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LINEAR DIFFERENTIAL EQUATIONS
WITH DELAYS:
ADMISSIBILITY AND CONDITIONAL
EXPONENTIAL STABILITY, II

by

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Linear Differential Equations with Delays: Admissibility and Conditional Exponential Stability, II

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1. Introduction.

In [2] and [3], C. V. Coffman and the present author reported on a first attempt at applying to linear differential equations with delays the methods of functional analysis developed for linear differential equations by Massera and Schäffer (see especially [4]) and for linear difference equations by themselves in [1]. The primary purpose of these investigations is to relate properties of the non-homogeneous equation such as "admissibility" ("for every second member in some given function space there is a solution in some given function space") and certain forms of conditional stability behaviour ("dichotomies") of the solutions of the homogeneous equation. The irreversibility of the process described by an equation with delays made it appear advisable to reduce the problem to the simplest kind of irreversible process, that described by a difference equation.

In a recent thesis (the substance of which is to appear in [5]), G. Pecelli has obtained results of this nature for certain equations with delays by constructing a theory parallelling that of differential and difference equations, without reduction to either.

In this paper we present a simpler and more powerful attack on the problem along the lines of [3]. Specifically, we consider on $[0, \infty)$ an equation of the form

$$\dot{u} + Mu = r \tag{1.1}$$

and the corresponding homogeneous equation

$$\dot{u} + Mu = 0 \tag{1.2}$$

in a Banach space E, ultimately to be assumed finite-dimensional; r is a continuous vector-valued function; the "solution" u is defined on $[-1, \infty)$, and M, the "memory functional", takes a continuous function u linearly into a continuous function Mu in such a way that the value of Mu at any given value t of the argument depends on the values of u on $[t-1,t]$ only.

The assumptions of our main result (Theorem 7.3) are that M transforms bounded functions "boundedly" into bounded functions, and that (1.1) has at least one bounded solution for each bounded r--in the tradition of

[1], [2], [3], [4], "$(\underset{\sim}{C},\underset{\sim}{C})$ is admissible for (1.1)". The conclusion describes the behaviour of "slices" of length 1 of solutions of (1.2) and its restrictions to $[m, \infty)$ for real $m \geq 0$: roughly speaking, the slices of bounded solutions tend uniformly exponentially to 0, and there exists a complementary finite-dimensional manifold of solutions of (1.2) whose slices tend uniformly exponentially to infinity and stay away uniformly from those of bounded solutions: this behaviour is a kind of "exponential dichotomy", in the sense of [1].

This paper can be read entirely independently of [3]. This entails some repetition--indeed, the sequence of ideas follows [3] rather faithfully--but appears justified by the following remarks.

In [2] and [3] it was assumed that the memory functional, apart from a term depending on the "present" value of u, had a gap behind the "present". This permitted an inchwise explicit construction of the solutions of (1.1) and (1.2) from the theory of ordinary differential equations. In abandoning the assumption of the presence of this gap, we have to do without the explicit construction; however, not only does our present approach completely subsume the results of [3] (to show this requires some messy routine computation, plus an obvious adjustment of scale), but it also simplifies the technical aspects considerably, in doing away with overlapping slices of different lengths and complicated computational verifications. Although we still rely heavily on [1], the final results no longer exalt slices with integral endpoints above the others.

Reliance on the theory of difference equations allows us to avoid all consideration of possibly unbounded operators and all explicit representations of M --say as a Stieltjès integral--and other, more technical complications of [5]; the use of a compactness argument first presented in [6] allows us, for finite-dimensional E, to achieve the description of the behaviour of the solutions of (1.2) with no extra assumptions.

As in [3], we have dealt here only with a concrete example of the "continuous case"; however, the same method is also applicable to the "Carathéodory case", where (1.1), (1.2) only hold locally in $\underset{\sim}{L}^1$, and where boundedness is replaced by membership in translation-invariant spaces of measurable functions. This and other matters, related to the present paper as [2] is to [3], will be dealt with elsewhere.

The author wishes to record his indebtedness to Professor C. V. Coffman for his valuable suggestions during our detailed discussion of all aspects of this work.

