A GENERALIZATION OF THE METHOD OF INTERIOR PARALLELS, AND ISOPERIMETRIC INEQUALITIES "WITH PARTIALLY FREE BOUNDARIES

by

Catherine Bandle

Research Report 70-48

November, 1970

A Generalization of the Method of Interior Parallels, and Isoperimetric Inequalities with Partially Free Boundaries

by

Catherine Bandle

Introduction

Makai [7], [8] and Pólya [10] introduced the method of interior parallels to construct trial functions for the Rayleigh quotient of a vibrating membrane. Payne and Weinberger [9] sharpened these results by using some geometrical inequalities of Sz.-Nagy [11]. They proved the following theorem: Among all homogeneous membranes of given area which are fixed along the outer boundary curve of given length and are free otherwise, the annulus has the highest first eigenvalue. Hersch [4] considered multiply-connected membranes which are fixed on one inner curve and are free otherwise. He showed that for these membranes with an inner boundary of given length the first eigenvalue is not greater than that for an annulus of the same area. The aim of the present is to prove similar results for membranes which are partially free on their outer boundary, and to generalize them to the case of inhomogeneous membranes. In the first section we extend the method of interior parallels to curves which are not closed. It will be that the inequalities of Sz.-Nagy remain valid in a slightly modified form. These inequalities will be used in the second part to derive isoperimetric inequalities for the membrane. The proofs will be similar to those in [9] and [4]. The principal results of this paper were obtained during a stay at the Advanced Studies Center of the Battelle Institute, Geneva.

§1. Inequalities of Sz.-Nagy for parallel sets in sectors.

1*1. Let S be the sector O ^ _ r < a o , 0 <^ 9 <L a (r,9 polar coordinates). Let G̃(G̃ 0 S ^ 0) be a bounded simply-connected domain which contains the origin and which is bounded by two linear segments an Q = 0 and 0 = a, and by a Jordan arc T_o. We assume that r_o OS consists of only one arc. if B is a domain such that B 0 F_o / (6, and if c c B is a rectifiable curve joining a point PcB and r_o then $d_B(P, 1" _o) = \inf_{c c B} \int_c ds$ denotes the distance from P to V_o with respect to B. The sets

$\tilde{G}_{-t}$ = fP€G̃; $d_g(P,r_Q)$ <L t] (0 £ t £ t_Q) will be called the ~~interior parallel sets~~ of V , and r̃ . = (PeG̃; d-(P,r) = t} are the corresponding interior parallels. For domains G̃ lying entirely in S we also define the exterior parallel sets

$\tilde{G}_{+t}$ = [P€S\G̃; d_g^g(P,r_o) {t l (0 £ t < GD) and the exterior parallels V^ = (PGS\G̃; d ,ft»(P,r) = t}. A(B) is the area of B and L(r) the length of I\ We shall use the abbreviations Ã(t) = A(G^$_t$) •

$$\tilde{A}_+(t) = A(\tilde{G}_{+t}),\ \tilde{L}(t) = L(f_{-t}) \text{ and } \tilde{L}_+(t) = L(f_{+t}) .$$

The following lemma is a generalization of a result obtained by Sz.-Nagy [11].

Lemma 1: (a) Let $\tilde{G}$ be contained in S. Then $\tilde{A}_+(t) - \frac{\alpha}{2} t^2$ is a continuous and concave function of $t (0 \le t < \infty)$.

(b) For $\alpha \le \pi$, $\tilde{A}_-(t) + (\alpha/2) t^2$ is a continuous and concave function of $t (0 \le t \le t_0$, where t_0 is the largest value of t such that $\tilde{G}_{-t_0}$ is not empty).

Proof: 1° Following [11] we consider first a domain $\tilde{G}$ whose boundary arc Γ_0 consists of circular arcs which are all convex with respect to $\tilde{G}$. We suppose further that their centers M_i $i = 1,2,\ldots,n$ are distinct, and that at the point of intersection of two arcs their tangents do not coincide.

