THERMODYNAMIC RESTRICTIONS
ON THE CONSTITUTIVE EQUATIONS OF
ELECTROMAGNETIC THEORY

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Thermodynamic Restrictions on the Constitutive Equations of Electromagnetic Theory

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Summary

We consider general materials for which the energy density, entropy density, electric induction, magnetic induction, electric current, and heat flux are determined when the temperature gradient and the histories of the electric intensity, magnetic intensity, and temperature are specified. Assuming that the functionals expressing such dependence on histories have smoothness properties of the "fading memory" type, we find the restrictions placed on the functionals by the second law of thermodynamics. We point out that the general theory has implications for the elementary theory of isotropic and anisotropic materials without memory. We also discuss application of the general theory to the problem of finding thermodynamic restrictions on the kernels occurring in the third-order theory of non-linear isotropic dielectrics with memory of integral type.

1. Admissible Processes

We here discuss the restrictions which the second law of thermodynamics places on the constitutive equations of electromagnetic theory, emphasizing non-linear materials which have long range, gradually fading memory. In our theory each process is described by a collection of nine functions of time t and place $\underset{\sim}{x}$ in a region $\mathcal{R}$ of three-dimensional Euclidean space $\mathcal{E}$; these functions are (1) the electric intensity $\underset{\sim}{E}$, (2) the electric induction $\underset{\sim}{D}$, (3) the magnetic intensity $\underset{\sim}{H}$, (4) the magnetic induction $\underset{\sim}{B}$, (5) the electric current# $\underset{\sim}{j}$, (6) the heat flux $\underset{\sim}{q}$,

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#also called the "free current" or "conduction current".

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(7) the energy density## ϵ, (8) the entropy density### η, and (9) the

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##per unit volume.

###also per unit volume.

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temperature θ. When these functions are sufficiently smooth, Faraday's law, Ampere's law, and the law of balance of energy (i.e. the first law of thermodynamics), have, respectively, the forms

$$\text{curl } \underset{\sim}{E} = -\dot{\underset{\sim}{B}}, \tag{1.1}$$

$$\text{curl } \underset{\sim}{H} = \dot{\underset{\sim}{D}} + \underset{\sim}{j}, \tag{1.2}$$

$$\text{div}(\underset{\sim}{q} + \underset{\sim}{E} \times \underset{\sim}{H}) = -\dot{\epsilon}, \tag{1.3}$$

and the rate γ of production of entropy (per unit volume and unit time) is given by

$$\gamma = \dot{\eta} + \operatorname{div}(\underset{\sim}{q}/\theta). \tag{1.4}$$

It follows from (1.1) and (1.2) that

$$-\operatorname{div}(\underset{\sim}{E} \times \underset{\sim}{H}) = \underset{\sim}{H}\cdot\dot{\underset{\sim}{B}} + \underset{\sim}{E}\cdot\dot{\underset{\sim}{D}} + \underset{\sim}{E}\cdot\underset{\sim}{j}, \tag{1.5}$$

and hence the law of balance of energy (1.3) can be written

$$\dot{\epsilon} = -\operatorname{div}\underset{\sim}{q} + \underset{\sim}{H}\cdot\dot{\underset{\sim}{B}} + \underset{\sim}{E}\cdot\dot{\underset{\sim}{D}} + \underset{\sim}{E}\cdot\underset{\sim}{j}. \tag{1.6}$$

Thus, for γ in (1.4) we have

$$\begin{aligned}\gamma &= \dot{\eta} + \frac{1}{\theta}\operatorname{div}\underset{\sim}{q} - \frac{1}{\theta^2}\underset{\sim}{q}\cdot\underset{\sim}{g} \\ &= \dot{\eta} - \frac{1}{\theta}\left[\dot{\epsilon} - \underset{\sim}{H}\cdot\dot{\underset{\sim}{B}} - \underset{\sim}{E}\cdot\dot{\underset{\sim}{D}} - \underset{\sim}{E}\cdot\underset{\sim}{j}\right] - \frac{1}{\theta^2}\underset{\sim}{q}\cdot\underset{\sim}{g},\end{aligned} \tag{1.7}$$

where

$$\underset{\sim}{g} \overset{\text{def}}{=} \operatorname{grad}_{\underset{\sim}{x}}\theta(\underset{\sim}{x},t) \tag{1.8}$$

is the <u>temperature gradient</u> (at $\underset{\sim}{x}$ at time t). In terms of the quantity,

$$\zeta \overset{\text{def}}{=} \epsilon - \theta\eta - \underset{\sim}{H}\cdot\underset{\sim}{B} - \underset{\sim}{E}\cdot\underset{\sim}{D}, \tag{1.9}$$

which we call the <u>free enthalpy density</u>, the rate of production of entropy can be written

$$\gamma = -\frac{1}{\theta}[\dot{\zeta} + \eta\dot{\theta} + \underset{\sim}{B}\cdot\dot{\underset{\sim}{H}} + \underset{\sim}{D}\cdot\dot{\underset{\sim}{E}}] + \frac{1}{\theta}\underset{\sim}{E}\cdot\underset{\sim}{j} - \frac{1}{\theta^2}\underset{\sim}{q}\cdot\underset{\sim}{g}. \tag{1.10}$$

Although for most of the systems considered in continuum physics θ can be only positive, there are systems which interact with electromagnetic fields and which exhibit negative temperatures.[#] The

[#] Purcell & Pound (1951), Ramsey (1956); see also Coleman & Noll (1959).

theory we develop here is independent of assumptions of positivity for θ. We do assume, however, that θ is not zero.

Given a process for the region $\mathcal{R}$ and a time interval $(-\infty,T]$, we may take a fixed point $\underset{\sim}{x}$ in $\mathcal{R}$ and consider, for each t in $(-\infty,T]$, the functions $\underset{\sim}{E}^t$, $\underset{\sim}{H}^t$, θ^t defined by

$$\underset{\sim}{E}^t(s) = \underset{\sim}{E}(\underset{\sim}{x},t-s), \quad \underset{\sim}{H}^t(s) = \underset{\sim}{H}(\underset{\sim}{x},t-s), \quad \theta^t(s) = \theta(\underset{\sim}{x},t-s), \quad s\in[0,\infty). \tag{1.11}$$

These functions, which map $[0,\infty)$ into sets of finite dimension, are called <u>histories</u>; e.g. $\underset{\sim}{E}^t$ is the <u>history up to</u> t <u>of the electric intensity</u> at the point $\underset{\sim}{x}$.

The basic constitutive assumption of our theory is the following: The <u>material</u> at each place $\underset{\sim}{x}$ in $\mathcal{R}$ is specified by listing six functions, $\hat{\zeta}$, $\hat{\underset{\sim}{D}}$, $\hat{\underset{\sim}{B}}$, $\hat{\eta}$, $\hat{\underset{\sim}{j}}$, and $\hat{\underset{\sim}{q}}$, called "constitutive functionals", which give the values of ζ, $\underset{\sim}{D}$, $\underset{\sim}{B}$, η, $\underset{\sim}{j}$, and $\underset{\sim}{q}$ at $\underset{\sim}{x}$ at time t, as functions of the value of $\underset{\sim}{g}$ at $(\underset{\sim}{x},t)$ and the histories up to t of $\underset{\sim}{E}$, $\underset{\sim}{H}$, and θ at $\underset{\sim}{x}$:

$$\left.\begin{aligned}
\zeta &= \hat{\zeta}(\underset{\sim}{E}^t, \underset{\sim}{H}^t, \theta^t; \underset{\sim}{g}),\\
\underset{\sim}{D} &= \hat{\underset{\sim}{D}}(\underset{\sim}{E}^t, \underset{\sim}{H}^t, \theta^t; \underset{\sim}{g}),\\
\underset{\sim}{B} &= \hat{\underset{\sim}{B}}(\underset{\sim}{E}^t, \underset{\sim}{H}^t, \theta^t; \underset{\sim}{g}),\\
\eta &= \hat{\eta}(\underset{\sim}{E}^t, \underset{\sim}{H}^t, \theta^t; \underset{\sim}{g}),\\
\underset{\sim}{j} &= \hat{\underset{\sim}{j}}(\underset{\sim}{E}^t, \underset{\sim}{H}^t, \theta^t; \underset{\sim}{g}),\\
\underset{\sim}{q} &= \hat{\underset{\sim}{q}}(\underset{\sim}{E}^t, \underset{\sim}{H}^t, \theta^t; \underset{\sim}{g}).
\end{aligned}\right\} \tag{1.12}$$