2. Spaces.

Throughout this paper, E will denote a given real or complex Banach space; in Section 7 we shall assume that its dimension is finite. The norm in E, as in all normed spaces for which no other symbol is prescribed, is denoted by $\| \ \|$. If X, Y are Banach spaces, $[X \to Y]$ denotes the Banach space of operators (bounded linear mappings) from X to Y, and we set $\tilde{X} = [X \to X]$.

We shall be dealing with sequences and with functions defined on intervals of the real line. We denote by ω the set $\{0,1,\ldots\}$ of all natural numbers, and set $\omega_{[m]} = \{n \in \omega : n \geqq m\}$, $m = 0,1,\ldots$. The notation for intervals of the real line is the usual one.

If m, m' are real numbers [natural numbers] with $m' \geqq m$, and f is a function defined on $[m, \infty)$ [on $\omega_{[m]}$], then $f_{[m']}$ shall denote the restriction of f to $[m', \infty)$ [to $\omega_{[m']}$].

Assume that X is a Banach space. For each natural number m we denote by $\underset{\sim}{s}_{[m]}(X)$ the linear space of all functions $f : \omega_{[m]} \to X$ and by $\underset{\sim}{\ell}^{\infty}_{[m]}(X)$ the Banach space of all bounded ones, with the norm $|\!|\!| f |\!|\!| = \sup\{\|f(n)\| : n \in \omega_{[m]}\}$. For each real m we denote by $\underset{\sim}{K}_{[m]}(X)$ the linear space of all continuous functions $f : [m, \infty) \to X$ and by $\underset{\sim}{C}_{[m]}(X)$ the Banach space of all bounded ones among them, with the norm $\| f \| = \sup\{\|f(t)\| : t \in [m, \infty)\}$. In all these notations the subscript is omitted when $m = 0$.

Finally, we denote by $\underset{\sim}{E}$ the Banach space of all continuous functions $f : [-1,0] \to E$, with the norm $[\![f]\!] = \max\{\|f(t)\| : t \in [-1,0]\}$.

The following example illustrates some obvious notational conventions. Suppose that $g \in \underset{\sim}{\ell}^{\infty}(\underset{\sim}{E})$; then $[\![g]\!]$ is the element of $\underset{\sim}{\ell}^{\infty}(R)$ given by $[\![g]\!](n) = [\![g(n)]\!]$, $n = 0,1,\ldots$; and $\| g \| = \| [\![g]\!] \|$ is the norm of g as an element of $\underset{\sim}{\ell}^{\infty}(\underset{\sim}{E})$.

3. Slicing operations.

Let $m \geq 0$ be a given real number. For each $t \geq m$ we define the linear mapping $\Pi(t) : \underset{\sim}{K}_{[m-1]}(E) \to \underset{\sim}{E}$ by

$$(3.1) \quad (\Pi(t)f)(s) = f(t+s) \qquad s \in [-1,0], \quad f \in \underset{\sim}{K}_{[m-1]}(E).$$

Thus $\Pi(t)$ maps f into the "slice" of f between $t-1$ and t, transplanted to $[-1,0]$ for convenience. (Note that indication of m is omitted; this will not cause any confusion.)

When m is an integer and $f \in \underset{\sim}{K}_{[m-1]}(E)$, we define $\varpi f \in \underset{\sim}{s}_{[m]}(\underset{\sim}{E})$ by

$$(3.2) \qquad (\varpi f)(n) = \Pi(n)f \qquad n \in \omega_{[m]}\ , \quad f \in \underset{\sim}{K}_{[m-1]}(E)\ .$$

Thus ϖ is a linear injective mapping of $\underset{\sim}{K}_{[m-1]}(E)$ into $\underset{\sim}{S}_{[m]}(\underset{\sim}{E})$. We record the obvious description of its range.

3.1. *Lemma*. *Assume the integer* $m \geq 0$ *and* $g \in \underset{\sim}{S}_{[m]}(\underset{\sim}{E})$ *given*. *Then* $g = \varpi f$ *for some* $f \in \underset{\sim}{K}_{[m-1]}(E)$ *if and only if* $(g(n))(0) = (g(n+1))(-1)$, $n=m,m+1,\ldots$; *if so, then* f *is bounded if and only if* g *is bounded, and then* $\|f\| = \|g\|$.

4. *The memory functional*.