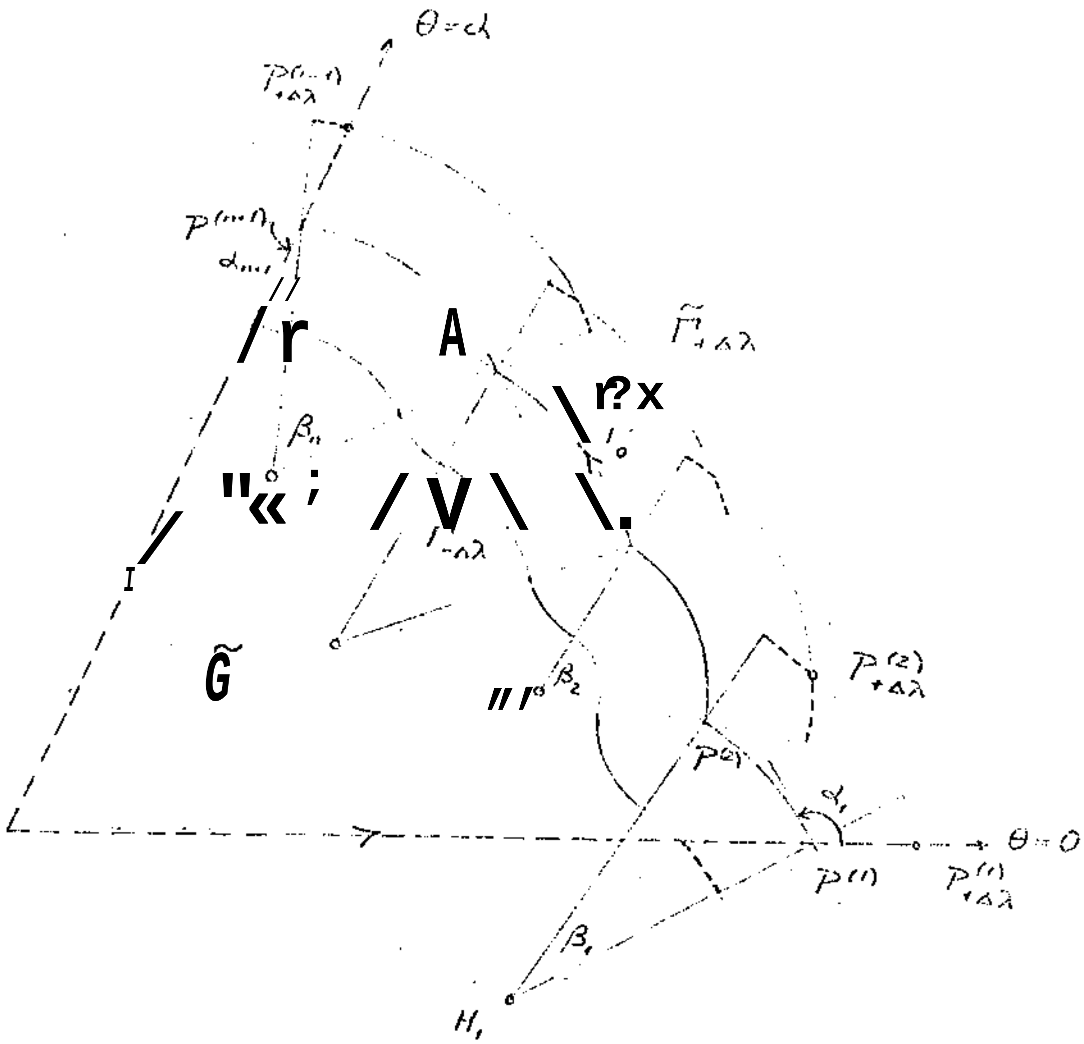

Figure 1

$P^{(i)}$ or $p|^{\wedge}_t$ $(i = l,2,...,n+l)$ are the corners of $\Gamma_o(\tilde{T}_{+\Delta t})$, including its endpoints. $a^{\wedge}$ denotes the external angles at P^{*1J}. It follows from the convexity that $a_i < 0$ for $i = 2,...,,n$. For the sake of brevity, only the case $ou_{\perp} > IT/2$ and $^a{}_{n+i}{}_{\perp} < TT/2$ will be considered; it is clear that the results hold also for the other cases. $(3_i$ is the angle of the i^{th} circular arc.