A collection of the nine functions, $\underset{\sim}{E}$, $\underset{\sim}{D}$, $\underset{\sim}{H}$, $\underset{\sim}{B}$, $\underset{\sim}{j}$, $\underset{\sim}{q}$, ϵ, η, θ (each one a function on $\mathcal{R}\times(-\infty,T]$), is here said to be an <u>admissible process on</u> $\mathcal{R}\times(-\infty,T]$, if it is compatible with both the field equations (1.1)-(1.3) and the constitutive equations (1.12).

The second law of thermodynamics is rendered precise by laying down the following postulate:

Dissipation Principle. <u>In each admissible process on</u> $\mathcal{R}\times(-\infty,T]$, <u>the relation</u>

$$\gamma(\underset{\sim}{x}, t) \geq 0 \tag{1.13}$$

<u>must hold at every point</u> $\underset{\sim}{x}$ <u>in</u> $\mathcal{R}$ <u>and for every time</u> t <u>in</u> $(-\infty,T]$.

This principle places restrictions upon the constitutive functionals $\hat{\zeta}$, $\hat{\underset{\sim}{D}}$, $\hat{\underset{\sim}{B}}$, $\hat{\eta}$, $\hat{\underset{\sim}{j}}$, and $\hat{\underset{\sim}{q}}$ in (1.12), and our problem is that of finding these restrictions.

Thus, we take the point of view of Coleman & Noll (1963) and interpret the second law of thermodynamics to be the assertion that constitutive equations must be so chosen that the rate of production of entropy is everywhere and always positive in all processes compatible with those equations.

2. Principle of Fading Memory

The nature of the restrictions which the dissipation principle places on constitutive functionals depends on the properties of smoothness assumed for those functionals. We here use the principle of fading memory as formulated by Coleman and Noll (1960, 1961, 1962) and modified by Coleman (1964). To state the principle in our present context, we let h be an influence function, i.e. a positive, monotone-decreasing#, Lebesgue

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#Coleman & Mizel (1966) have shown that conditions on h weaker than monotonicity suffice for most of the theory of fading memory.

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measurable function on $(0,\infty)$ decaying to zero fast enough to be integrable over $(0,\infty)$, and we define the norm $\|\Gamma^t\|$ of the triplet of histories $\Gamma^t = (\underset{\sim}{E}^t, \underset{\sim}{H}^t, \theta^t)$ by

$$\|\Gamma^t\|^2 = E^t(0)^2 + H^t(0)^2 + \theta^t(0)^2 + \int_0^\infty E^t(s)^2 h(s)ds + \int_0^\infty H^t(s)^2 h(s)ds + \int_0^\infty \theta^t(s)^2 h(s)ds, \tag{2.1}$$

with $E^t(s)^2 = \underset{\sim}{E}^t(s)\cdot\underset{\sim}{E}^t(s)$, etc. We let the common domain of $\hat{\zeta}, \hat{\underset{\sim}{D}}, \ldots, \hat{\underset{\sim}{q}}$ in (1.12) be the set $\mathfrak{D} = \mathfrak{C}\times\mathcal{V}_{(3)}$, where $\mathfrak{C}$ is the set of all triplets $\Gamma^t = (\underset{\sim}{E}^t, \underset{\sim}{H}^t, \theta^t)$ with $\theta^t(s) \neq 0$ and $\|\Gamma^t\| < \infty$, while $\mathcal{V}_{(3)}$ is a vector space of dimension 3 (i.e. $\underset{\sim}{g}\in\mathcal{V}_{(3)}$). [If, as is usually the case, the material is such that negative temperatures are not possible, then we take $\mathfrak{C}$ to be the set of all triplets $\Gamma^t = (\underset{\sim}{E}^t, \underset{\sim}{H}^t, \theta^t)$ with $\theta^t(s) > 0$ and $\|\Gamma^t\| < \infty$.] Clearly, regardless of whether we have $\theta > 0$ or the weaker restriction $\theta \neq 0$, $\mathfrak{C}$ is a cone in a Hilbert space $\mathfrak{H}$ formed from functions mapping $[0,\infty)$ into $\mathcal{V}_{(7)} = \mathcal{V}_{(3)} \times \mathcal{V}_{(3)} \times \mathcal{V}_{(1)}$, and $\mathfrak{C}$ spans $\mathfrak{H}$.

We assume that all the constitutive functionals $\hat{\zeta}$, $\hat{\underset{\sim}{D}}$, $\hat{\underset{\sim}{B}}$, $\hat{\eta}$, $\hat{\underset{\sim}{j}}$, and $\hat{\underset{\sim}{q}}$ are continuous on $\mathfrak{D} = \mathfrak{C} \times \mathcal{V}_{(3)}$. In addition, we assume that, for each fixed value of $\underset{\sim}{g}$, the functionals in $(1.12)_{1-4}$ are continuously Fréchet-differentiable on $\mathfrak{C}$ in the sense that if we let $\mathfrak{f}$ stand for $\hat{\zeta}$, $\hat{\underset{\sim}{D}}$, $\hat{\underset{\sim}{B}}$, or $\hat{\eta}$, then, for each Γ^t in $\mathfrak{C}$, $\mathfrak{f}$ has a Fréchet differential $d\mathfrak{f}(\Gamma^t;\underset{\sim}{g}|\cdot)$; this differential is a bounded linear form on $\mathfrak{H}$ with the property that for functions Φ in $\mathfrak{H}$ with $\Gamma^t + \Phi$ in $\mathfrak{C}$,

$$\mathfrak{f}(\Gamma^t+\Phi;\underset{\sim}{g}) = \mathfrak{f}(\Gamma^t;\underset{\sim}{g}) + d\mathfrak{f}(\Gamma^t;\underset{\sim}{g}|\Phi) + o(\|\Phi\|). \tag{2.2}$$

The functional $d\mathfrak{f}(\cdot;\cdot|\cdot)$ is assumed to be jointly continuous in all its arguments. We further assume that $\hat{\zeta}$ is continuously differentiable on $\mathfrak{D}$, considered a cone in $\mathfrak{H} \oplus \mathcal{V}_{(3)}$; in other words, there exists a continuous $\mathcal{V}_{(3)}$-valued function $\partial_{\underset{\sim}{g}}\hat{\zeta}$ on $\mathfrak{D} = \mathfrak{C} \times \mathcal{V}_{(3)}$, such that, for each $(\Gamma^t;\underset{\sim}{g})$ in $\mathfrak{D}$, if $(\Gamma^t+\Phi;\ \underset{\sim}{g}+\underset{\sim}{v})$ is in $\mathfrak{D}$, then

$$\hat{\zeta}(\Gamma^t+\Phi;\ \underset{\sim}{g}+\underset{\sim}{v}) = \hat{\zeta}(\Gamma^t;\underset{\sim}{g}) + d\hat{\zeta}(\Gamma^t;\underset{\sim}{g}|\Phi) + \partial_{\underset{\sim}{g}}\hat{\zeta}(\Gamma^t;\underset{\sim}{g})\cdot\underset{\sim}{v} + o\left(\|\Phi\| + |\underset{\sim}{g}|\right). \tag{2.3}$$

The functions $\underset{\sim}{E}^t$, $\underset{\sim}{H}^t$, θ^t have for their domain the interval $[0,\infty)$; the restrictions $\underset{\sim}{E}^t_r$, $\underset{\sim}{H}^t_r$, θ^t_r of these functions to the open interval $(0,\infty)$ are called <u>past</u> <u>histories</u>. We write Γ^t_r for the triplet $(\underset{\sim}{E}^t_r, \underset{\sim}{H}^t_r, \theta^t_r)$.