We now make precise the assumptions on the "memory functional" M that appears in (1.1). It is linear and maps continuous functions into continuous functions, and the value of Mu at t is to depend only on the slice of u between $t-1$ and t . Specifically, we assume the following:

(M_1) : $M : \underset{\sim}{K}_{[-1]}(E) \rightarrow \underset{\sim}{K}(E)$ *is a linear mapping such that if* $t \in [0,\infty)$ *and* $u, u' \in \underset{\sim}{K}_{[-1]}(E)$ *satisfy* $\Pi(t)u = \Pi(t)u'$, *then* $(Mu)(t) = (Mu')(t)$.

Assumption (M_1) permits, for each real $m \geq 0$, the "cutting down" of M to a linear mapping $M_{[m]}$: $\underset{\sim}{K}_{[m-1]}(E) \rightarrow \underset{\sim}{K}_{[m]}(E)$: Each $u \in \underset{\sim}{K}_{[m-1]}(E)$ satisfies

$u = v_{[m-1]}$ for some $v \in \underset{\sim}{K}_{[-1]}(E)$, and we may set $M_{[m]}u = (Mv)_{[m]}$; since $t \geqq m$ implies $\Pi(t)v = \Pi(t)u$, assumption (M_1) shows that $M_{[m]}u$ thus defined does not depend on the choice of v. If $m' \geqq m \geqq 0$, these cut-down memory functionals then satisfy

$$(4.1) \qquad M_{[m']}u_{[m'-1]} = (M_{[m]}u)_{[m']} \qquad u \in \underset{\sim}{K}_{[m]}(E).$$

It is obvious that (M_1) implies the existence, for every $t \in [0, \infty)$, of a linear mapping $\hat{M}(t) : \underset{\sim}{E} \to E$ such that

$$(4.2) \qquad (M_{[m]}u)(t) = \hat{M}(t)\Pi(t)u \qquad t \geqq m \geqq 0, \quad u \in \underset{\sim}{K}_{[m-1]}(E).$$

We shall generally impose the following additional assumption:

(M_2): <u>The restriction of</u> M <u>to</u> $\underset{\sim}{C}_{[-1]}(E)$ <u>is a bounded linear mapping</u> $M_{\underset{\sim}{C}} : \underset{\sim}{C}_{[-1]}(E) \to \underset{\sim}{C}(E)$.

If M satisfies (M_1) and (M_2) it follows at once that $\hat{M}(t)$ is bounded, i.e., in $[\underset{\sim}{E} \to E]$, for each t, with

$$(4.3) \qquad \|M_{\underset{\sim}{C}}\| = \sup\{\|\hat{M}(t)\| : t \in [0, \infty)\}.$$

We note in passing that $t \mapsto \hat{M}(t) : [0, \infty) \to [\underset{\sim}{E} \to E]$ is then continuous in the strong operator topology; $t \mapsto \|\hat{M}(t)\|$

is lower semi-continuous, but need not be continuous. Conversely, given a strong-operator-continuous and uniformly bounded function $t \mapsto \hat{M}(t) : [0, \infty) \to [\underset{\sim}{E} \to E]$, formula (4.2) with $m = 0$ defines a linear mapping M that satisfies (M_1) and (M_2).

5. Solutions.

Henceforth we assume given the space E and the memory functional M satisfying conditions (M_1) and (M_2).

For every $r \in \underset{\sim}{K}(E)$, a solution of (1.1) is a function $u \in \underset{\sim}{K}_{[-1]}(E)$ whose restriction $u_{[0]}$ to $[0, \infty)$ is continuously differentiable (the derivative is $\dot{u}_{[0]} \in \underset{\sim}{K}(E)$) and that satisfies $\dot{u}_{[0]} + Mu = r$ on $[0, \infty)$. More generally, for every real $m \geq 0$, a solution of $(1.1)_{[m]}$ is a function $u \in \underset{\sim}{K}_{[m-1]}(E)$ whose restriction $u_{[m]}$ is continuously differentiable and that satisfies $\dot{u}_{[m]} + M_{[m]}u = r_{[m]}$ in $[m, \infty)$. In particular, if $m' \geq m \geq 0$ and u is a solution of $(1.1)_{[m]}$, then $u_{[m'-1]}$ is a solution of $(1.1)_{[m']}$ on account of (4.1). These definitions and statements of course also apply to the homogeneous equation (1.2).