The point $P\_[V^{i)}$ $i = 2,...,n$ lies on a hyperbola through P^{*1} with the focal points $M._{1-i}$ and $M.._I$. If At is small, $P^{\wedge}|$ is close to the tangent to this hyperbola at $p^{\backslash 1)}$. Since this tangent besects the angle between $P^{(i)}M._{1}$ and $P^{(i)}M._{1-1}$, it follows that

$$(1)\quad \pounds_+(At) - \tilde{L}(0) = At \sum_{i=l}^{n} 0-_x - At \sum_{i=2}^{n} 2tg\ (-a_{\perp}/2) + o(At) +$$

$$+ \quad At(a_1 - i//2) - At\ tg\ (u/2 - a_{n+1})$$

and therefore $d^a = \lim\limits_{At\backslash O} \frac{\tilde{L}_+(At)-\tilde{L}(O)}{At}$ exists. Because of the inequality $tg\ y^2 \wedge v^2$ and the relation

$$(2)\quad \sum^{n+1} a. + \int Kds = 2\pi - (r/ - a)$$

(K is the curvature of dG, and s the arc length of $S\tilde{G})$,

we conclude that

$$(3)\quad d^a \leq 1\ a.$$

Since $(f_{+fc})_{\frac{t}{r}} = \wedge_{+(}/_{t+r)}w$ (3) is valid for all exterior parallels. Under our assumptions on Γ_{o}, $f(t) = \acute{L}_+(t) - at$ is continuous and has a right-hand side derivative $f^1(t)$ £ 0. Since $\tilde{L}_+(t)$ = A|(t),

we can write $f(t) = (\tilde{A}_+(t) - (a/2)t^2)$». The function

$F(t) = \tilde{A}_+(t) - (a/2)t^2$ is concave, because $F''(t) = f'(t) \leq 0$.

This proves assertion (a) of lemma 1 for this particular case.

The proof of (b) is analogous. We have

$$(4) \quad \tilde{L}(At) - \tilde{L}(0) = - At \sum_{i=1}^{n} p_i - At \sum_{i=2}^{n} a_i - At'tg(a_n - IT/2) +$$

$$+ At(Tf/2 - \%_{+1}) + o(At) .$$

Hence $d^1 = \lim_{At \searrow 0} \frac{L(At) - L(0)}{At}$ exists and satisfies the inequality

$$(5) \quad d^1 \geq - a$$

Because of the assumption $a \leq u'$, (5) holds for all interior parallels, in particular for those which consist of different arcs. As before, we conclude that $J(t) = \tilde{A}(t) + (a/2)t^2$ is a continuous and concave function of t.

2° In the general case we approximate $\tilde{G}$ by domains of the type described in 1°. The proof is exactly the same as in [11] and will therefore be omitted.

From now on, let T_o be a Jordan arc containing a finite number of arcs of class C^2. The concavity of F(t) or J(t) guarantees the existence of $\tilde{A}'(t)$ or $\tilde{A}^{\wedge}(t)$. In [2], [3][1] it is shown that, except for a finite number of corners, and for almost all t, the $\tilde{F}_t$... are of class C^2. Hence

[1] The results of Hartman are valid for parallels on more general Riemannian manifolds.

$$d\tilde{A}(t) = \int_{\tilde{L}_{+t}} dsdt + o(dt) \quad \text{or} \quad d\tilde{A}(t) = \int_{-t} dsdt + o(dt)$$

It follows therefore that

(6) $\tilde{A}|(t) = \tilde{L}_+(t)$ and $\hat{A}(t) = \tilde{L}(t)$.