Since a knowledge of the triplet $\Gamma^t = (\mathbf{E}^t, \mathbf{H}^t, \theta^t)$ is equivalent to knowledge of Γ^t_r and the present values $\mathbf{E}^t(0) = \mathbf{E}(t)$, $\mathbf{H}^t(0) = \mathbf{H}(t)$, $\theta^t(0) = \theta(t)$, each function $\mathfrak{f}$ of Γ^t and $\mathbf{g}(t)$ can be regarded as a function of Γ^t_r, $\mathbf{E}(t)$, $\mathbf{H}(t)$, $\theta(t)$, and $\mathbf{g}(t)$:

$$\mathfrak{f}(\Gamma^t;\mathbf{g}(t)) = \mathfrak{f}(\mathbf{E}^t,\mathbf{H}^t,\theta^t;\mathbf{g}(t)) = \mathfrak{f}\Big(\Gamma^t_r;\mathbf{E}(t),\mathbf{H}(t),\theta(t),\mathbf{g}(t)\Big). \tag{2.4}$$

Thus, if we put simply $\mathbf{E}$ for $\mathbf{E}(t)$, $\mathbf{H}$ for $\mathbf{H}(t)$, etc., we can write the constitutive equations $(1.12)_{1-4}$ in the forms

$$\left.\begin{aligned}
\zeta &= \hat{\zeta}(\Gamma^t_r;\mathbf{E},\mathbf{H},\theta,\mathbf{g}),\\
\mathbf{D} &= \hat{\mathbf{D}}(\Gamma^t_r;\mathbf{E},\mathbf{H},\theta,\mathbf{g}),\\
\mathbf{B} &= \hat{\mathbf{B}}(\Gamma^t_r;\mathbf{E},\mathbf{H},\theta,\mathbf{g}),\\
\eta &= \hat{\eta}(\Gamma^t_r;\mathbf{E},\mathbf{H},\theta,\mathbf{g}).
\end{aligned}\right\} \tag{2.5}$$

Among the implications of the smoothness assumption (2.2) is the existence of linear differential operators, $\underline{\underline{D}}_{\mathbf{E}}$, $\underline{\underline{D}}_{\mathbf{H}}$, $\underline{\underline{D}}_{\theta}$, and δ, which, in the case $\mathfrak{f} = \hat{\zeta}$, are defined by the formulae[#]

[#]See Coleman (1964).

$$\left.\begin{aligned}
\underline{\underline{D}}_{\mathbf{E}}\hat{\zeta}(\Gamma^t;\mathbf{g})\cdot\mathbf{v} &= \frac{\partial}{\partial\nu}\,\hat{\zeta}(\Gamma^t_r;\mathbf{E}+\nu\mathbf{v},\mathbf{H},\theta,\mathbf{g})\Big|_{\nu=0},\\
\underline{\underline{D}}_{\mathbf{H}}\hat{\zeta}(\Gamma^t;\mathbf{g})\cdot\mathbf{v} &= \frac{\partial}{\partial\nu}\,\hat{\zeta}(\Gamma^t_r;\mathbf{E},\mathrm{H}+\nu\mathbf{v},\theta,\mathbf{g})\Big|_{\nu=0},\\
\underline{\underline{D}}_{\theta}\hat{\zeta}(\Gamma^t;\mathrm{g}) &= \frac{\partial}{\partial\nu}\,\hat{\zeta}(\Gamma^t_r;\mathbf{E},\mathbf{H},\theta+\nu,\mathbf{g})\Big|_{\nu=0},\\
\delta\hat{\zeta}(\Gamma^t;\mathbf{g}|\Psi_r) &= \frac{\partial}{\partial\nu}\,\zeta(\Gamma^t_r+\nu\Psi_r;\mathbf{E},\mathbf{H},\theta,\mathbf{g})\Big|_{\nu=0}.
\end{aligned}\right\} \tag{2.6}$$

These equations hold for each pair $(\Gamma^t;\underset{\sim}{g})$ in $\mathfrak{D}$, each vector $\underset{\sim}{v}$ in $\mathcal{V}_{(3)}$, and each function Ψ_r which is the restriction to $(0,\infty)$ of some function Ψ in $\mathfrak{H}$ with the property that $\Gamma^t + \Psi$ is in $\mathfrak{C}$. We call $\underset{\sim}{\underline{\underline{D}}_E}\hat{\zeta}$, $\underset{\sim}{\underline{\underline{D}}_H}\hat{\zeta}$, and $\underline{\underline{D}}_\theta\hat{\zeta}$ <u>instantaneous</u> <u>derivatives</u> of $\hat{\zeta}$, for they are derivatives with respect to present values keeping past histories fixed. The functionals $\underset{\sim}{\underline{\underline{D}}_E}\hat{\zeta}$ and $\underset{\sim}{\underline{\underline{D}}_H}\hat{\zeta}$ are vector-valued, while $\underline{\underline{D}}_\theta\hat{\zeta}$ is scalar-valued. Of course, $\underset{\sim}{\underline{\underline{D}}_E}$, $\underset{\sim}{\underline{\underline{D}}_H}$, and $\underline{\underline{D}}_\theta$ can operate not only on $\hat{\zeta}$ but also on $\hat{\underset{\sim}{D}}$, $\hat{\underset{\sim}{B}}$, and $\hat{\eta}$. As examples, we may consider the tensor-valued# functional $\underset{\sim}{\underline{\underline{D}}_E}\hat{\underset{\sim}{B}}$ and the vector-valued

#Here <u>tensor</u> means "second-order tensor"; i.e. an element of $\mathrm{Lin}(\mathcal{V}_{(3)},\mathcal{V}_{(3)})$, the set of linear transformations of $\mathcal{V}_{(3)}$ into $\mathcal{V}_{(3)}$.

functional $\underline{\underline{D}}_\theta\hat{\underset{\sim}{B}}$:

$$\left.\begin{aligned} \underset{\sim}{\underline{\underline{D}}_E}\hat{\underset{\sim}{B}}(\Gamma^t;\underset{\sim}{g})\underset{\sim}{v} &= \frac{\partial}{\partial\nu}\hat{\underset{\sim}{B}}(\Gamma^t_r;\underset{\sim}{E}+\nu\underset{\sim}{v},\underset{\sim}{H},\theta,\underset{\sim}{g})\Big|_{\nu=0}, \qquad \underset{\sim}{v}\in\mathcal{V}_{(3)}, \\ \underline{\underline{D}}_\theta\hat{\underset{\sim}{B}}(\Gamma^t;\underset{\sim}{g}) &= \frac{\partial}{\partial\nu}\hat{\underset{\sim}{B}}(\Gamma^t_r;\underset{\sim}{E},\underset{\sim}{H},\theta+\nu,\underset{\sim}{g})\Big|_{\nu=0}. \end{aligned}\right\} \tag{2.7}$$

