TRIP AND JUNC

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T R I P and J U N C

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Introduction

Adjoint functors [3] and triples (also known as standard constructions [8], monads, and triads) were born at about the same time, and it was noticed very soon that every adjunction induces a triple and a cotriple. Again at about the same time — some years later — Eilenberg and Moore [2] and Kleisli [4] showed that every triple (and dually every cotriple) is induced by an adjunction. In fact, a given triple is induced by many adjunctions, and Kleisli provided in a sense the finest, and Eilenberg and Moore the coarsest, adjunction which induces a given triple. Maranda [6] and Linton [5] extended the universal properties of the Eilenberg-Moore and Kleisli constructions to natural transformations between triples on the same category. Recently, Pumplün [7] constructed categories with all adjunctions and all triples respectively as objects, and he extended the construction of a triple from an adjunction, and the Eilenberg-Moore and Kleisli constructions of adjunctions from triples, to functors between these categories. The universal properties of the constructions then became adjointness properties of the functors constructed from them.

In the present report, we modify Pumplün's theory by introducing a more natural category of adjunctions. The main tool for this is a theorem on conjugate natural transformations which we believe to be new, but which may well have been in the folklore — at least of some folk — for some time. Special versions of it have been in print for a long time. The category of adjunctions — which we call JUNC — is in fact a double category in the sense of Ehresmann [1], and as such it acts transversally on the two categories of triples — called TRIP in this report — introduced by Pumplün.

We usually present proofs in this report up to the point of drawing the diagrams, but we leave the chasing to the reader. Our diagrams are mostly diagrams of natural transformations, and the Five Rules of Godement, and in particular Règle V, will be used very often.

1. The double category JUNC

1.1. Conjugate natural transformations

An adjunction $(F,U;\eta,\varepsilon)$ from a category $\mathcal{A}$ to a category $\mathcal{C}$, called source and target of the adjunction, consists of two functors $U : \mathcal{A} \to \mathcal{C}$ and $F : \mathcal{C} \to \mathcal{A}$, and of two natural transformations $\eta : \mathrm{Id}\,\mathcal{C} \longrightarrow U F$ and $\varepsilon : F U \longrightarrow \mathrm{Id}\,\mathcal{A}$, subject to the conditions

$$U\varepsilon \cdot \eta U = \mathrm{id}\, U \quad , \qquad \varepsilon F \cdot F\eta = \mathrm{id}\, F \ .$$

We call η the unit, and ε the counit, of the adjoint situation.

Note that $\cdot$ denotes the "transversal" composition of natural transformations of functors between the same category, and that we do not use a symbol for the "lateral" composition of functors, or of a natural transformation preceded or followed by a functor. This will be a consistent practice.

We consider now two functors and two adjunctions, with sources and targets indicated by the following diagram. No commutativity is implied.

$$\begin{array}{ccc} \mathcal{A}_0 & \xrightarrow{\quad P \quad} & \mathcal{A}_1 \\ F_0 \uparrow \big\downarrow U_0 & & F_1 \uparrow \big\downarrow U_1 \\ \mathcal{C}_0 & \xrightarrow{\quad Q \quad} & \mathcal{C}_1 \end{array}$$

With these notations, we prove the following theorem.

<u>Theorem</u>. <u>The following three conditions for natural transformations</u> μ_0 <u>and</u> μ_1 <u>are logically equivalent, and each of them defines a bijection between natural transformations</u> $\mu_0 : Q\, U_0 \longrightarrow U_1\, P$ <u>and</u> $\mu_1 : F_1\, Q \longrightarrow P\, F_0$.

(i) $\mu_0 F_0 \cdot Q\,\eta_0 = U_1 \mu_1 \cdot \eta_1 Q$.

(ii) $P\,\varepsilon_0 \cdot \mu_1 U_0 = \varepsilon_1 P \cdot F_1 \mu_0$.

(iii) <u>The diagram</u>

$$\begin{array}{ccc} \mathcal{A}_0(F_0\, C,\ A) & \xrightarrow{\ \cong\ } & \mathcal{C}_0(C,\ U_0\, A) \\ \big\downarrow P & & \big\downarrow Q \\ \mathcal{A}_1(P\, F_0\, C\ ,\ P\, A) & & \mathcal{C}_1(Q\, C\ ,\ Q\, U_0\, A) \\ \big\downarrow \mathcal{A}_1(\mu_{1_C}\ ,\ P\, A) & & \big\downarrow \mathcal{C}_1(Q\, C\ ,\mu_{0_A}) \\ \mathcal{A}_1(F_1\, Q\, C\ ,\ P\, A) & \xrightarrow{\ \cong\ } & \mathcal{C}_1(Q\, C\ ,\ U_1\, P\, A) \end{array}$$

<u>is commutative for every pair of objects</u> A <u>of</u> $\mathcal{A}_0$ <u>and</u> C <u>of</u> $\mathcal{C}_0$.

<u>Proof</u>. The top of diagram (iii) relates $f : F_0\, C \longrightarrow A$ in $\mathcal{A}_0$ to g :

$C \longrightarrow U_o A$ in $\mathcal{C}_o$ if and only if $f = \varepsilon_{o_A} (F_o g)$ and $g = (U_o f) \eta_{o_C}$. The vertical arrows transform f and g into $f_1 = (P f) \mu_{1_C}$ and $g_1 = \mu_{o_A} (Q g)$ which must be related in the same way as f and g . If (ii) is satisfied, then

$$\begin{aligned} \varepsilon_{1_{PA}} (F_1 g_1) &= (\varepsilon_1 P)_A (F_1 \mu_o)_A (F_1 Q g) \\ &= (P \varepsilon_o)_A (\mu_1 U_o)_A (F_1 Q U_o f)(F_1 Q \eta_o)_C \\ &= (P \varepsilon_o)_A (P F_o U_o f) (\mu_1 U_o F_o)_C (F_1 Q \eta_o)_C \\ &= (P f) (P \varepsilon_o F_o)_C (P F_o \eta_o)_C \mu_C = (P f) \mu_{1_C} = f_1 , \end{aligned}$$

and diagram (iii) always commutes.

Conversely, if all diagrams (iii) commute, put $C = U_o A$, $g = \mathrm{id}\, U_o A$. Then $f = \varepsilon_{o_A}$, $g_1 = \mu_{o_A}$, and

$$f_1 = (P \varepsilon_o)_A (\mu_1 U_o)_A = (\varepsilon_1 P)_A (F_1 \mu_o)_A .$$

Thus (ii) is valid. We must also have $g_1 = (U_1 f_1) (\eta_1 Q U_o)_C$. Thus

$$\mu_o = U_1 P \varepsilon_o \cdot U_1 \mu_1 U_o \cdot \eta_1 Q U_o ,$$

and μ_1 determines μ_o uniquely.

If only μ_1 is given, and we define μ_o by the formula just obtained, then diagram (iii) commutes for $C = U_o A$, $g = \mathrm{id}\, U_o A$, and (ii) is valid by the preceding paragraph. But then diagram (iii) always commutes.