1.2. An oriented arc will be called convex if it lies everywhere on the left of its tangent (or half-tangent). Let p be an arc from A to B and let C = {c} be the class of all convex arcs lying on the right side of P and joining A and B. P* denotes the arc with the property L(P*) = min $\int$ ds. For short, we shall

say that p* is the "right convex hull" of P. We shall assume that G is a bounded simply-connected plane domain with the positively oriented boundary $r = T_o \cup y$. T_o is an arc which, except for a finite number of corners, is in the class C^2.
y* denotes the right convex hull of y, and A and B are the endpoints of y. The half-tangents of y* in A and B will be called t_1 and t_2, and cp is the oriented angle between t_1 and t_2. If $0 \leq u: < \pi$, wo denote by G the domain with the boundary r_o, t_1 and t_2. Because of the convexity of y*, G is contained in $\tilde{G}$. Let the interior parallels $\tilde{T}_{-t}$ as well as $\tilde{G}_{-t}$ be defined the same way as in 1.1. We set $r_{-t} = f_{-t} \cap G'$ and $G_{-t} = \tilde{G}_{-t} \cap G$, and we shall use the notations $L(t) = L(T_{-t})$, $L = L(r_o)$, $A(t) = A(G_{-t})$ and $A_o = A(G)$.

Lemma 2: If 0 <i cp < IT and a = IT - cp, then

(7) $L^2(t) = ({}^{il}f\wedge)^2 \wedge L^2 - 2aA(t)$

Equality holds only for the circular sector of angle a.

The proof is the same as in [4,5], Lemma 1(b) and formula (6) lead to

(8) L(t) £ L̃(t) £ - at + L and

(9) dA/dt = L(t)

and thus,

(10) $A(t)\ 1 - (a/2)t^2 + Lt$

(11) $(L/a - t)^2 \wedge (L/a)^2 - (2/a)A(t)$.

By (8) and (9) we have L' (t)L(t)dt <^ - adA(t), and integration yields

(12) $L^2(t) - L^2$ £ - 2aA(t) and $L^2 \wedge 2aA(t)$.

This inequality was obtained by a different method in [1].

Because of (12), (11) yields

(13) $L/a - t\ 1\ f\ (L/a)^2 - (2/a)A(t)\}^{1/2}$.

In view of (8), this proves the assertion.

We now consider the following case: Let G be contained in the sector S = {0<^9<^_g, O<^r<ao}, and let A be on 9 = 0 and B on 0 = 8. r thus divides S into two components G_o and G_∞. We suppose that G belongs to the unbounded component G_∞. The exterior parallels are defined as

f_{+t} = (PeG^; $<i_{G_{QD}}$ (P,r_Q) = t); we set $r_{+t} = f_{+t}$ n G

and $L_+(t) = L(F_{+t})$. The definition of G_{+fc} and $A_+(t)$ is analogous. Following [4] and [5], we have

Lemma 3: $L_+(t) £ Le^{RT/p}$, where $T = \int_o^t \frac{dt}{A_+(t)}$.

Proof: By lemma l(a), formula (6) and $L_+(t) <^L_+(t)$, it follows that

$$T= \int_o^t \frac{dt}{I \cdot T(+)} \wedge \int_o^t \frac{dt}{} \quad \frac{1}{}\ln \frac{L+3t}{} \quad \frac{1}{} \ln \frac{L_+(t)}{}$$

Equality holds only for the circular sector.

§2. Applications: Upper bounds for the first eigenvalue of membranes with partially free boundaries.

2.1. Let G be a simply-connected domain whose boundary $F = F_o \cup \gamma$ is subject to the conditions of §1.2. We consider the homogeneous membrane

Me: $Au + Au = 0$ in G, $\frac{\partial u}{\partial n} + k(s)u = 0$ on T_o (n the outer normal, $k(s) \geq 0$, s the arc length), $\frac{\partial u}{\partial n} = 0$ on γ.

The first eigenvalue is characterized by

$$(14) \qquad 7\backslash\wedge = \text{Min } R[w] = \text{Min} \frac{ft(w) + \int_{r_o} k(s)\, w^2\, ds}{\iint_G w^2\, dxdy}, \text{ wliere}$$

$$\&(w) = \iint_G \text{grad}^2 w\, dxdy \,\#$$

Theorem I: Let $a = IT - cp$, where cp is the angle formed by the half-tangents t_1 and $?_2$ of y^* in A and B (see §1.2).