We say that a process is <u>regular</u> <u>at</u> $\underset{\sim}{x}$ <u>at</u> <u>time</u> t if (1) the time-derivatives $\dot{\underset{\sim}{E}}$, $\dot{\underset{\sim}{H}}$, $\dot{\theta}$, and $\dot{\underset{\sim}{g}}$ exist at $\underset{\sim}{x}$ at time t, (2) the triplet of histories up to t, $\Gamma^t = (\underset{\sim}{E}^t,\underset{\sim}{H}^t,\theta^t)$, is, at $\underset{\sim}{x}$, an absolutely continuous function in $\mathfrak{C}$, and (3) $\mathfrak{H}$ contains the function $\dot{\Gamma}^t = (\dot{\underset{\sim}{E}}^t,\dot{\underset{\sim}{H}}^t,\dot{\theta}^t)$, defined by the equation

$$\dot{\Gamma}^t(s) \overset{\text{def}}{=\!=} -\frac{d}{ds}\Gamma^t(s) = \left(\dot{\underset{\sim}{E}}(\tau),\dot{\underset{\sim}{H}}(\tau),\dot{\theta}(\tau)\right)\Big|_{\tau=t-s}, \tag{2.8}$$

which gives a value to $\dot{\Gamma}^t(s)$ at $s = 0$ and for almost all $s \in (0,\infty)$. A function Γ^t with the properties (2) and (3) is called a regular total history. The function $\dot{\Gamma}^t$ in (2.8) may be called the time-derivative of the history Γ^t. We write $\dot{\Gamma}^t_r$ for the restriction of $\dot{\Gamma}^t$ to $(0,\infty)$. Arguments given by Coleman & Mizel here yield#

#See also Coleman (1964) and Mizel & Wang (1966).

Remark 1 (Chain Rule). It follows from our assumption of smoothness for $\hat{\zeta}$ that in each admissible process that is regular at a point $\underset{\sim}{x}$ at time t, the time-derivative of ζ exists at that point at time t and obeys the formula

$$\dot{\zeta}(t) = \frac{d}{dt}\hat{\zeta}(\Gamma^t;\underset{\sim}{g}(t)) = \delta\hat{\zeta}\left(\Gamma^t;\underset{\sim}{g}(t)\,|\,\dot{\Gamma}^t_r\right) + \underset{\approx}{D}_{\underset{\sim}{E}}\hat{\zeta}(\Gamma^t;\underset{\sim}{g}(t))\cdot\dot{\underset{\sim}{E}}(t) + \underset{\approx}{D}_{\underset{\sim}{H}}\hat{\zeta}(\Gamma^t;\underset{\sim}{g}(t))\cdot\dot{\underset{\sim}{H}}(t) + $$

$$+ D_\theta\hat{\zeta}(\Gamma^t;\underset{\sim}{g}(t))\dot{\theta}(t) + \partial_{\underset{\sim}{g}}\hat{\zeta}(\Gamma^t;\underset{\sim}{g}(t))\cdot\dot{\underset{\sim}{g}}(t). \tag{2.9}$$

3. Consequences of the Dissipation Principle

If we substitute the constitutive equations (1.12) into the field equations (1.1)-(1.3), we obtain a system of seven functional-differential equations for the seven independent, real-valued, components of $\underset{\sim}{E}$, $\underset{\sim}{H}$, and θ, regarded as time-dependent fields over a region. It therefore appears reasonable to assert that one can independently choose $\underset{\sim}{E}^t$, $\underset{\sim}{H}^t$, θ^t, $\underset{\sim}{g}$, and $\dot{\underset{\sim}{g}}$ at a given material point and rest assured that there exists at least one# admissible process corresponding to this choice. The following

#Actually, we expect that there should be many.

assumption renders this assertion precise.

Assumption of solvability. Let a point $\underset{\sim}{x}$ and an instant t be given. For each pair of vectors $\underset{\sim}{v}$ and $\underset{\sim}{w}$ in $\mathcal{V}_{(3)}$ and each regular total history Ψ, there exists a $T > t$, a region $\mathcal{R}$ containing $\underset{\sim}{x}$, and an admissible process on $\mathcal{R}\times(-\infty,T]$ which is regular at $(\underset{\sim}{x},t)$ and such that

$$\underset{\sim}{g}(\underset{\sim}{x},t) = \underset{\sim}{v}, \qquad \dot{\underset{\sim}{g}}(\underset{\sim}{x},t) = \underset{\sim}{w}, \qquad \Gamma^t = \Psi, \tag{3.1}$$

where Γ^t is the history up to t of the triplet $(\underset{\sim}{E},\underset{\sim}{H},\theta)$.

It follows from (1.10) and (2.9) that in any process that is regular at $(\underset{\sim}{x},t)$,

$$\gamma = -\frac{1}{\theta}\Big\{[\underset{\approx}{D}_{\underset{\sim}{E}}\hat{\zeta}(\Gamma^t;\underset{\sim}{g}) + \hat{\underset{\sim}{D}}(\Gamma^t;\underset{\sim}{g})]\cdot\dot{\underset{\sim}{E}} + [\underset{\approx}{D}_{\underset{\sim}{H}}\hat{\zeta}(\Gamma^t;\underset{\sim}{g}) + \hat{\underset{\sim}{B}}(\Gamma^t;\underset{\sim}{g})]\cdot\dot{\underset{\sim}{H}}$$

$$+ [\underset{\approx}{D}_{\theta}\zeta(\Gamma^t;\underset{\sim}{g}) + \hat{\eta}(\Gamma^t;\underset{\sim}{g})]\dot{\theta} + \partial_{\underset{\sim}{g}}\hat{\zeta}(\Gamma^t;\underset{\sim}{g})\cdot\dot{\underset{\sim}{g}}\Big\}$$

$$+ \frac{1}{\theta}\Big\{\underset{\sim}{E}\cdot\hat{\underset{\sim}{j}}(\Gamma^t;\underset{\sim}{g}) - \frac{1}{\theta}\,\underset{\sim}{g}\cdot\hat{\underset{\sim}{q}}(\Gamma^t;\underset{\sim}{g}) - \delta\zeta(\Gamma^t;\underset{\sim}{g}|\dot{\Gamma}^t_r)\Big\}, \tag{3.2}$$

where $\gamma = \gamma(\underset{\sim}{x},t)$, $\theta = \theta(\underset{\sim}{x},t)$, $\dot{\underset{\sim}{E}} = \dot{\underset{\sim}{E}}(\underset{\sim}{x},t)$, etc.

Arguments completely analogous to those employed by Coleman (1964)# here yield

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#See also Coleman & Mizel (1967).