One proves dually that (i) $\Longleftrightarrow$ (iii), and that there is a unique μ_1 for which (i) and (iii) are valid if only μ_o is given.

Definition. We call $\mu_o : Q U_o \longrightarrow U_1 P$ and $\mu_1 : F_1 Q \longrightarrow P F_o$ conjugate natural transformations, from the adjunction (F_o, U_o) to the adjunction (F_1, U_1) , if they satisfy (i), (ii), (iii) in the theorem just proved.

We note that our theorem is well known, and the definition well established, for the case that P and Q are identity functors.

1.2. Double categories

Definition. A double category $\mathbb{D}$ consists of the following.

(i) Categories $\mathbb{D}^{\mathrm{lat}}$ and $\mathbb{D}^{\mathrm{tr}}$ with the same class of morphisms. These morphisms are called cells of $\mathbb{D}$.

(ii) Categories $\mathcal{A}^{\mathrm{lat}}$ and $\mathcal{A}^{\mathrm{tr}}$ with the same class $\mathrm{Ob}\,\mathbb{D}$ of objects. Morphisms in $\mathcal{A}^{\mathrm{lat}}$ and $\mathcal{A}^{\mathrm{tr}}$ are called lateral and transversal arrows of $\mathbb{D}$.

(iii) Functors $d_i^{\mathrm{lat}} : \mathbb{D}^{\mathrm{tr}} \to \mathcal{A}^{\mathrm{tr}}$ and $d_i^{\mathrm{tr}} : \mathbb{D}^{\mathrm{lat}} \to \mathcal{A}^{\mathrm{lat}}$ $(i = 0, 1)$, and functors $\mathrm{id}^{\mathrm{lat}} : \mathcal{A}^{\mathrm{tr}} \to \mathbb{D}^{\mathrm{tr}}$ and $\mathrm{id}^{\mathrm{tr}} : \mathcal{A}^{\mathrm{lat}} \to \mathbb{D}^{\mathrm{lat}}$.

Composition in $\mathbb{D}^{\mathrm{lat}}$ and in $\mathcal{A}^{\mathrm{lat}}$ is called lateral composition of $\mathbb{D}$ and denoted by $\perp$ or no symbol, and composition in $\mathbb{D}^{\mathrm{tr}}$ and in $\mathcal{A}^{\mathrm{tr}}$ is called transversal composition of $\mathbb{D}$ and denoted by $\top$ or just $\cdot$.

These data are subject to the following conditions.

(iv) $d_i^{\mathrm{lat}}\,\mathrm{id}^{\mathrm{lat}} = \mathrm{Id}\,\mathcal{A}^{\mathrm{tr}}$ and $d_i^{\mathrm{tr}}\,\mathrm{id}^{\mathrm{tr}} = \mathrm{Id}\,\mathcal{A}^{\mathrm{lat}}$ $(i = 0, 1)$. Every cell $\mathrm{id}^{\mathrm{lat}}\,U$ is an identity morphism of $\mathbb{D}^{\mathrm{lat}}$, and every cell $\mathrm{id}^{\mathrm{tr}}\,P$ is an identity morphism of $\mathbb{D}^{\mathrm{tr}}$.

(v) $\beta \perp \alpha$ is defined for cells α, β if and only if $d_0^{\mathrm{lat}}\beta = d_1^{\mathrm{lat}}\alpha$, and $\gamma \perp \alpha$ is defined for cells α, γ if and only if $d_0^{\mathrm{tr}}\gamma = d_1^{\mathrm{tr}}\alpha$.

(vi) If $\beta \perp \alpha$, $\delta \perp \gamma$, $\gamma \top \alpha$, $\delta \top \beta$ are defined, then

$$(\delta \top \beta) \perp (\gamma \top \alpha) = (\delta \perp \gamma) \top (\beta \perp \alpha) .$$

We call this the middle interchange law.

The data and conditions are somewhat redundant; the purpose of this paper is not served by parsimony in this respect.

If a cell $\mathcal{E}$ is an identity morphism of $\mathbb{D}^{lat}$ or of $\mathbb{D}^{tr}$, then clearly $\mathcal{E} = id^{lat}\, d_i^{lat}\mathcal{E}$ or $\mathcal{E} = id^{tr}\, d_i^{tr}\mathcal{E}$ respectively by (iv) and (v). Thus we may identify the objects of $\mathbb{D}^{lat}$ with the transversal arrows of $\mathbb{D}$, and the objects of $\mathbb{D}^{tr}$ with the lateral arrows of $\mathbb{D}$. We shall always do this.

Every cell α of $\mathbb{D}$ induces a diagram

$$\begin{array}{ccc} A_{oo} & \xrightarrow{d_o^{tr}\alpha} & A_{o1} \\ \Big\downarrow{\scriptstyle d_o^{lat}\alpha} & & \Big\downarrow{\scriptstyle d_1^{lat}\alpha} \\ A_{1o} & \xrightarrow{d_1^{tr}\alpha} & A_{11} \end{array}$$

composed of objects and arrows (of both kinds) of $\mathbb{D}$. We call this diagram the <u>frame</u> of α. Frames without cells form a double category in an obvious way, and mapping each cell of $\mathbb{D}$ into its frame defines a functor of double categories.

An obvious example of a double category is the double category of commutative squares over a category $\mathcal{C}$. A less trivial example is presented below.

1.3. <u>The double category</u> JUNC

We define a double category JUNC, with functors as lateral arrows and adjunctions as transversal arrows, as follows. A frame of JUNC is a diagram

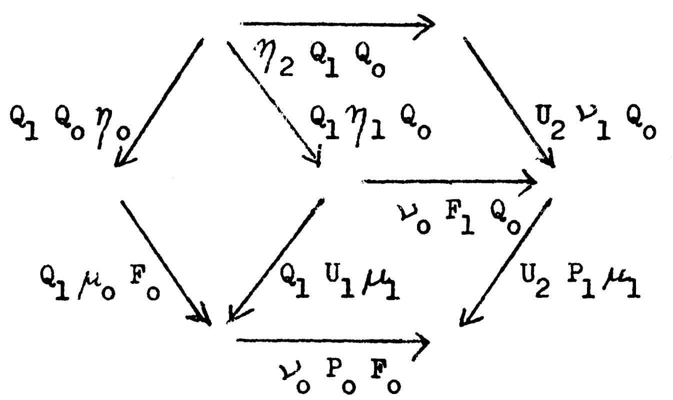
$$\begin{array}{ccc} \mathcal{A}_o & \xrightarrow{P} & \mathcal{A}_1 \\ {\scriptstyle F_o}\Big\uparrow\Big\downarrow{\scriptstyle U_o} & & {\scriptstyle F_1}\Big\uparrow\Big\downarrow{\scriptstyle U_1} \\ \mathcal{C}_o & \xrightarrow{Q} & \mathcal{C}_1 \end{array}$$

of functors and adjunctions (units and counits not shown). A cell of JUNC consists of a frame and of two conjugate natural transformations $\mu_o : Q\,U_o \longrightarrow U_1\,P$ and $\mu_1 : F_1\,Q \longrightarrow P\,F_o$ which fit into the frame. We may use (μ_o, μ_1) or more elaborate notations to denote such a cell.