A function $u \in \underset{\sim}{K}_{[m-1]}(E)$ is a solution of (1.1) if and only if

$$u(t) = u(m) - \int_m^t ((M_{[m]}u)(s) - r(s))ds, \qquad t \geqq m .$$

Existence and uniqueness theorems for the initial value problem follow as usual from Banach's Contractive Mapping Principle, and inequalities for the solutions from Gronwall's Inequality. The inequality $\|(M_{[m]}u)(t)\| \leqq \|\underset{\sim}{M_C}\| \, [\![\Pi(t)u]\!]$, an immediate consequence of (4.2) and (4.3), plays a basic role here. We omit the details. In view of the linearity of the equation, the results are summarized as follows.

5.1. Lemma. <u>For each real</u> $m \geq 0$ <u>there exist linear mappings</u> $P(m) : \underset{\sim}{E} \to \underset{\sim}{K}_{[m-1]}(E)$ <u>and</u> $Q(m) : \underset{\sim}{K}(E) \longrightarrow \underset{\sim}{K}_{[m-1]}(E)$ <u>such that, for every</u> $v \in \underset{\sim}{E}$ <u>and every</u> $r \in \underset{\sim}{K}(E)$, <u>the function</u> $u = P(m)v + Q(m)r$ <u>is the unique solution of</u> $(1.1)_{[m]}$ <u>with</u> $\Pi(m)u = v$; <u>and</u>

$$\|(P(m)v)(t)\| \leqq [\![v]\!] \exp(\|\underset{\sim}{M_C}\|(t-m)) \qquad t \geqq m, \ v \in \underset{\sim}{E}$$

(5.1)

$$\|(Q(m)r)(t)\| \leqq \left(\int_m^t \|r(s)\|ds\right)\exp(\|\underset{\sim}{M_C}\|(t-m)) \quad t \geqq m, \quad r \in \underset{\sim}{K}(E).$$

We note the following corollary of Lemma 5.1 and the preceding discussion on "cutting down" the domain of the equation.

5.2. Lemma. *If* u *is a solution of* $(1.2)_{[m]}$ *for some* $m \geq 0$, *then*

$$\| \Pi(t)u \| \leq \| \Pi(t_o)u \| \exp(\|\underset{\sim}{M}_C\|(t-t_o)), \qquad t \geq t_o \geq m .$$

6. The associated difference equation.

We construct a certain difference equation in $\underset{\sim}{E}$ in such a way that the values of a solution are the slices of a solution of (1.1). For this purpose, we define the linear mappings

$$(6.1) \qquad \begin{aligned} A(n) &= -\Pi(n)P(n-1): \underset{\sim}{E} \to \underset{\sim}{E} \\ B(n) &= \Pi(n)Q(n-1): \underset{\sim}{K}(E) \to \underset{\sim}{E} \end{aligned} \qquad n = 1,2,\ldots$$

and observe that (5.1) implies

$$(6.2) \qquad \begin{aligned} &A(n) \in \tilde{\underset{\sim}{E}}, \quad \|A(n)\| \leq \exp\|\underset{\sim}{M}_C\| && n = 1,2,\ldots \\ &\| B(n)r \| \leq \| (\varpi r)(n) \| \exp \|\underset{\sim}{M}_C\| && n = 1,2,\ldots, \; r \in \underset{\sim}{K}(E) \end{aligned}$$

We set $A = (A(n)) \in \underset{\sim}{\ell}^{\infty}_{[1]}(\underset{\sim}{\tilde{E}})$ and define the linear mapping $B : \underset{\sim}{K}(E) \rightarrow \underset{\sim}{s}_{[1]}(\underset{\sim}{E})$ by $(Br)(n) = B(n)r$, $n = 1,2,\ldots$.

With A thus defined, we consider the following difference equations in $\underset{\sim}{E}$:

$$(6.3) \qquad x(n) + A(n)x(n-1) = f(n) \qquad n = 1,2,\ldots$$

$$(6.4) \qquad x(n) + A(n)x(n-1) = 0 \qquad n = 1,2,\ldots$$

and their restrictions $(6.3)_{[m]}$, $(6.4)_{[m]}$ to $n = m+1, m+2, \ldots$ for $m \in \omega$. Here $f \in \underset{\sim}{s}_{[1]}(E)$.