Theorem 1. It follows from the Dissipation Principle (1.13) that

(i) ζ, $\underset{\sim}{D}$, $\underset{\sim}{B}$, and η are given by functions of $\underset{\sim}{E}^t$, $\underset{\sim}{H}^t$, and θ^t alone; i.e. $(1.12)_1$-$(1.12)_4$ must reduce to

$$\left.\begin{aligned} \zeta &= \hat{\zeta}(\underset{\sim}{E}^t,\underset{\sim}{H}^t,\theta^t), & \underset{\sim}{D} &= \hat{\underset{\sim}{D}}(\underset{\sim}{E}^t,\underset{\sim}{H}^t,\theta^t),\\ \underset{\sim}{B} &= \hat{\underset{\sim}{B}}(\underset{\sim}{E}^t,\underset{\sim}{H}^t,\theta^t), & \eta &= \hat{\eta}(\underset{\sim}{E}^t,\underset{\sim}{H}^t,\theta^t), \end{aligned}\right\} \tag{3.3}$$

(ii) the functional $\hat{\zeta}$ must determine the functionals $\hat{\underset{\sim}{D}}$, $\hat{\underset{\sim}{B}}$, and $\hat{\eta}$ through the relations,

$$\hat{\underset{\sim}{D}} = -\underset{\approx}{D}_{\underset{\sim}{E}}\hat{\zeta}, \qquad \hat{\underset{\sim}{B}} = -\underset{\approx}{D}_{\underset{\sim}{H}}\hat{\zeta}, \qquad \hat{\eta} = -\underset{\approx}{D}_{\theta}\hat{\zeta}, \tag{3.4}$$

which hold throughout $\mathfrak{C}$, and

(iii) when $\Gamma^t = (\underset{\sim}{E}^t,\underset{\sim}{H}^t,\theta^t)$ is a regular total history, the inequality

$$\theta\delta\hat{\zeta}(\Gamma^t|\dot{\Gamma}^t_r) \le \theta\underset{\sim}{E}\cdot\hat{\underset{\sim}{j}}(\Gamma^t;\underset{\sim}{g}) - \underset{\sim}{g}\cdot\hat{\underset{\sim}{q}}(\Gamma^t;\underset{\sim}{g}) \tag{3.5}$$

must hold for all $\underset{\sim}{g}$.

Remark 2. It follows from (3.3) and (3.4) that (3.2) reduces to

$$\theta\gamma = \underset{\sim}{E}\cdot\hat{\underset{\sim}{j}}(\Gamma^t;\underset{\sim}{g}) - \frac{1}{\theta}\,\underset{\sim}{g}\cdot\hat{\underset{\sim}{q}}(\Gamma^t;\underset{\sim}{g}) - \delta\hat{\zeta}(\Gamma^t|\dot{\Gamma}^t_r). \qquad (3.6)$$

Hence, the conditions (i), (ii), and (iii), taken together, are not only necessary, but are also sufficient to have $\gamma(\underset{\sim}{x},t) \geq 0$ in each admissible process that is regular at the point $\underset{\sim}{x}$ and time t.

Remark 3. On setting $\underset{\sim}{g} = \underset{\sim}{0}$ in (3.5) we obtain the relation,

$$\theta\delta\hat{\zeta}(\Gamma^t|\dot{\Gamma}^t_r) \leq \theta\hat{\underset{\sim}{j}}(\Gamma^t;\underset{\sim}{0})\cdot\underset{\sim}{E}, \qquad (3.7)$$

which we call the internal dissipation inequality. For materials which do not conduct electricity, i.e. for materials which have $\hat{\underset{\sim}{j}} \equiv \underset{\sim}{0}$, this reduces to

$$\theta\delta\hat{\zeta}(\Gamma^t|\dot{\Gamma}^t_r) \leq 0. \qquad (3.8)$$

For materials which do conduct electricity, we have, instead, merely the implication,

$$\underset{\sim}{E} = \underset{\sim}{0} \implies \theta\delta\hat{\zeta}(\Gamma^t|\dot{\Gamma}^t_r) \leq 0. \qquad (3.9)$$

Of course, in (3.5)-(3.9), $\Gamma^t = (\underset{\sim}{E}^t,\underset{\sim}{H}^t,\theta^t)$, and we have written $\underset{\sim}{E}$ for $\underset{\sim}{E}^t(0)$ and θ for $\theta^t(0)$.

Since we have assumed that $\hat{\underset{\sim}{D}}$, $\hat{\underset{\sim}{B}}$, and $\hat{\eta}$ are Fréchet-differentiable on $\mathfrak{C}$, it follows from (3.4) that the second instantaneous derivatives of

$\hat{\zeta}$ exist and obey symmetry relations of the type $D_{\mathbf{H}}D_{\mathbf{E}}\hat{\zeta} = D_{\mathbf{E}}D_{\mathbf{H}}\hat{\zeta}$; hence,

$$D_{\mathbf{H}}\hat{\mathbf{D}} = -D_{\mathbf{H}}D_{\mathbf{E}}\hat{\zeta} = -D_{\mathbf{E}}D_{\mathbf{H}}\hat{\zeta} = D_{\mathbf{E}}\hat{\mathbf{B}},$$

etc., and Theorem 1 has the following important corollary.

Theorem 2. *The Dissipation Principle implies that*

(i) *the "cross relations"*

$$D_{\mathbf{E}}\hat{\mathbf{B}} = D_{\mathbf{H}}\hat{\mathbf{D}}, \qquad D_{\mathbf{E}}\hat{\eta} = D_{\theta}\hat{\mathbf{D}}, \qquad D_{\mathbf{H}}\hat{\eta} = D_{\theta}\hat{\mathbf{B}}, \tag{3.10}$$

hold throughout $\mathfrak{C}$, *and*

(ii) *the values of* $D_{\mathbf{E}}\hat{\mathbf{D}}$ *and* $D_{\mathbf{H}}\hat{\mathbf{B}}$ *are symmetric tensors*; i.e. *at each* $\Gamma^t = (\mathbf{E}^t, \mathbf{H}^t, \theta^t)$ *in* $\mathfrak{C}$,

$$D_{\mathbf{E}}\hat{\mathbf{D}}(\Gamma^t) = [D_{\mathbf{E}}\hat{\mathbf{D}}(\Gamma^t)]^T, \tag{3.11}$$

$$D_{\mathbf{H}}\hat{\mathbf{B}}(\Gamma^t) = [D_{\mathbf{H}}\hat{\mathbf{B}}(\Gamma^t)]^T, \tag{3.12}$$

where ^{T} *denotes the transpose*[#].

[#] i.e., if $\mathbf{A} \in \mathrm{Lin}(\mathcal{V}_{(3)}, \mathcal{V}_{(3)})$, then $\mathbf{v}\cdot\mathbf{A}\mathbf{u} = \mathbf{u}\cdot\mathbf{A}^T\mathbf{v}$ for all $\mathbf{u}, \mathbf{v} \in \mathcal{V}_{(3)}$.

The hypotheses of smoothness we have employed here for constitutive functionals are direct analogues of those used by Coleman (1964) in his theory of the thermodynamics of deformable materials with memory. Theorems 1 and 2 remain valid, however, under the axiomatic, and more general, approach to the theory of fading memory developed by Coleman & Mizel (1967) (1968). Theorem 1 is provable also if one uses the appropriate analogues for $\hat{\zeta}$, $\hat{D}$, etc. of the general postulates of regularity recently explored by Coleman & Owen (1970). In an essay to appear soon in the Archive for Rational Mechanics & Analysis, we discuss the thermodynamics of electromagnetic phenomena from a point of view closer to that taken by Coleman & Owen. There we derive analogues of Coleman's "integrated dissipation inequalities" and explore at length the relation of equilibrium properties to dynamical properties.

Note: It is evident from Coleman & Gurtin's essay on the principle of equipresence (1967) that Theorem 2 and the conclusions (ii) and (iii) of Theorem 1 remain valid if one puts $\Gamma^t = (\underset{\sim}{E}^t, \underset{\sim}{H}^t, \theta^t, \underset{\sim}{g}^t)$, that is, if one includes in (1.12) a dependence of $\hat{\zeta}$, $\hat{\underset{\sim}{D}}$, $\hat{\underset{\sim}{B}}$, etc. on the past history $\underset{\sim}{g}_r^t$ of the temperature gradient $\underset{\sim}{g}$, adds to the right side of (2.1) a corresponding term $|g^t(0)|^2 + \int_0^\infty g^t(s)^2 h(s)ds$, and replaces $\partial_g\hat{\zeta}$ by $D_{\underset{\sim}{g}}\hat{\zeta}$. In such a case, however, Remark 3 no longer holds in its present form, and the conclusion (i) of Theorem 1 should be modified to read:#

#Cf. Coleman & Gurtin (1967, Thm. 6, p. 205).