Composition of lateral arrows of JUNC is of course the usual composition of functors. Lateral composition of cells $(P_o, Q_o; \mu_o, \mu_1) : (F_o, U_o) \longrightarrow (F_1, U_1)$ and $(P_1, Q_1; \nu_o, \nu_1) : (F_1, U_1) \longrightarrow (F_2, U_2)$ is defined by

$$(\nu\mu)_o = \nu_o\,P_o \cdot Q_1\,\mu_o \ , \qquad (\nu\mu)_1 = P_1\mu_1 \cdot \nu_1\,Q_o \ .$$

This is clearly associative, and $\mathrm{id}^{lat}\,(F,U) = (\mathrm{id}\,U\ ,\ \mathrm{id}\,F)$ acts as identity cell. d_o^{tr} and d_1^{tr} are functors for lateral composition, and it remains only to verify that $(\nu\mu)_o$ and $(\nu\mu)_1$ are in fact conjugate natural transformations. The diagram



shows this by verifying condition (i) for $(\nu\mu)_o$ and $(\nu\mu)_1$.

We compose adjunctions by putting

$$(F', U', \eta', \varepsilon') \circ (F, U, \eta, \varepsilon) = (F\,F',\ U'\,U, \bar{\eta}, \bar{\varepsilon}) \ ,$$

with $\bar{\eta} = U'\,\eta\,F' \cdot \eta'$ and $\bar{\varepsilon} = \varepsilon \cdot F\,\varepsilon'\,U$. We have

$$\bar{\varepsilon}\,F\,F' \circ F\,F'\bar{\eta} = \varepsilon\,F\,F' \circ F\,\varepsilon'\,U\,F\,F' \circ F\,F'\,U'\,\eta\,F' \circ F\,F'\eta'$$

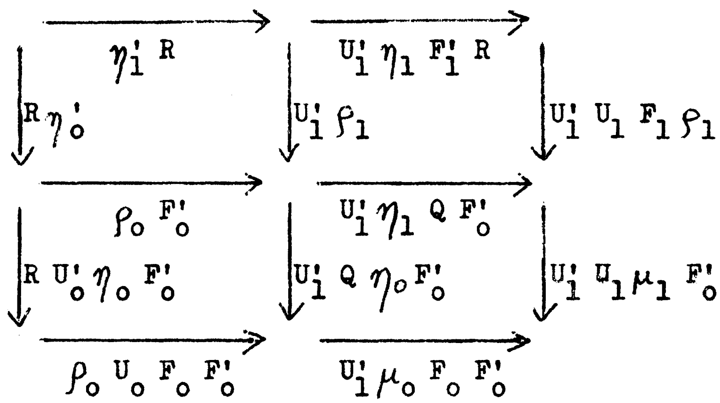
$$= \varepsilon\,F\,F' \circ F\,\eta\,F' \circ F\,\varepsilon'\,F' \circ F\,F'\eta' = \mathrm{id}\,F\,F' \ ,$$

and dually $U'U\bar{\varepsilon} \cdot \bar{\eta}U'U = \mathrm{id}\,U'U$. Thus the composition of adjunctions is an adjunction. Composition clearly is associative, and $(\mathrm{Id}\,\mathscr{C}, \mathrm{Id}\,\mathscr{C})$, with unit and counit $\mathrm{id}\,\mathrm{Id}\,\mathscr{C}$, is the identity adjunction on $\mathscr{C}$. Thus adjunctions are the morphisms of a category, with categories as objects.

Now let $(P,Q;\mu_0,\mu_1) : (F_0,U_0) \longrightarrow (F_1,U_1)$ and $(Q,R;\rho_0,\rho_1) : (F'_0,U'_0) \longrightarrow (F'_1,U'_1)$ be cells of JUNC which fit together transversally. We define their transversal composition by putting

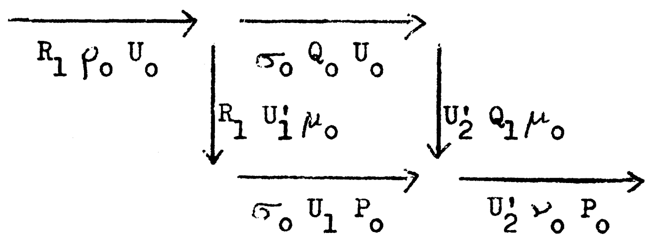
$$(\rho\cdot\mu)_0 = U'_1\mu_0 \cdot \rho_0 U_0 \ , \qquad (\rho\cdot\mu)_1 = \mu_1 F'_0 \cdot F_1\rho_1 \ .$$

This is clearly associative, and $\mathrm{id}^{tr}\,P = (P,P;\,\mathrm{id}\,P,\,\mathrm{id}\,P)$, with identity adjunctions as lateral source and target, acts as transversal identity cell. The commutative diagram



shows that $(\rho\cdot\mu)_0$ and $(\rho\cdot\mu)_1$ are conjugate natural transformations, by verifying condition (i). Thus we have a category JUNC^{tr}.

The commutative diagram



and the corresponding diagram for μ_1 , ν_1 , ρ_1 , σ_1 prove the middle interchange law $\sigma\rho \cdot \nu\mu = (\sigma \cdot \nu)(\rho \cdot \mu)$, and the remaining conditions for a double category are easily verified for JUNC.

2. The categories TRIP

2.1. Categories $TRIP_o$ and $TRIP_1$.

We recall that a triple $\mathbb{T} = (T, \varkappa, \mu)$ on a category $\mathscr{C}$ consists of a functor $T : \mathscr{C} \longrightarrow \mathscr{C}$ and natural transformations $\varkappa : \mathrm{Id}\,\mathscr{C} \longrightarrow T$ and $\mu : T\,T \longrightarrow T$ such that $\mu \cdot T\varkappa = \mathrm{id}\,T = \mu \cdot \varkappa T$ and $\mu \cdot T\mu = \mu \cdot \mu T$. We call $\varkappa$ the unit and μ the multiplication of $\mathbb{T}$.