6.1. Lemma. *Let $m \in \omega$ and $r \in \underset{\sim}{K}(E)$ be given. A function $x \in \underset{\sim}{s}_{[m]}(\underset{\sim}{E})$ is a solution of $(6.3)_{[m]}$ with $f = Br$ if and only if $x = \varpi u$ for some solution u of $(1.1)_{[m]}$. In particular, x is a solution of $(6.4)_{[m]}$ if and only if $x = \varpi u$ for some solution u of $(1.2)_{[m]}$.*

Proof. If u is a solution of $(1.1)_{[m]}$ and $n > m$, then $u_{[n-2]}$ is a solution of $(1.1)_{[n-1]}$; by Lemma 5.1 we have

$$(\varpi u)(n) = \Pi(n)u = \Pi(n)u_{[n-2]} = \Pi(n)(P(n-1)\Pi(n-1)u_{[n-2]}$$

$$+ Q(n-1)r) = -A(n)\Pi(n-1)u + B(n)r$$

$$= -A(n)(\varpi u)(n-1) + (Br)(n) ,$$

and so $x = \varpi u$ is a solution of $(6.3)_{[m]}$ with $f = Br$. Conversely, if x is a solution of $(6.3)_{[m]}$ with $f = Br$, let u be the solution of $(1.1)_{[m]}$ with $\Pi(m)u = x(m)$. By the first part of the proof, ϖu is a solution of $(6.3)_{[m]}$; but $(\varpi u)(m) = \Pi(m)u = x(m)$; therefore $\varpi u = x$.

It is clear that not every $f \in \underset{\sim}{S}_{[1]}(E)$ is of the form $f = Br$. It is still possible, however, to relate equation (6.3) with arbitrary f to equation (1.1).

6.2. *Theorem. For each* $f \in \underset{\sim}{S}_{[1]}(\underset{\sim}{E})$ *there exists* $r \in \underset{\sim}{K}(E)$ *such that*

$$(6.5) \qquad \|(\varpi r)(n)\| \leq k^2(\|f(n-2)\| + \|f(n-1)\|) \qquad n = 1,2,\ldots$$

and such that the solution w *of*

$$(6.6) \qquad w(n) + A(n)w(n-1) = f(n) - (Br)(n) \qquad n = 1,2,\ldots$$

with $w(0) = 0$ *satisfies*

$$(6.7) \qquad \|w(n)\| \leq k(\|f(n-1)\| + \|f(n)\|), \qquad n = 1,2,\ldots,$$

where $f(-1) = f(0) = 0$ *and* $k = \frac{3}{2} + \|\underset{\sim}{M}_C\|$.

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<u>Proof</u>. We define $g \in \underset{\sim}{s}_{[1]}(\underset{\sim}{E})$ by

$$(g(n))(s) = -6s(1+s)(f(n))(0) + s(3s+2)(\hat{M}(n)f(n)),$$

$$-1 \leqq s \leqq 0, \quad n = 1,2,\ldots .$$

Obviously,

$$(6.8)\quad (g(n))(-1) = \hat{M}(n)f(n) \qquad (g(n))(0) = 0 \qquad n = 1,2,\ldots ,$$

$$(6.9)\quad \int_{-1}^{0} (g(n))(s)ds = (f(n))(0) \qquad n = 1,2,\ldots ,$$

and, using (4.3),

$$(6.10)\quad [g(n)] \leqq \frac{3}{2}[f(n)] + \|\underset{\sim}{M}_C\|[f(n)] = k[f(n)]$$

$$n = 1,2,\ldots .$$

We extend f and g by setting $f(-1) = f(0) = g(-1) = g(0) = 0$; then (6.8), (6.9), (6.10) remain valid for $n=-1,0$, except that the first formula of (6.8) becomes meaningless for $n = -1$.

We now define $w \in \underset{\sim}{s}(E)$ by

$$(6.11)\quad (w(n))(s) = (f(n))(s) + \int_{s}^{0} (g(n-1))(\sigma)d\sigma$$

$$-1 \leqq s \leqq 0, \quad n = 0,1,\ldots .$$

It is obvious that each $w(n)$ is indeed continuous. We find $w(0) = 0$, as required, and, from (6.9),

$$(6.12)\qquad \begin{aligned} (w(n))(-1) &= (f(n))(-1) + (f(n-1))(0) \\ (w(n))(0) &= (f(n))(0) \end{aligned} \qquad n = 0, 1, \ldots ;$$

further, (6.11) and (6.10) yield

$$(6.13)\qquad \| w(n) - f(n) \| \leqq \| g(n-1) \| \leqq k \| f(n-1) \| \qquad n=0,1,\ldots ,$$

and (6.7) follows, since $k > 1$.