(i) ζ, $\underset{\sim}{D}$, $\underset{\sim}{B}$, and η are given by functions of $\underset{\sim}{E}^t$, $\underset{\sim}{H}^t$, θ^t, and $\underset{\sim}{g}_r^t$ alone; i.e.

$$\underline{\underline{D}}_{\underset{\sim}{g}}\hat{\zeta} = \underset{\sim}{0}, \quad \underline{\underline{D}}_{\underset{\sim}{g}}\hat{\underset{\sim}{D}} = \underset{\sim}{0}, \quad \underline{\underline{D}}_{\underset{\sim}{g}}\hat{\underset{\sim}{B}} = \underset{\sim}{0}, \quad \text{and} \quad \underline{\underline{D}}_{\underset{\sim}{g}}\hat{\eta} = \underset{\sim}{0}.$$

4. Materials Without Memory as a Special Case

A physically important, albeit mathematically elementary, special case of the present theory is obtained by considering materials for which the present values of ϵ, $\underset{\sim}{D}$, $\underset{\sim}{B}$, η, $\underset{\sim}{j}$, and $\underset{\sim}{q}$ are given by functions of the present values of $\underset{\sim}{E}$, $\underset{\sim}{H}$, θ, and $\underset{\sim}{g}$ and are independent of the past histories $\underset{\sim}{E}^t_r$, $\underset{\sim}{H}^t_r$, and θ^t_r. Such "materials without memory" obey constitutive equations of the form#

#Of course, if ϵ, $\underset{\sim}{D}$, $\underset{\sim}{B}$, and η are functions of $\underset{\sim}{E}$, $\underset{\sim}{H}$, θ, and $\underset{\sim}{g}$, then so also is ζ.

$$\left.\begin{aligned}
\zeta &= \tilde{\zeta}(\underset{\sim}{E},\underset{\sim}{H},\theta,\underset{\sim}{g}), \\
\underset{\sim}{D} &= \tilde{\underset{\sim}{D}}(\underset{\sim}{E},\underset{\sim}{H},\theta,\underset{\sim}{g}), \\
\underset{\sim}{B} &= \tilde{\underset{\sim}{B}}(\underset{\sim}{E},\underset{\sim}{H},\theta,\underset{\sim}{g}), \\
\eta &= \tilde{\eta}(\underset{\sim}{E},\underset{\sim}{H},\theta,\underset{\sim}{g}), \\
\underset{\sim}{j} &= \tilde{\underset{\sim}{j}}(\underset{\sim}{E},\underset{\sim}{H},\theta,\underset{\sim}{g}), \\
\underset{\sim}{q} &= \tilde{\underset{\sim}{q}}(\underset{\sim}{E},\underset{\sim}{H},\theta,\underset{\sim}{g}).
\end{aligned}\right\} \tag{4.1}$$

When (4.1) is regarded as a special case of (1.12), the functions $\tilde{\zeta}$, $\tilde{\underset{\sim}{D}}$, $\tilde{\underset{\sim}{B}}$, $\tilde{\underset{\sim}{j}}$, and $\tilde{\underset{\sim}{q}}$ are continuous on a cone## $\mathcal{C}$ in $\mathcal{V}_{(1)} = \mathcal{V}_{(3)} \oplus \mathcal{V}_{(3)} \oplus \mathcal{V}_{(1)} \oplus \mathcal{V}_{(3)}$;

##i.e. either the cone corresponding to $\theta > 0$ or that corresponding to $\theta \neq 0$.

$\tilde{\zeta}(\cdot,\cdot,\cdot,\underset{\sim}{g})$, $\tilde{\underset{\sim}{D}}(\cdot,\cdot,\cdot,\underset{\sim}{g})$, and $\tilde{B}(\cdot,\cdot,\cdot,\underset{\sim}{g})$ are differentiable (i.e. have gradients in the ordinary sense) on the corresponding set in $\mathcal{V}_{(7)} = \mathcal{V}_{(3)} \oplus \mathcal{V}_{(3)} \oplus \mathcal{V}_{(1)}$; and $\tilde{\zeta}$ is differentiable on $\mathcal{C}$. We write $\partial_{\underset{\sim}{E}}\tilde{\zeta}$ for the $\mathcal{V}_{(3)}$-valued gradient of the function $\tilde{\zeta}(\cdot,\underset{\sim}{H},\theta,\underset{\sim}{g})$, $\partial_\theta\tilde{\zeta}$ for the scalar-valued gradient of $\tilde{\zeta}(\underset{\sim}{E},\underset{\sim}{H},\cdot,\underset{\sim}{g})$, $\partial_{\underset{\sim}{H}}\tilde{\underset{\sim}{D}}$ for the $\mathrm{Lin}(\mathcal{V}_{(3)} \times \mathcal{V}_{(3)})$-valued gradient of $\tilde{\underset{\sim}{D}}(\underset{\sim}{E},\cdot,\theta,\underset{\sim}{g})$. Theorem 1 has the following corollary.

Theorem 3. The Dissipation Principle places the following restrictions on the functions $\tilde{\zeta}$, $\tilde{\underset{\sim}{D}}$, $\tilde{\underset{\sim}{B}}$, $\tilde{\eta}$, $\tilde{\underset{\sim}{j}}$, and $\tilde{\underset{\sim}{q}}$ in (4.1):

(i) $\tilde{\zeta}$, $\tilde{\underset{\sim}{D}}$, and $\tilde{\underset{\sim}{B}}$ must be constant in $\underset{\sim}{g}$; i.e. $(4.1)_1$-$(4.1)_4$ reduce to

$$\left.\begin{aligned} \zeta &= \tilde{\zeta}(\underset{\sim}{E},\underset{\sim}{H},\theta), & \underset{\sim}{D} &= \tilde{\underset{\sim}{D}}(\underset{\sim}{E},\underset{\sim}{H},\theta), \\ \underset{\sim}{B} &= \tilde{\underset{\sim}{B}}(\underset{\sim}{E},\underset{\sim}{H},\theta), & \eta &= \tilde{\eta}(\underset{\sim}{E},\underset{\sim}{H},\theta); \end{aligned}\right\} \tag{4.2}$$

(ii) $\tilde{\zeta}$ must determine $\tilde{\underset{\sim}{D}}$, $\tilde{\underset{\sim}{B}}$, and $\tilde{\eta}$ through the relations,

$$\tilde{\underset{\sim}{D}} = -\partial_{\underset{\sim}{E}}\tilde{\zeta}, \qquad \tilde{\underset{\sim}{B}} = -\partial_{\underset{\sim}{H}}\tilde{\zeta}, \qquad \tilde{\eta} = -\partial_\theta\tilde{\zeta}; \tag{4.3}$$

(iii) $\tilde{\underset{\sim}{q}}$ and $\tilde{\underset{\sim}{j}}$ must obey the inequality

$$\underset{\sim}{g}\cdot\tilde{\underset{\sim}{q}}(\underset{\sim}{E},\underset{\sim}{H},\theta,\underset{\sim}{g}) \le \theta\underset{\sim}{E}\cdot\tilde{\underset{\sim}{j}}(\underset{\sim}{E},\underset{\sim}{H},\theta,\underset{\sim}{g}) \tag{4.4}$$

throughout $\mathcal{C}$.