We define categories $TRIP_o$ and $TRIP_1$ with triples as objects as follows. If $\mathbb{T}_o$ and $\mathbb{T}_1$ are triples on categories $\mathscr{C}_o$ and $\mathscr{C}_1$, then a morphism $(P,\pi) : \mathbb{T}_o \longrightarrow \mathbb{T}_1$ of $TRIP_o$ consists of a functor $P : \mathscr{C}_o \longrightarrow \mathscr{C}_1$ and a natural transformation $\pi : P\,T_o \longrightarrow T_1\,P$ which satisfies the conditions

$$\pi \cdot P\varkappa_o = \varkappa_1 P \ , \qquad \pi \cdot P\mu_o = \mu_1 P \cdot T_1\pi \cdot \pi T_o \ .$$

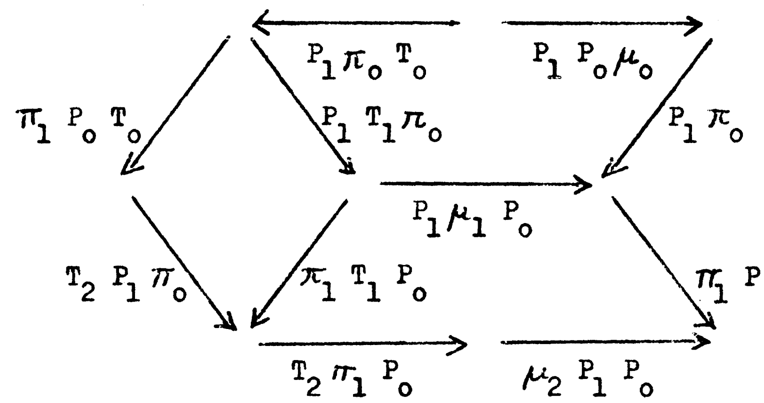
If $(P_o, \pi_o) : \mathbb{T}_o \longrightarrow \mathbb{T}_1$ and $(P_1, \pi_1) : \mathbb{T}_1 \longrightarrow \mathbb{T}_2$ are morphisms of $TRIP_o$, then $(P_1, \pi_1)(P_2, \pi_2)$ is defined by

$$(P_1, \pi_1)(P_o, \pi_o) = (P_1\,P_o\,, \pi_1\,P_o \cdot P_1\pi_o) \ .$$

This is clearly associative, and $(\mathrm{Id}\,\mathscr{C}, \mathrm{id}\,T)$ is the identity morphism of a triple $(T, \varkappa, \mu)$ on $\mathscr{C}$. We have

$$\pi_1\,P_o \cdot P_1\pi_o \cdot P_1\,P_o\varkappa_o = \pi_1\,P_o \cdot P_1\varkappa_1\,P_o = \varkappa_2\,P_1\,P_o \ ,$$

and the following diagram is commutative.



This shows that $(P_1, \pi_1)(P_o, \pi_o)$ is in $TRIP_o$. **Thus** $TRIP_o$ is a category.

The category $TRIP_1$ can be considered as a conjugate of $TRIP_o$. Objects are again triples. A morphism $(P, \pi) : \bar{T}_o \longrightarrow \bar{T}_1$ of $TRIP_1$ consists of a functor $P : \mathcal{C}_o \longrightarrow \mathcal{C}_1$ of the underlying categories and a natural transformation $\pi : T_1 P \longrightarrow P T_o$, and we require that

$$\pi \bullet \varkappa_1 P = P \varkappa_o \ , \qquad \pi \bullet \mu_1 P = P \mu_o \bullet \pi T_o \circ T_1 \pi \ .$$

The composition of morphisms $(P_o, \pi_o) : \bar{T}_o \longrightarrow \bar{T}_1$ and $(P_1, \pi_1) : \bar{T}_1 \longrightarrow \bar{T}_2$ is given for $TRIP_1$ by

$$(P_1, \pi_1)(P_o, \pi_o) = (P_1 P_o \ , \ P_1 \pi_o \bullet \pi_1 P_o) \ .$$

The proof that $TRIP_1$ is a category is exactly analogous to that for $TRIP_o$; we omit it.

2.2. <u>Transversal action of</u> $JUNC_i$ <u>on</u> $TRIP_i$

We do not define in general the action of a double category on a category; the following discussion will make it clear what is meant by this.

When considering the interaction of JUNC and TRIP , we must always restrict ourselves to two subcategories $JUNC_o$ and $JUNC_1$ of JUNC which correspond to the two categories $TRIP_0$ and $TRIP_1$.

$JUNC_0$ consists of all cells $(P,Q;\nu_0,\nu_1) : (F_0,U_0) \longrightarrow (F_1,U_1)$ of JUNC for which $F_1\, Q = P\, F_0$ and $\nu_1 = \mathrm{id}\, F_1\, Q = \mathrm{id}\, P\, F_0$. $JUNC_0$ is clearly closed under transversal and lateral composition, and all lateral and transversal identity cells of JUNC are in $JUNC_0$. Thus $JUNC_0$ is a double subcategory of JUNC with the same arrow categories.

Dually, $JUNC_1$ consists of all cells of JUNC for which $Q\, U_0 = U_1\, P$ and $\nu_0 = \mathrm{id}\, Q\, U_0 = \mathrm{id}\, U_1\, P$. This is also a double subcategory of JUNC.

The intersection of $JUNC_0$ and $JUNC_1$ consists of all commutative frames of functors and adjunctions, with identity transformations inside.

An adjunction (F,U,η,ε) with source $\mathcal{A}$ acts transversally on a triple $(T,\varkappa,\mu)$ on $\mathcal{A}$, from the left, by the law

$$(F,U) \circ (T,\varkappa,\mu) = (U\,T\,F,\bar\varkappa,\bar\mu) ,$$

with $\bar\varkappa = U\varkappa F \cdot \eta$ and $\bar\mu = U\mu F \cdot U\,T\varepsilon\,T\,F$. We have

$$\bar\mu \cdot U\,T\,F\bar\varkappa = U\mu F \cdot U\,T\varepsilon\,T\,F \cdot U\,T\,F\,U\varkappa F \cdot U\,T\,F\eta$$

$$= U\mu F \cdot U\,T\varkappa F \circ U\,T\varepsilon F \circ U\,T\,F\eta = \mathrm{id}\, U\,T\,F ,$$

and similarly for $\bar\mu \cdot \bar\varkappa U\,T\,F$. The diagram

$$\begin{array}{ccccc}
 & \xrightarrow{UTFUT\varepsilon TF} & & \xrightarrow{UTFU\mu F} & \\
{\scriptstyle UT\varepsilon TFUTF}\downarrow & & {\scriptstyle UT\varepsilon TTF}\downarrow & & \downarrow{\scriptstyle UT\varepsilon TF} \\
 & \xrightarrow{UTT\varepsilon TF} & & \xrightarrow{UT\mu F} & \\
{\scriptstyle U\mu FUTF}\downarrow & & {\scriptstyle U\mu TF}\downarrow & & \downarrow{\scriptstyle U\mu F} \\
 & \xrightarrow{UT\varepsilon TF} & & \xrightarrow{U\mu F} &
\end{array}$$

commutes, completing the proof that $(U\,T\,F, \overline{\varkappa}, \overline{\mu})$ is a triple. This is the action of a category on a set with target function, with the formal laws which one expects. The transversal target of a triple is of course the category on which it acts. The formal laws are easily verified; we shall not discuss them.

