a polynomial operator $e(P;p)$ on C(X;B) by setting

$$e(P,p)(h) = \sum_{i=1}^{n} T_i(h^i(p)) .$$

Let now the norm in C(X;B) be K-times F-differentiable at f, $\|f\| = 1$ so that there exists a polynomial operator P, $P(h) = \sum_{i=1} T_i(h^i)$ such that $\frac{\|f+h\| - \|f\| - P(h)}{\|h\|} \to 0$

as $\|h\| \to 0$. Since $K \ge 2$, the norm in C(X;B) is once F-differentiable. Hence there is a unique point $p \in X$ such that $\|f(p)\| = \|f\| = 1$ and such a point p is isolated in X. Further the conclusion (*) of the preceding paragraph is valid and there exists a $\delta > 0$ such that $\|h\| < \delta$ implies $\|f+h\| = \|f(p) + h(p)\|$ and it follows that if $x \in B$ of sufficiently small norm and $C_x$ is the constant function on X to B with range {x} then

$$\frac{\|f(p)+x\| - \|f(p)\| - e(P;p)(x)}{\|x\|^K} = \frac{\|f + c_x\| - \|f\| - P(c_x)}{\|c_x\|^K}$$

Hence the K-times F-differentiability of the norm in C(X;B) at f implies that the norm in B is K-times F-differentiable at f(p) .

Conversely suppose a function $f \in C(X;B)$, $\|f\| = 1$, has the property (1). Further let the norm in $B$ be K-times F-differentiable at $f(p)$. Let $P$ be the polynomial operator on $B$ such that

$$\frac{\|f(p) + x\| - \|f(p)\| - P(x)}{\|x\|^K} \to 0.$$

Then arguing as in the proof of the preceding part (see paragraph 2) it is verified that $\| \ \|$ is K-times F-differentiable at $f$ with $e(P^p)$ as the approximating polynomial. This completes the proof of the theorem.

Remark 2. Noting that the absolute value function is $C^\infty$ on $R \sim \{0\}$ it follows from the preceding theorem that if the norm in $C(X;R)$ is once F-differentiable at a function $f$ in $C(X;R)$ then the norm is K-times F-differentiable for all $K \geq 1$.

In conclusion we note that by arguments very similar to those employed in the proof of Theorem 2 one can obtain the following characterization of functions $f$ in $C(X;B)$ such that the norm in $C(X;B)$ is p-times (infinitely) continuously differentiable at $f$.

Theorem 3. The norm in $C(X;B)$ is p-times (infinitely) continuously differentiable in a neighborhood of $f$ if and only if (1) there exists exactly one point $p \in X$ such that $||f(p)|| = ||f||$ and such a point $p$ is isolated in $X$ and (2) the norm in $B$ is $p$-times (infinitely) continuously differentiable in a neighborhood of $f(p)$.

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