Remark 4. It follows from (4.4) that, for $\underset{\sim}{g} = \underset{\sim}{0}$,

$$\theta \underset{\sim}{E} \cdot \underset{\sim}{\tilde{j}}(\underset{\sim}{E}, \underset{\sim}{H}, \theta, \underset{\sim}{0}) \geq 0, \tag{4.5}$$

and, for $\underset{\sim}{E} = \underset{\sim}{0}$,

$$\underset{\sim}{g} \cdot \underset{\sim}{\tilde{q}}(\underset{\sim}{0}, \underset{\sim}{H}, \theta, \underset{\sim}{g}) \leq 0. \tag{4.6}$$

An argument given by Coleman & Noll (1963, pp. 175-176) may here be used to show that if $\underset{\sim}{\tilde{j}}(\cdot, \underset{\sim}{H}, \theta, \underset{\sim}{0})$ and $\underset{\sim}{\tilde{q}}(\underset{\sim}{0}, \underset{\sim}{H}, \theta, \cdot)$ have gradients at $\underset{\sim}{0}$ in $\mathcal{V}_{(3)}$, then (4.5) yields

$$\underset{\sim}{\tilde{j}}(\underset{\sim}{0}, \underset{\sim}{H}, \theta, \underset{\sim}{0}) = \underset{\sim}{0}, \tag{4.7}$$

and (4.6) yields

$$\underset{\sim}{\tilde{q}}(\underset{\sim}{0}, \underset{\sim}{H}, \theta, \underset{\sim}{0}) = \underset{\sim}{0}; \tag{4.8}$$

i.e. in a material without memory, whenever both the electric intensity and the temperature gradient are zero, neither heat nor current flows.#

#In a forthcoming work, we show that for the materials with memory discussed in Sections 1-3 one can make only the following weaker assertion: neither heat nor current flows in a *steady state* with zero electric intensity and zero temperature gradient.

The analogue of Theorem 2 here is

Remark 5. It follows from (4.3) that the functions $\tilde{D}$, $\tilde{B}$, $\tilde{\eta}$ in (4.2) must be such that

(i) the cross relations

$$\partial_{E}\tilde{B} = \partial_{H}\tilde{D}, \qquad \partial_{E}\tilde{\eta} = \partial_{\theta}\tilde{D}, \qquad \partial_{H}\tilde{\eta} = \partial_{\theta}\tilde{B}, \tag{4.9}$$

hold, and

(ii) the values of $\partial_{E}\tilde{D}$ and $\partial_{H}\tilde{B}$ are symmetric tensors.

Frequently, in works on electromagnetic theory, one ignores thermodynamic variables and, if only "weak fields" are involved, employs linear constitutive equations of the form

$$D = \kappa E, \qquad B = \mu H, \qquad j = \sigma E, \tag{4.10}$$

where κ, μ, and σ are constant tensors (i.e. fixed elements of $\mathrm{Lin}(\mathcal{V}_{(3)}, \mathcal{V}_{(3)})$); κ is called the dielectric tensor, μ the (tensorial) magnetic permeability, and σ the conductivity tensor. The equations (4.10) are occasionally referred to as the "classical constitutive equations of electromagnetic theory". For many materials, including isotropic solids and several crystal classes, it follows from material symmetry that κ, μ, and σ must be symmetric tensors; however, for some materials, including the triclinic crystals, arguments based on material symmetry alone place no restrictions on these tensors. Nonetheless, there are thermodynamic restrictions, and, as the following remark asserts, $\underset{\sim}{\kappa}$ and $\underset{\sim}{\mu}$ must be symmetric tensors, even if such symmetry does not follow from a consideration of isotropy groups.

Remark 6. For materials which obey (4.10),

$$\partial_{\underset{\sim}{E}}\underset{\sim}{\tilde{D}} = \underset{\sim}{\kappa}, \qquad \partial_{\underset{\sim}{H}}\underset{\sim}{\tilde{B}} = \underset{\sim}{\mu}, \tag{4.11}$$

and item (ii) of Remark 5 reduces to the assertion that $\underset{\sim}{\kappa}$ and $\underset{\sim}{\mu}$ are symmetric tensors; i.e.

$$\underset{\sim}{\kappa} = \underset{\sim}{\kappa}^T, \qquad \underset{\sim}{\mu} = \underset{\sim}{\mu}^T.$$

Furthermore, when θ is positive (4.5) implies that $\underset{\sim}{\sigma}$ is positive semi-definite.

5. An Application to a Third-Order Theory of Isotropic, Non-linear, Dielectrics with Memory

G. Pettini (1970) has recently proven a uniqueness theorem for solutions of Maxwell's equations (1.1) and (1.2) in bounded regions of isotropic materials without thermodynamic influences but with fading memory. She employs constitutive assumptions of the form

$$\left.\begin{aligned} \underset{\sim}{D} = \hat{\underset{\sim}{D}}(\underset{\sim}{E}^t) &= \hat{\underset{\sim}{D}}_*(\underset{\sim}{E}^t) + \hat{\underset{\sim}{D}}_0(\underset{\sim}{E}^t) + \hat{\underset{\sim}{D}}_1(\underset{\sim}{E}^t) + \hat{D}_2(\underset{\sim}{E}^t) + \hat{\underset{\sim}{D}}_3(\underset{\sim}{E}^t), \\ \underset{\sim}{B}(t) &= \mu_\circ \underset{\sim}{H}(t), \qquad \mu_\circ > 0, \\ \hat{\underset{\sim}{j}} &\equiv \underset{\sim}{0}, \end{aligned}\right\} \quad (5.1)$$

with

$$\left.\begin{aligned} \hat{\underset{\sim}{D}}_*(\underset{\sim}{E}^t) &= g(0)\underset{\sim}{E}(t) + \int_0^\infty g'(s)\underset{\sim}{E}^t(s)ds, \qquad g(0) > 0, \\ \hat{\underset{\sim}{D}}_0(\underset{\sim}{E}^t) &= \alpha E(t)^2 \underset{\sim}{E}(t), \\ \hat{\underset{\sim}{D}}_1(\underset{\sim}{E}^t) &= \left[\int_0^\infty \phi_1(s)\underset{\sim}{E}^t(s)ds\right] E(t)^2 + \left[\underset{\sim}{E}(t)\cdot\int_0^\infty \phi_1^*(s)\underset{\sim}{E}^t(s)ds\right]\underset{\sim}{E}(t), \\ \hat{\underset{\sim}{D}}_2(\underset{\sim}{E}^t) &= \left[\int_0^\infty\int_0^\infty \phi_2(s_1,s_2)\underset{\sim}{E}^t(s_1)\cdot\underset{\sim}{E}^t(s_2)ds\right]\underset{\sim}{E}(t) + \\ &\quad + \int_0^\infty\int_0^\infty \left[\phi_2^*(s_1,s_2)\underset{\sim}{E}(t)\cdot\underset{\sim}{E}^t(s_1)\right]\underset{\sim}{E}^t(s_2)ds_1 ds_2, \\ \hat{D}_3(\underset{\sim}{E}^t) &= \int_0^\infty\int_0^\infty\int_0^\infty \left[\phi_3(s_1,s_2,s_3)\underset{\sim}{E}^t(s_1)\cdot\underset{\sim}{E}^t(s_2)\right]\underset{\sim}{E}^t(s_3)ds_1 ds_2 ds_3. \end{aligned}\right\} \quad (5.2)$$

Here α is a material constant, g, ϕ_1, ϕ_1^*, ϕ_2, ϕ_2^*, and ϕ_3 are real-valued, differentiable material functions, $\lim_{s\to\infty} g(s)$ exists, $g'(s) = dg(s)/ds$,[#]