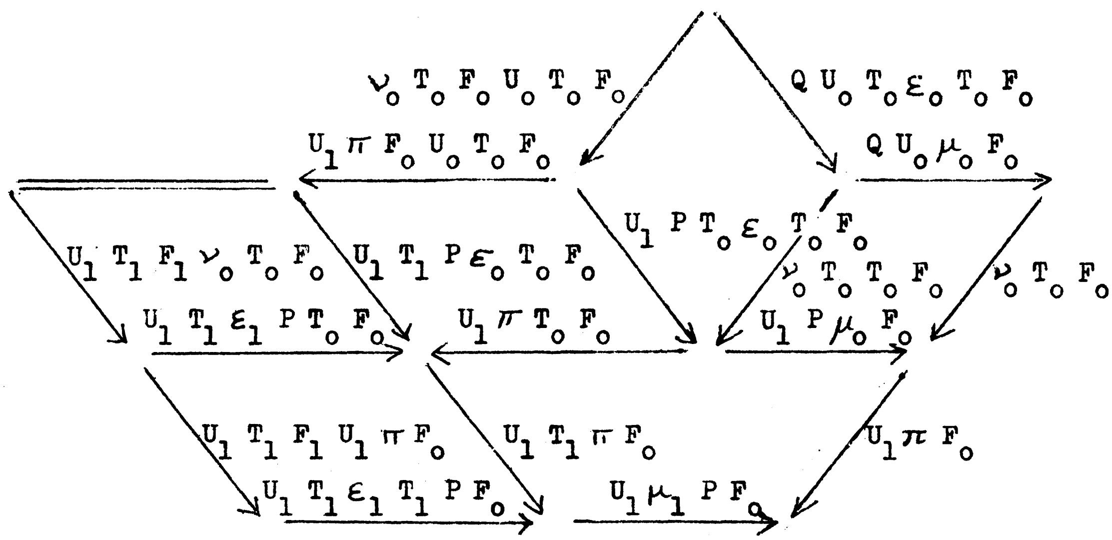
We extend the action of adjunctions on triples to a transversal left action of $JUNC_o$ on $TRIP_o$ by putting

$$(P, Q, \nu_o, \mathrm{id}) \cdot (P, \pi) = (Q\,,\, U_1 \pi F_o \circ \nu_o T_o F_o) \;,$$

for (P, π) in $TRIP_o$ and $(P, Q; \nu_o, \mathrm{id}) : (F_o, U_o) \longrightarrow (F_1, U_1)$ in $JUNC_o$. If $(T_i, \varkappa_i, \mu_i)$ are source and target of (P, π) in $JUNC_o$ $(i = 0, 1)$, then we must verify that $(\nu_o, \mathrm{id}) \circ (P, \pi)$ is again in $TRIP_o$, with source and target $(F_i, U_i) \circ (T_i, \varkappa_i, \mu_i)$ $(i = 0, 1)$. We have

$$\begin{aligned} \overline{\pi} \cdot Q \overline{\varkappa}_o &= U_1 \pi F_o \cdot \nu_o T_o F_o \cdot Q U_o \varkappa_o F_o \cdot Q \eta_o \\ &= U_1 \pi F_o \circ U_1 P \varkappa_o F_o \cdot \nu_o F_o \circ Q \eta_o \\ &= U_1 \varkappa_1 P F_o \cdot \eta_1 Q = \overline{\varkappa}_1 Q \;, \end{aligned}$$

using $\nu_1 = \mathrm{id}\, P F_o$ and $P F_o = F_1 Q$ in the last two steps. The diagram

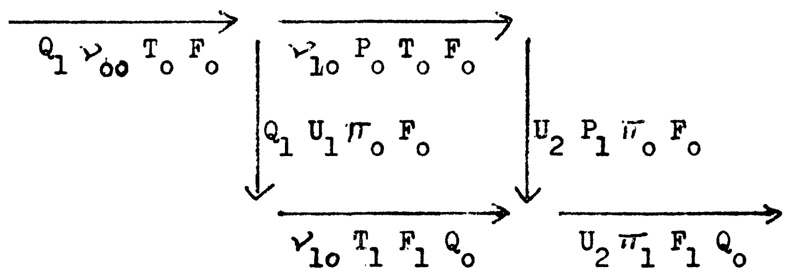


commutes, and this completes the proof.

Transversally, this is again the action of a category on a set with target function. The transversal target of (P, π) in $TRIP_o$ is P, and thus we have a transversal target functor from $TRIP_o$ to CAT, the lateral arrow category of JUNC. With respect to lateral composition in $JUNC_o$ and the composition of $TRIP_o$, the action of $JUNC_o$ on $TRIP_o$ is a functor of two variables. We have verified the proper behavior for objects, and $(\nu_o, id) \cdot \pi$ clearly is an identity morphism if (ν_o, id) and π are identity morphisms. It remains to verify the middle interchange law. Thus let $(P_i, \pi_i) : (T_i, \varkappa_i, \mu_i) \longrightarrow (T_{i+1}, \varkappa_{i+1}, \mu_{i+1})$ in $TRIP_o$ and $(P_i, Q_i, \nu_{io}, id) : (F_i, U_i) \longrightarrow (F_{i+1}, U_{i+1})$ in $JUNC_o$, for $i = 0, 1$. We must show that, in shorthand notation,

$$((\nu_{1o}, id) \cdot \pi_1)((\nu_{oo}, id) \cdot \pi_o) = ((\nu_{1o}, id)(\nu_{oo}, id)) \cdot (\pi_1 \pi_o) .$$

This follows immediately from the fact that the diagram



commutes, and that $F_1 Q_o = P_o F_o$.

The action of $JUNC_1$ on $TRIP_1$ is defined dually; we put

$$(P, Q; id, \nu_1) \cdot (P, \pi) = (Q, U_1 \pi F_o \cdot U_1 T_1 \nu_1)$$

for $(P, \pi) : (T_o, \varkappa_o, \mu_o) \longrightarrow (T_1, \varkappa_1, \mu_1)$ in $TRIP_1$ and $(P, Q; id, \nu_1) : (F_o, U_o) \longrightarrow (F_1, U_1)$ in $JUNC_1$.

2.3. The functors $D_i : \mathrm{JUNC}_i \longrightarrow \mathrm{TRIP}_i$.

For an adjunction (F,U,η,ε) , we put

$$D_o\,(F,U) \;=\; D_1\,(F,U) \;=\; (U\,F, \eta\,, U\,\varepsilon\,F)\ .$$

This is a triple on the target category of (F,U) .

For a morphism $(P,Q;\nu_o, \mathrm{id}) : (F_o,U_o) \longrightarrow (F_1,U_1)$ of JUNC_o , we put

$$D_o\,(P,Q;\nu_o, \mathrm{id}) \;=\; (Q, \nu_o\,F_o) \;:\; D_o\,(F_o,U_o) \longrightarrow D_o\,(F_1,U_1)\ .$$

We claim that this defines a functor $D_o : \mathrm{JUNC}_o \longrightarrow \mathrm{TRIP}_o$ for the lateral composition of JUNC_o , and that D_o preserves transversal composition, i.e. that

$$D_o\,(\rho\cdot\mu) \;=\; \rho\circ(D_o\mu)\ ,$$

in abbreviated notation, if the lefthand side is defined.