In order to construct r we define, for each $n \in \omega_{[1]}$, a function $z_n \in \underset{\sim}{K}_{[n-2]}(E)$ satisfying

$$(6.14)\qquad \Pi(n-1) z_n = -w(n-1), \qquad \Pi(n) z_n = f(n) - w(n) ,$$

but otherwise arbitrary; it is possible to find such a continuous function because $-(w(n-1))(0) = -(f(n-1))(0) = (f(n))(-1) - (w(n))(-1)$, by (6.12). We define $h \in \underset{\sim}{S}_{[1]}(\underset{\sim}{E})$ by

$$(6.15)\qquad h(n) = g(n-1) + \Pi(n) M_{[n-1]} z_n \qquad n=1,2,\ldots .$$

We note that, on account of assumption (M_1), h_n depends on z_n only through its values on $[n-2,n]$, hence is determined by (6.14). By (6.15), (6.8), (4.2), (6.14), we have, for $n = 1,2,\ldots$,

$$h(n)(0) = (g(n-1))(0) + (M_{[n-1]}z_n)(n)$$

$$= \hat{M}(n)\Pi(n)z_n = \hat{M}(n)(f(n) - w(n)) ,$$

$$h(n+1)(-1) = (g(n))(-1) + (M_{[n]}z_{n+1})(n)$$

$$= \hat{M}(n)f(n) + \hat{M}(n)\Pi(n)z_{n+1} = \hat{M}(n)(f(n) - w(n)) ,$$

so these elements are equal for each n. By Lemma 3.1 there exists $r \in \underset{\sim}{K}(E)$ such that

$$\varpi r = h . \tag{6.16}$$

For $n \in \omega_{[1]}$ and $n-1 \leqq t \leqq n$, (4.2), (4.3), (6.14), (6.7), (6.13) imply

$$\begin{aligned} \|(M_{[n-1]}z_n)(t)\| &= \|\hat{M}(t)\Pi(t)z_n\| \leqq \|\underset{\sim}{M_C}\| \| \Pi(t)z_n \| \\ &\leqq \|\underset{\sim}{M_C}\| \max\{ \|\Pi(n-1)z_n\|, \|\Pi(n)z_n\| \} \\ &= \|\underset{\sim}{M_C}\| \max\{ \|w(n-1)\|, \|f(n) - w(n)\| \} \\ &\leqq k\|\underset{\sim}{M_C}\| (\|f(n-2)\| + \|f(n-1)\|) . \end{aligned} \tag{6.17}$$

Then (6.16) and (6.15), (6.17), (6.10) yield

$$\|(\varpi r)(n)\| \leqq \|g(n-1)\| + k\|M_{\underset{\sim}{C}}\|(\|f(n-2)\| + \|f(n-1)\|)$$

$$\leqq k^2(\|f(n-2)\| + \|f(n-1)\|) ,$$

i.e., (6.5), since $k > 1 + \|M_{\underset{\sim}{C}}\|$.

It remains to be proved that w and r thus constructed satisfy (6.6). For this purpose, let $n \in \omega_{[1]}$ be fixed and consider the solution u of $(1.1)_{[n-1]}$ with

$$(6.18) \qquad \Pi(n-1)u = -w(n-1) = \Pi(n-1)z_n .$$

Let t, $n-1 \leqq t < n$, be given. From (6.14), (6.11), (3.1),

$$\dot{z}_n(t) = \frac{d}{dt}(f(n) - w(n))(t-n) = (g(n-1))(t-n) ,$$

and from (6.16), (6.15), (3.1),

$$r(t) = (h(n))(t-n) = (g(n-1))(t-n) + (M_{[n-1]}z_n)(t) ;$$

thus $\dot{z}_n(t) + (M_{[n-1]}z_n)(t) = r(t)$, $n-1 \leqq t < n$, so that z_n satisfies $(1.1)_{[n-1]}$ on $[n-1,n)$; by (6.18) and uniqueness, z_n and u coincide on $[n-2,n]$. From

(6.14), Lemma 5.1, (6.18), (6.1) we have

$$f(n) - w(n) = \Pi(n)z_n = \Pi(n)u = \Pi(n)(P(n-1)\Pi(n-1)u + Q(n-1)r)$$

$$= A(n)w(n-1) + B(n)r ,$$

and (6.6) is satisfied for this arbitrary $n \in \omega_{[1]}$.