[#]Pettini does not exhibit the term in g′ but does remark (1970, p. 2, footnote 2) that "per brevità di esposizione, si è tralasciato un termine eredetario lineare, pero la nostra demostrazione [of uniqueness] rimane valida anche in presenza di questo termine,..."

and, of course, $\underset{\sim}{E}(t) = \underset{\sim}{E}^t(0)$, $E(t)^2 = \underset{\sim}{E}(t)\cdot\underset{\sim}{E}(t)$. It is clear that if one adds to $\phi_2(s_1,s_2)$ a function β with $\beta(s_1,s_2) = -\beta(s_1,s_2)$, then the value of $\hat{\underset{\sim}{D}}_2(\underset{\sim}{E}^t)$ is not changed; hence, without loss of generality, one can take $\phi_2(s_1,s_2)$ to be symmetric in s_1 and s_2; i.e.

$$\phi_2(s_2,s_1) = \phi_2(s_1,s_2), \tag{5.3}$$

as was observed by Pettini. She found, however, that her method of proving uniqueness required that she assume, further, that

$$\phi_1^*(s) = 2\phi_1(s) \tag{5.4}$$

and

$$\phi_2^*(s_1,s_2) = \phi_2^*(s_2,s_1), \tag{5.5}$$

for all s, s_1, and s_2.

Now, direct calculation shows that (5.2) yields

$$\left.\begin{aligned}
\underset{\approx}{\underline{\underline{D}}}_{\underset{\sim}{E}}\hat{\underset{\sim}{D}}_{*}(\underset{\sim}{E}^{t}) &= g(0)\underset{\sim}{1},\\
\underset{\approx}{\underline{\underline{D}}}_{\underset{\sim}{E}}\hat{\underset{\sim}{D}}_{0}(\underset{\sim}{E}^{t}) &= \alpha E(t)^{2}\underset{\sim}{1} + 2\alpha\underset{\sim}{E}(t)\otimes\underset{\sim}{E}(t),\\
\underset{\approx}{\underline{\underline{D}}}_{\underset{\sim}{E}}\hat{\underset{\sim}{D}}_{1}(\underset{\sim}{E}^{t}) &= 2\left[\int_{0}^{\infty}\phi_{1}(s)\underset{\sim}{E}^{t}(s)ds\right]\otimes\underset{\sim}{E}(t) + \underset{\sim}{E}(t)\otimes\left[\int_{0}^{\infty}\phi_{1}^{*}(s)\underset{\sim}{E}^{t}(s)ds\right] +\\
&\quad + \left[\underset{\sim}{E}(t)\cdot\int_{0}^{\infty}\phi_{1}^{*}(s)\underset{\sim}{E}^{t}(s)ds\right]\underset{\sim}{1},\\
\underset{\approx}{\underline{\underline{D}}}_{\underset{\sim}{E}}\hat{\underset{\sim}{D}}_{2}(\underset{\sim}{E}^{t}) &= \left[\int_{0}^{\infty}\int_{0}^{\infty}\phi_{2}(s_{1},s_{2})\underset{\sim}{E}^{t}(s_{1})\cdot\underset{\sim}{E}^{t}(s_{2})\right]\underset{\sim}{1} +\\
&\quad + \int_{0}^{\infty}\int_{0}^{\infty}\left[\phi_{2}^{*}(s_{1},s_{2})\underset{\sim}{E}^{t}(s_{2})\otimes\underset{\sim}{E}^{t}(s_{1})\right]ds_{1}ds_{2},\\
\underset{\approx}{\underline{\underline{D}}}_{\underset{\sim}{E}}\hat{\underset{\sim}{D}}_{3}(\underset{\sim}{E}^{t}) &= 0,
\end{aligned}\right\} \quad (5.6)$$

where $\otimes$ denotes the tensor product of two vectors, and $\underset{\sim}{1}$ is the unit tensor. As it is evident from (5.6) that $\underset{\approx}{\underline{\underline{D}}}_{\underset{\sim}{E}}\hat{\underset{\sim}{D}}_{*}(\underset{\sim}{E}^{t})$, $\underset{\approx}{\underline{\underline{D}}}_{\underset{\sim}{E}}\hat{\underset{\sim}{D}}_{0}(\underset{\sim}{E}^{t})$, and $\underset{\approx}{\underline{\underline{D}}}_{\underset{\sim}{E}}\hat{\underset{\sim}{D}}_{3}(\underset{\sim}{E}^{t})$ are symmetric tensors for each history $\underset{\sim}{E}^{t}$, we can assert that $\underset{\approx}{\underline{\underline{D}}}_{\underset{\sim}{E}}\hat{\underset{\sim}{D}}$ has all its values in the space of symmetric tensors if, and only if, $\underset{\approx}{\underline{\underline{D}}}_{\underset{\sim}{E}}\hat{\underset{\sim}{D}}_{1}(\underset{\sim}{E}^{t})$ and $\underset{\approx}{\underline{\underline{D}}}_{\underset{\sim}{E}}\hat{\underset{\sim}{D}}_{2}(\underset{\sim}{E}^{t})$ are always symmetric tensors. But, $\underset{\approx}{\underline{\underline{D}}}_{\underset{\sim}{E}}\hat{\underset{\sim}{D}}_{1}(\underset{\sim}{E}^{t})$ is a symmetric tensor for each history $\underset{\sim}{E}^{t}$ if, and only if, (5.4) holds, while $\underset{\approx}{\underline{\underline{D}}}_{\underset{\sim}{E}}\underset{\sim}{D}_{2}(\underset{\sim}{E}^{t})$

has this property if, and only if, (5.5) holds. This proves#

#D. Graffi and M. Fabrizio told one of us, Coleman, about Pettini's uniqueness theorem in a conversation held in Bologna in October, 1969. As we had already proven Theorems 1 & 2, Coleman was able to point out, in that conversation, the relation of (5.4) and (5.5) to our results. We understand that Fabrizio has recently employed the present Theorem 2 to obtain a broad generalization of Pettini's theorem.

Remark 7. The relations (5.4) and (5.5) follow from equation (3.11) of Theorem 2.

The following remark may be proven using the relations $(3.4)_1$ and (3.5) of Theorem 1 and an argument employed by Coleman (1964, pp. 249-251), but in the interest of brevity we omit details.

Remark 8. It follows from the Dissipation Principle that, for each material of the type (5.1), (5.2), the function g is such that

$$\lim_{s \to \infty} g(s) \leq g(0).$$

We discuss generalizations of this result in a forthcoming paper.

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Thermodynamic Restrictions on the Constitutive Equations of Electromagnetic Theory

Bernard D. Coleman and Ellis H. Dill

Zusammenfassung

Die Autoren betrachten allgemeine Stoffe, für die die Energiedichte, die Entropiedichte, die elektrische Induktion, die magnetische Induktion und der Wärmefluss durch den Temperatur-Gradienten und die Geschichten der elektrischen Intensität, der magnetischen Intensität und der Temperatur bestimmt sind. Unter der Annahme, dass die Funktionale, die diese Abhängigkeit von Geschichten beschreiben, Glattheitsbedingungen vom "schwindendem Gedächtnis"– Typ erfüllen, leiten die Autoren die Einschränkungen ab, die der zweite Hauptsatz der Thermodynamik den Funktionalen auferlegt. Es wird darauf hingewiesen, dass die allgemeine Theorie Konsequensen für die elementare Theorie der isotropen und anisotropen Stoffe ohne Gedächtnis hat. Ausserdem werden die Anwendungen der allgemeinen Theorie auf das Problem diskutiert, wie man die thermodynamischen Einschränkungen für die Kerne findet, die in der Theorie dritter Ordnung für isotrope Dielektrika mit Gedächtnis vom Integral-Typ vorkommen.