Dually, we define $D_1 : \mathrm{JUNC}_1 \longrightarrow \mathrm{TRIP}_1$, with the same properties, by

$$D_1\,(P,Q;\mathrm{id}, \nu_1) \;=\; (Q, U_1\,\nu_1) \;:\; D_1\,(F_o,U_o) \longrightarrow D_1\,(F_1,U_1)$$

for a morphism $(P,Q;\mathrm{id}, \nu_1) : (F_o,U_o) \longrightarrow (F_1,U_1)$ of JUNC_1 .

We prove our claims for D_o as follows. We denote by $S : \mathrm{JUNC}_o \longrightarrow \mathrm{CAT}$ the transversal source functor on JUNC_o , i.e. $S\,(P,Q;\nu_o, \mathrm{id}) = P$, and by $Z : \mathrm{TRIP}_o \longrightarrow \mathrm{CAT}$ the transversal target functor on TRIP_o . S is a functor for lateral composition, and $S\,(\rho\cdot\mu) = S\mu$ for transversal composition. We define a functor $I : \mathrm{CAT} \longrightarrow \mathrm{TRIP}_o$ by putting

$$I\mathcal{A} \;=\; (\mathrm{Id}\,\mathcal{A}, \varkappa, \mu) \qquad \text{with} \qquad \varkappa = \mu = \mathrm{id}\ \mathrm{Id}\,\mathcal{A}$$

for a category $\mathcal{A}$, and

$$I\,P \;=\; (P, \mathrm{id}\,P) \;:\; I\mathcal{A} \longrightarrow I\mathcal{B}$$

for a functor $P : \mathcal{A} \longrightarrow \mathcal{B}$. This obviously defines a functor I . We note that

$$D_o\,(F,U) \;=\; (F,U) \circ I\,S\,(F,U)$$

for an adjunction (F,U) , and

$$D_o\,\nu \;=\; \nu \cdot I\,S\,\nu$$

for a morphism $\nu = (\nu_o, \text{id})$ of JUNC_o . Now the desired formal properties of D_o follow immediately from the corresponding properties of the transversal action of JUNC_o on TRIP_o , and of the functor S .

We note that I is a left adjoint coretractor of **Z** . **If** $\mathcal{A}$ **is a category** and $(T,\varkappa,\mu)$ a triple on a category $\mathcal{C}$, then

$$(P, \varkappa P) \;:\; I\mathcal{A} \longrightarrow (T,\varkappa,\mu)$$

in TRIP_o for every functor $P : \mathcal{A} \longrightarrow \mathcal{C}$, and this is clearly the only morphism from $I\mathcal{A}$ to $(T,\varkappa,\mu)$ in TRIP_o with **transversal target** P . Thus I **is left adjoint to** Z , and the identity functors $\text{Id}\,\mathcal{A} : \mathcal{A} \longrightarrow Z\,I\mathcal{A}$ define the unit of an adjunction (I,Z) .

The functor $D_1 : \text{JUNC}_1 \longrightarrow \text{TRIP}_1$ is treated dually. If we define functors I and Z for TRIP_1 , then we note that I is a right adjoint coretractor of Z .

3. The functors E and K

3.1. Triple algebras and free triple algebras

We describe in this section the triple algebras of Eilenberg and Moore, and the free triple algebras of Kleisli, with the properties which we shall need, but without proofs. Let $(T,\varkappa,\mu)$ be a triple on a category $\mathcal{C}$.

Objects of the category $\mathcal{C}^T$ of triple algebras are all pairs (C,u) consisting of an object C of $\mathcal{C}$ and a morphism $u : T\,C \longrightarrow C$ of $\mathcal{C}$ such that $u\,\varkappa_C = \mathrm{id}\,C$ and $u\,\mu_C = u\,(T\,u)$. Morphisms $f : (A,u) \longrightarrow (B,v)$ of $\mathcal{C}^T$ are all morphisms $f : A \longrightarrow B$ of $\mathcal{C}$ such that $f\,u = v\,(T\,f)$. Composition in $\mathcal{C}^T$ is lifted from composition in T.

We define an adjunction $E\,(T,\varkappa,\mu) = (F^T, U^T, \varkappa, \varepsilon^T)$ from $\mathcal{C}^T$ to $\mathcal{C}$ as follows. $U^T\,(C,u) = C$ for objects, and $U^T\,f$ is the morphism $f : A \longrightarrow B$ for a morphism $f : (A,u) \longrightarrow (B,v)$ of algebras. This defines a functor $U^T : \mathcal{C}^T \longrightarrow \mathcal{C}$. We put $F^T\,C = (T\,C, \mu_C)$ for objects, and $F^T\,f = T\,F : F^T\,A \longrightarrow F^T\,B$ for morphisms $f : A \longrightarrow B$. This defines a functor $F^T : \mathcal{C} \longrightarrow \mathcal{C}^T$, with $U^T\,F^T = T$. For every algebra (C,u), we have $u : (T\,C, \mu_C) \longrightarrow (C,u)$ in $\mathcal{C}^T$, and we put $\varepsilon^T\,(C,u) = u : (T\,C, \mu_C) \longrightarrow (C,u)$. One verifies easily that these data, and $\eta^T = \varkappa$, define an adjunction as desired, and that

$$D_o\,E\,(T,\varkappa,\mu) = (T,\varkappa,\mu) .$$

We note that $u = (U\,\varepsilon^T)_A$ for an algebra (A,u).

Objects of the category $\mathcal{C}_T$ of free triple algebras are the objects of $\mathcal{C}$, and F_T on objects is the identity mapping. We find it convenient, however, to distinguish the object C of $\mathcal{C}$ from the object $F_T\,C$ of $\mathcal{C}_T$ in notation. A morphism in $\mathcal{C}_T\,(F_T\,A\,,\,F_T\,B)$ is a pair (f,B) with $f : A \longrightarrow T\,B$ in $\mathcal{C}$. Composition in $\mathcal{C}_T$ is defined by $(g,C)\,(f,B) = (\mu_C\,(T\,g)\,f\,,\,C)$, and $\mathrm{id}\,F_T\,C = (\varkappa_C\,,\,C)$ for an object C of $\mathcal{C}$.