If $0 < a <i\, ir$, then

$$(15)\quad A_1 \le \operatorname{Min}_r \frac{\int^{A_Q} (L^2 - 2aA) f'^2(A)\,dA + Kf^2(O)}{\int_{A=0}^{A} f(A)\,dA} = \frac{\Lambda}{T}$$

where $K = \int_{F_0} k(s)ds$ is the total elasticity and $A_Q = A(G)$ the total area. f ranges over all real functions of class $D^1[0,A_Q]$.

Equality holds if and only if G is the domain $0 \le 9 \le a$, $R_o \le r \le R_\pm$ $(R_\pm = L/a$, $(a/2)(R^2 - R_Q^2) = A_Q)$ and

$$k(s) = \begin{cases} K/L & \text{on } r = R \\ 0 & \text{elsewhere} \end{cases}$$

(see fig. 2). A^ is the first root of the equation

$$\sqrt{\Lambda}(J_o'(\sqrt{\Lambda} R_o) N_o'(\sqrt{\Lambda} R_1) - J_o'(\sqrt{\Lambda} R_1) N_o'(\sqrt{\Lambda} R_o)) +$$

$$+ K(J_Q'(\sqrt{\Lambda} R_o) N_o(\sqrt{\Lambda} R_1) - \wedge(\wedge T R \wedge\wedge M / \overline{A}\wedge)) = o,$$

where J_o is the Bessel function and N_o is the Neumann function of order zero.

Proof: We introduce in (14) trial functions w(P) which are constant along the interior parallels T_{xi}, i.e. $w(P) = v(t)$ if $P \in r_t$. It then follows that

(16) $$D(w) = \iint_{\Gamma_t} \left(\frac{\partial w}{\partial n}\right)^2 dn\,ds = \int_0^{A_0} \left(\frac{dv}{dt}\right)^2 dA = \int_0^{A_0} \left(\frac{dv}{dA}\right)^2 \left(\frac{dA}{dt}\right)^2 dA$$

and by lemma 2 (§1.2)

(17) $$D(w) \leq \int_0^{A_0} (L^2 - 2aA)\,v'^2(A)\,dA$$

Since

(18) $$\iint_G w^2\,dx\,dy = \int_0^{A_0} v^2\,dA$$ and

(19) $$\int k(s) w^2 ds = Kv^2(0),$$

it follows that

$$\lambda_1 \leq \frac{\int_0^{A_0} (L^2 - 2aA)v'^2 dA + Kv^2(0)}{\int_0^{A_0} v^2 dA}.$$ Since this

is true for all v, inequality (15), is proved. Because of the symmetry of the extremal domain, the level lines of the first eigenfunction coincide with the interior parallels. In this case equality holds in (17).

This method can also be used to construct upper bounds for the first eigenvalue of the inhomogeneous membrane M_e': $\Delta u + \lambda p u = 0$

in G $(p \geq 0)$ with the same boundary conditions as M_e. The

Rayleigh quotient is $R[w] = [\mathcal{D}(w) + \int_{\Gamma_0} k(s) w^2 ds] / \iint_G p w^2 dxdy$.

Choosing again $w(P) = v(t)$ for $P \in \Gamma_t$ as a trial function, the denominator becomes $\iint_G p w^2 dxdy = \int_0^{t_0} v^2 \int_{\Gamma_t} p\, ds\, dt = \int_0^{t_0} v^2 g(t) L(t) dt$.

If p is bounded ($0 \le p(x,y) \le H$), then $0 \le g(t) \le H$. We consider the first eigenfunction $v_0(A)$ of the problem

$$[(L^2 - 2aA)v']' + g\lambda v = 0 \quad \text{in } (0, M/H), \quad v'(0) - Kv(0) = 0,$$

$v'(M/H) = 0$, where $M = \iint_G p\, dxdy$. $v_0(A)$ is non-decreasing.

We set $V_0(A) = \begin{cases} v_0(A) & \text{in } (0, M/H) \\ v_0(M/H) & \text{in } [M/H, A_0] \end{cases}$. It than follows that $\int_0^{A_0} g(A) V_0^2(A)\, dA \le H \int_0^{M/H} v_0^2\, dA$. This establishes the following generalization of theorem I, which is similar to a result obtained by Krein [6].