We define an adjunction $K\,(T,\varkappa,\mu) = (F_T, U_T, \varkappa, \varepsilon_T)$ from $\mathcal{C}_T$ to $\mathcal{C}$ as follows. $F_T\,f = (\varkappa_B\,f, B)$ for a morphism $f : A \longrightarrow B$ of $\mathcal{C}$, and $U_T\,(f,B) = \mu_B\,(T\,f)$ for a morphism (f,B) of $\mathcal{C}_T$. One verifies easily that this defines indeed functors $U_T : \mathcal{C}_T \longrightarrow \mathcal{C}$ and $F_T : \mathcal{C} \longrightarrow \mathcal{C}_T$, and that

$U_T\ F_T = T$. We define $\varepsilon_T : F_T\ U_T \longrightarrow \mathrm{Id}\ \mathscr{C}_T$ by putting $(\varepsilon_T\ F_T)_A = (\mathrm{id}\ T\ A,\ A)$ for an object A of $\mathscr{C}$. These data, together with $\eta_T = \varkappa$, define an adjunction as desired, and

$$D_1\ K\ (T,\varkappa,\mu) = (T,\eta,\mu) \ .$$

If $(f,B) : F_T\ A \longrightarrow F_T\ B$ in $\mathscr{C}_T$ and $g : A \longrightarrow T\ B$ in $\mathscr{C}$, then we note that $(f,B) = (\varepsilon_T\ F_T)_B\ (F_T\ g)$ and $g = (U_T\ (f,B))\,\varkappa_A$ if and only if $f = g$.

3.2. The functor $E : \mathrm{TRIP}_1 \longrightarrow \mathrm{JUNC}_1$

Theorem. Let (F,U) be an adjunction and $(T,\varkappa,\mu)$ a triple. For every morphism $(Q,\pi) : D_1\ (F,U) \longrightarrow (T,\varkappa,\mu)$ in TRIP_1 , there is exactly one morphism $(P,Q;\ \mathrm{id},\nu_1) : (F,U) \longrightarrow E\ (T,\varkappa,\mu)$ in JUNC_1 such that $(Q,\pi) = D_1\ (P,Q;\ \mathrm{id},\nu_1)$.

Proof. A functor Q and a natural transformation $\pi : T\ Q \longrightarrow Q\ U\ F$ are given so that

$$\pi \cdot \varkappa Q = Q\eta \ , \qquad \pi \cdot \mu Q = Q\ U\ \varepsilon\ F \cdot \pi U\ F \cdot T\pi \ ,$$

and we must find a functor P and a natural transformation $\nu_1 : F^T\ Q \longrightarrow P\ F$ which satisfy

$$U^T\ P = Q\ U \ , \qquad \pi = U^T\ \nu_1 \ , \qquad Q\eta = U^T\ \nu_1 \cdot \varkappa Q \ ,$$

and hence also $\varepsilon^T\ P = P\ \varepsilon \cdot \nu_1\ U$.

We must put $P\ A = (Q\ U\ A,\ \varphi_A)$ for an object A of the source of (F,U) , where $\varphi_A : T\ Q\ U\ A \longrightarrow Q\ U\ A$ defines an algebra. But then

$$\varphi_A = (U^T\ \varepsilon^T\ P)_A \ ,$$

and we must put $\varphi_A = (Q\,U\varepsilon)_A\,(\pi U)_A$ in order to satisfy all requirements. For $f : A \longrightarrow B$, we must put

$$P\,f \;=\; Q\,U\,f \;:\; (Q\,U\,A, \varphi_A) \longrightarrow (Q\,U\,B, \varphi_B) \;.$$

Thus the requirements determine P uniquely, but we must verify that we have defined a functor.

The morphisms φ_A define a natural transformation

$$\varphi \;=\; Q\,U\,\varepsilon \cdot \pi\,U \;:\; T\,Q\,U \longrightarrow Q\,U \;.$$

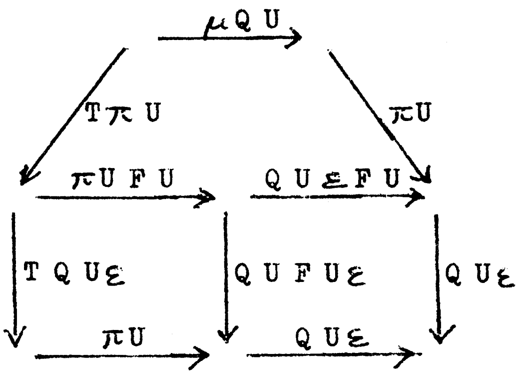
Thus each $P\,f$ is a morphisms of algebras, and hence P a functor, if each $P\,A$ is an algebra. Each $P\,A$ is an algebra if and only if

$$\varphi \cdot \varkappa\,Q\,U \;=\; \mathrm{id}\,Q\,U \;, \qquad \varphi \cdot \mu\,Q\,U \;=\; \varphi \cdot T\varphi \;.$$

For this, we note that

$$\varphi \cdot \varkappa\,Q\,U \;=\; Q\,U\,\varepsilon \cdot \pi\,U \cdot \varkappa\,Q\,U \;=\; Q\,U\,\varepsilon \cdot Q\,\eta\,U \;=\; \mathrm{id}\,Q\,U \;,$$

and that the diagram



commutes.

There is a unique natural transformation $\nu_1 : F^T\,Q \longrightarrow P\,F$ such that $U^T\nu_1 \cdot \varkappa\,Q = Q\,\eta$. If always $\pi_C : (T\,Q\,C, \mu_{QC}) \longrightarrow (Q\,U\,F\,C, \varphi_{FC})$, then $\pi = U^T\nu_1$, since we also have $\pi \cdot \varkappa\,Q = Q\,\eta$. Thus we must show that

$\pi \cdot \mu Q = \varphi F \cdot T\pi$. But this says $\pi \cdot \mu Q = Q U \varepsilon F \cdot \pi U F \cdot T\pi$ which π must satisfy.

Corollary. The adjunctions $E(T, \varkappa, \mu)$ define a right adjoint coretractor $E : TRIP_1 \longrightarrow JUNC_1$ of the functor $D_1 : JUNC_1 \longrightarrow TRIP_1$.

This follows immediately from the Theorem. In order to obtain E for morphisms, use the proof of the Theorem for $(F,U) = E(T', \varkappa', \mu')$.

3.3. The functor $K : TRIP_o \longrightarrow JUNC_o$

Theorem. Let (F,U) be an adjunction and $(T, \varkappa, \mu)$ a triple. For every morphism $(Q,\pi) : (T, \varkappa, \mu) \longrightarrow D_o(F,U)$ in $TRIP_o$, there is exactly one morphism $(P,Q; \nu_o, id) : K(T, \varkappa, \mu) \longrightarrow (F,U)$ in $JUNC_o$ such that $(Q,\pi) = D_o(P,Q; \nu_o, id)$.

Proof. A functor Q and a natural transformation $\pi : Q T \longrightarrow U F Q$ are given so that

$$\pi \cdot Q\varkappa = \eta Q , \quad \pi \cdot Q\mu = U \varepsilon F Q \cdot U F \pi \cdot \pi T ,$$

and we must find a functor P and a natural transformation $\nu_o : Q U_T \longrightarrow U P$ which satisfy

$$P F_T = F Q , \quad \pi = \nu_o F_T , \quad \eta Q = \nu_o F_T \cdot Q \varkappa$$

and hence also $P \varepsilon_T = \varepsilon P \cdot F \nu_o$.

If $(f,B) : F_T A \longrightarrow F_T B$, then $(f,B) = (\varepsilon_T F_T)_B (F_T f)$, and thus

$$P(f,B) = (P \varepsilon_T F_T)_B (P F_T f) = (\varepsilon F Q)_B (F\pi)_B (F Q f)$$

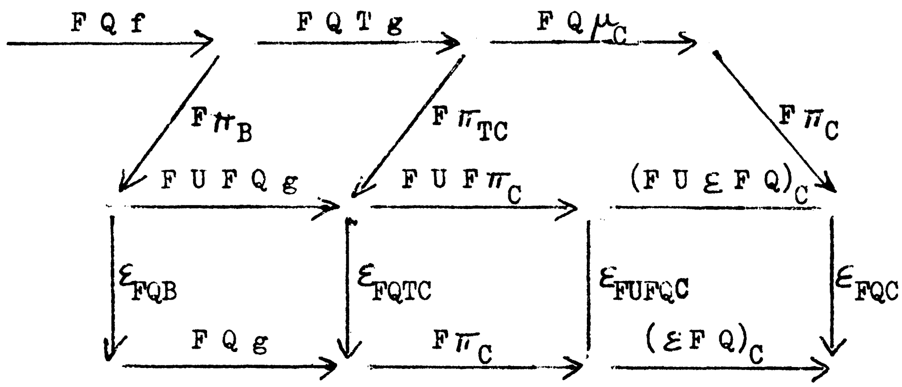
if the requirements are satisfied. Thus P is unique, but we have to verify that P is a functor and that $P F_T = F Q$.

If $f : A \longrightarrow B$, then

$$P F_T f = (\varepsilon F Q)_B (F \pi)_B (F Q \varkappa)_B (F Q f)$$

$$= (\varepsilon F Q)_B (F \eta Q)_B (F Q f) = F Q f .$$

Thus $P F_T = F Q$, and $P (\mathrm{id}\, F_T B) = \mathrm{id}\, (P F_T B)$ follows. The diagram



commutes for $f : A \longrightarrow T B$ and $g : B \longrightarrow T C$, and thus

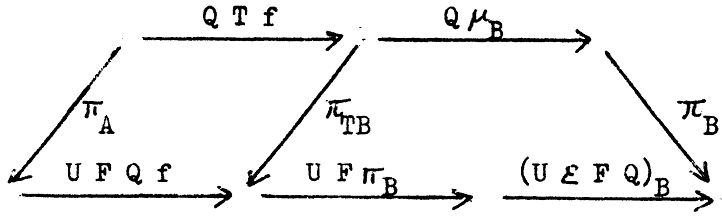
$$(P (g,C)) (P (f,B)) = P (\mu_C (T g) f , C) .$$

Now all properties of P are verified.

There is a unique natural transformation $\nu_0 : Q U_T \longrightarrow U P$ such that $\nu_0 F_T \cdot Q \varkappa = \eta Q$, and we have $\pi \cdot Q \varkappa = \eta Q$. Thus $(\nu_0 F_T)_B = \pi_B$ for all objects, and $\pi = \nu_0 F_T$, if

$$\pi_B (Q U_T (f,B)) = (U P (f,B)) \pi_A$$

whenever $(f,B) : F_T A \longrightarrow F_T B$. The commutative diagram



shows that this is indeed the case.

Corollary. The adjunctions $K(T, \eta, \mu)$ define a left adjoint coretractor $K : TRIP_o \longrightarrow JUNC_o$ of the functor $D_o : JUNC_o \longrightarrow TRIP_o$.

This follows immediately from the Theorem.

4. Duality

Dual double categories do not come in pairs, but in quadruples. If we reverse the transversal arrows, but not the lateral arrows, of a double category $\mathbb{D}$, then we obtain a conjugate double category $\mathbb{D}^c$, with the same cells, but with the order of transversal composition reversed. Thus $\mathbb{D}^{trans}$ and $(\mathbb{D}^c)^{trans}$ are dual categories, while $\mathbb{D}^{lat}$ and $(\mathbb{D}^c)^{lat}$ are the same category. This carries over to the two arrow categories. If we reverse lateral arrows, but not transversal arrows, then we obtain a transpose double category $\mathbb{D}^t$ with $\mathbb{D}^{lat}$ and $(\mathbb{D}^t)^{lat}$ dual, and $\mathbb{D}^{trans}$ and $(\mathbb{D}^t)^{trans}$ the same. Finally, we can reverse all arrows to obtain a symmetric double category $\mathbb{D}^s = (\mathbb{D}^c)^t = (\mathbb{D}^t)^c$.

The usual categorical duality replaces every category $\mathcal{C}$ with a dual category $\mathcal{C}^{op}$, every functor $F : \mathcal{A} \to \mathcal{B}$ with a dual functor $F^{op} : \mathcal{A}^{op} \longrightarrow \mathcal{B}^{op}$, and every natural transformation $\mu : F \longrightarrow G$ with its dual $\mu^{op} : G^{op} \longrightarrow F^{op}$. The dual of an adjunction $(F, U; \eta, \varepsilon) : \mathcal{A} \to \mathcal{B}$ is an adjunction

$$(F, U; \eta, \varepsilon)^{op} = (U^{op}, F^{op}, \varepsilon^{op}, \eta^{op}) : \mathcal{B}^{op} \longrightarrow \mathcal{A}^{op} .$$

The arrow is reversed because left and right are interchanged. The dual of a triple (T, η, μ) on a category $\mathcal{C}$ is the cotriple $(T^{op}, \eta^{op}, \mu^{op})$ on $\mathcal{C}^{op}$.

The behavior of arrows indicates that categorical duality induces an isomorphism of the double category JUNC and its conjugate $JUNC^c$. It is easily seen that this is indeed the case. Since left and right are interchanged in adjunctions, duality interchanges $JUNC_o$ and $JUNC_1$, carrying either of these double categories into the conjugate of the other.

Duality also provides an isomorphism of $TRIP_o$ with a category $COTR_1$ with cotriples as objects, and of $TRIP_1$ with a category $COTR_o$ of cotriples. The left action of $JUNC_i$ on $TRIP_i$ becomes a left action of $(JUNC_j)^c$ on $COTR_j$ for $j \neq i$, and hence a right action of $JUNC_j$ on $COTR_j$. The functors D_i become functors $D^*_j : (JUNC_j)^{lat} \longrightarrow COTR_j$ which preserve the right transversal action, i.e. $D^*_j(\alpha \cdot \beta) = (D^*_j \alpha) \cdot \beta$ for appropriate α and β . The Eilenberg-Moore construction of coalgebras for cotriples leads to a functor E^* : $COTR_o \longrightarrow JUNC_o$, a right adjoint coretract of $D^*_o : JUNC_o \longrightarrow COTR_o$, and the Kleisli construction leads to a functor $K^* : COTR_1 \longrightarrow JUNC_1$, a left adjoint coretract of D^*_1 .

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