<u>Theorem</u> I': Let G and V satisfy the conditions of theorem I. For given M, L, a, H and K we have

$$\lambda_1 \le \min_{f \in D(0, M/H)} \frac{\int_0^{M/H} (L^2 - 2aA) f'^2 dA + K f^2(0)}{H \int_0^{M/H} f^2 dA},$$

i.e., the membrane with the largest possible first eigenvalue is that covering the domain $0 \le \theta \le \alpha$, $\bar{R}_0 \le r \le R_1$ $[R_1 = L/\alpha,\ \frac{\alpha}{2}(R_1^2 - \bar{R}_0^2) = M]$, and elastically supported along $r = R_1$ $[k(s) = K/L]$ and free on the rest of the boundary.

Figure 2

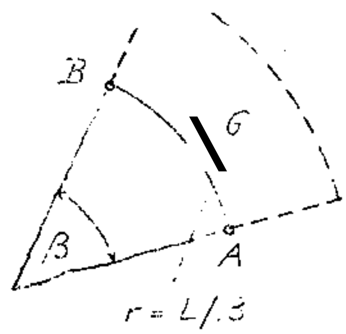

Figure 3

Remarks: 1) This theorem holds also for multiply-connected membranes which are free along the interior boundaries.

2) It is always possible to extend $k(s)$ and r_0 in such a way that the conditions of theorem I are satisfied.

3) For the homogeneous membrane fixed along Γ_0, we have $\lambda_1 < \frac{3L^2}{A_0}$. This inequality is similar to that of Makai [7] and is obtained immediately by setting $v(t) = t$.

4) If $k(s) = GO$ and $a = 0$, then $A_1 \leq^\wedge (\wedge)^2$.

Equality holds if and only if G is a rectangle and p is constant. This result was obtained by Pólya [10] for a fixed, homogeneous membrane.

2.2 Let G be a simply-connected domain with the boundary X" = T_o y y (see §1.2) subject to the following conditions:

G is contained in the sector S O ^ 9 i ^ O { _ r < o o . A and B 1ie on 9 = 0 and 9 = B_K. We assume further that T_o divides S into a bounded component G^o containing the origin, and an unbounded component G^. Let G be in G^. In this case r_{+t} (see §1.2) and $T^{+t} = G$ fl V^{+t} are defined. The following result holds for the membrane Me.

Theorem II: The first eigenvalue satisfies the inequality

$$(20) \quad A_X \; 1 \; \mathrm{Min}_f \frac{\int_0^{A_o} (L^2 + 2pA_+) f^2 (A_+) dA_+ + Kf^2(0)}{\int_0 f^2(A_+) dA_+}$$

Equality holds if and only if G is the domain $0 \leq 9 \leq \beta$, $R_1 \leq r \leq R_2$ $(R_1 = L/\beta$, $P(R_2^2 - R_1^2) = 2A_0)$, and

$$Ms) = \begin{cases} K/L & \text{on } F_o \\ 0 & \text{elsewhere} \end{cases} \text{ (see fig. 3).}$$

The proof is the same as in [4], [5]. We introduce in the Raleigh principle functions $w(P)$ with the level times F_{+t}, i.e. $w(P) = v(t)$ if $P \in r_{+t}$. It follows that

$$(21)\qquad \lambda_1 \le \frac{\int_0^{A_0} \left(\frac{dv}{dt}\right)^2 dA_+ + K v^2(0)}{\int_0^{A_0} v^2 dA_+}$$

If we set $T(t) = \int_0^t \frac{dt}{[illegible]}$ and $p(t) = [A'(t)]^2$, (21) takes the form

$$(22)\qquad \lambda_1 \le \frac{\int_0^{T_1} \left(\frac{dv}{dT}\right)^2 dT + K v^2(0)}{\int_0^{T_1} p v^2(T)\, dT}, \quad \text{where}$$

T_1 is determined by the equation $\int_0^{T_1} p(T)\, dT = A_0$. Let T_2 be such that $\int_0^{T_2} L^2 e^{2pT} dT = A_0$. We define $\rho(T) = \begin{cases} L^2 e^{2pT} & \text{if } T \le T_2 \\ 0 & \text{elsewhere} \end{cases}$.

Because of lemma 3, we have $T_2 \le T_1$. Let $\hat{u}(T)$ be the first eigenfunction of the extremal domain, and set

$$v(T) = \hat{U}(T) = \begin{cases} \hat{u}(T) & \text{if } 0 \le T \le T_2 \\ \hat{u}(T_2) & \text{if } T \ge T_2 \end{cases}.$$

$\hat{u}(T)$ is a non-decreasing function. It then follows that

$$(23)\quad \int_0^{T_1} \frac{1}{p}\hat{u}^2(T)\,dT \geq \int_0^{T_1} \frac{1}{H}\hat{u}^2 dT = \int_0^{T_2} L^2 e^{2\beta T}\hat{u}^2 dT.$$

By (22) and (23) we have

$$\lambda_1 \leq \frac{\int_0^{T_2} \hat{u}'^2 dT + K\hat{u}^2(0)}{\int_0^{T_2} L^2 e^{2\beta T}\hat{u}^2 dT},$$ which is

equivalent to (20).

The same argument as in Theorem I' shows that this result can be generalized. For the inhomogeneous membrane M' with bounded mass distribution $0 \leq p(x,y) \leq H$ we have

<u>Theorem</u> II' : Let G and T satisfy the conditions of Theorem II. For given M,L,p,H and K we have

$$\lambda_1 \leq \underset{f \in D^1(0,M/H)}{\text{Min}} \frac{\int_0^{M/H} (L^2 + 2\beta A_+) f'^2(A_+)dA_+ + Kf^2(0)}{H\int_0^{M/H} f^2 dA_+},$$

i.e., the membrane with the largest possible first eigenvalue is that covering the domain $0 \leq \theta \leq \beta$, $R_1 \leq r \leq \bar{R}_2$ $[R_1 = 1/\beta,$ $\frac{\beta H}{2}(\bar{R}_2^{\,2} - R_1^{\,2}) = M]$, and elastically supported along $r = R_1$ $[k(s) = K/L]$ and free on the rest of the boundary.

The remarks (1) and (4) under Theorem I' remain also valid in this case.

References

[1] C. Bandle, "Extremaleigenschaften von Kreissektoren und Halbkugeln, (to appear).

[2] G. Bol, "Isoperimetrische Ungleichungen für Bereiche auf Flächen", Jahresbericht der Deutschen Math. Vereinigung, 51 (1941), pp. 219-257.

[3] P. Hartman, "Geodesic Parallel Coordinates in the Large", Amer. J. Math., 86 (1964), pp. 705-727.

[4] J. Hersch, "Contribution to the Method of interior Parallels Applied to Vibrating Membranes", Studies in Mathematical Analysis and Related Topics, Stanford University Press (1962), pp. 132-139.

[5] J. Hersch, "The Method of Interior Parallels Applied to Polygonal or Multiply Connected Membranes", Pac. J. Math. 12 (1963), pp. 1229-1238.

[6] M. G. Krein, "On Certain Problems on the Maximum and Minimum of Characteristic Values and on Lyapunov Zones of Stability", Amer. Math. Soc. Transl. Ser. 2,1 (1955), pp. 163-187.

[7] E. Makai, "Bounds for the Principal Frequency of a Membrane and the Torsional Rigidity of a Beam", Acta Sci. Math. (= Acta Szeged), 20. (1959), pp. 33-35.

[8] E. Makai, "On the Principal Frequency of a Convex Membrane and Related Problems", Czechosl. Math. J., 9 (1959), pp. 66-70.

[9] L. E. Payne and H. F. Weinberger, "Some Isoperimetric Inequalities for Membrane Frequencies and Torsional Reigidity", J. Math. Analysis and Appl., 2 (1961), pp. 210-216.

[10] G. Pólya, "Two More Inequalitites Between Physical and Geometrical Quantities", J. Indian Math. Soc., 24 (1960) pp. 413-419.

[11] B. Sz.-Nagy, "Ueber Parallelmengen Nichtkonvexer Ebener Bereiche", Acta Sci. Math. (= Acta Szeged), 20 (1959), pp. 36-47.