7. Admissibility and the solutions of the homogeneous equation.

The discussion in the preceding section enables us to reduce the consideration of equations (1.1) and (1.2) to analysis of the difference equations (6.3) and (6.4) by means of the theory in [1]. We shall indeed have to rely heavily on that paper for the crucial steps in the proof of Theorem 7.3. M is still assumed to satisfy (M_1) and (M_2), and A, B are defined by (6.1).

We begin with the non-homogeneous equations. We say that $(\underset{\sim}{C},\underset{\sim}{C})$ is admissible with respect to M--more loosely, with respect to (1.1)--if for every $r \in \underset{\sim}{C}(E)$ there is a bounded solution u of (1.1). We recall ([1; p. 154]) that, similarly, $(\underset{\sim}{\ell}^{\infty},\underset{\sim}{\ell}^{\infty})$ is admissible with respect to A--or with respect to (6.3)--if for every $f \in \underset{\sim}{\ell}^{\infty}_{[1]}(\underset{\sim}{E})$ there is a bounded solution x of (6.3).

7.1. <u>Theorem</u>. $(\mathbf{C},\mathbf{C})$ <u>is admissible with respect to</u> M <u>if and only if</u> $(\ell^{\infty},\ell^{\infty})$ <u>is admissible with respect to</u> A.

<u>Proof</u>. 1. Assume that $(\mathbf{C},\mathbf{C})$ is admissible with respect to M. Let $f \in \ell^{\infty}_{[1]}(\mathbf{E})$ be given, and choose r and w as provided by Theorem 6.2. Now (6.5) and (6.7) imply that r and w are bounded (with $\|r\| \leqq 2k^2\|f\|$, $\|w\| \leqq 2k\|f\|$). There exists, by assumption, a bounded solution u of (1.1). Then ϖu is bounded, and satisfies $(\varpi u)(n) + A(n)(\varpi u)(n-1) = (Br)(n)$, $n = 1,2,\ldots$; since w is a bounded solution of (6.6), we conclude that $\varpi u + w$ is a bounded solution of (6.3). Thus $(\ell^{\infty},\ell^{\infty})$ is admissible with respect to A.

2. Assume, conversely, that $(\ell^{\infty},\ell^{\infty})$ is admissible with respect to A, and let $r \in \mathbf{C}(E)$ be given. By (6.2), $Br \in \ell^{\infty}_{[1]}(\mathbf{E})$ (with $\|Br\| \leqq \|r\| \exp\|M_{\mathbf{C}}\|$); by the assumption, there exists a bounded solution x of (6.3) with $f = Br$. By Lemma 6.1, $x = \varpi u$ for some solution u of (1.1); and this u is bounded. Thus $(\mathbf{C},\mathbf{C})$ is admissible with respect to M.

The admissibility of $(\ell^{\infty},\ell^{\infty})$ with respect to A implies, under certain additional conditions, an <u>exponential dichotomy</u> of the solutions of the homogeneous equations (6.4) [m] (see [1; Section 7]): roughly speaking, the bounded solutions tend uniformly exponentially to 0, there exists a "complementary" manifold of solutions of (6.4) tending uniformly exponentially to infinity, the two kinds of solutions remain uniformly apart, and together they span all solutions. Since Lemma 6.1 provides a bijective correspondence between solutions of $(1.2)_{[m]}$ and $(6.4)_{[m]}$ (for integral m), Theorem 7.1 will allow us to translate that result into an analogous implication for differential equations with delays. We shall restrict ourselves here to finite-dimensional E; this will allow us to make use of the following compactness result.

7.2. Lemma. If E is finite-dimensional, then $A(n)$ is a compact operator for $n = 1,2,\ldots$.

Proof. Let $n \in \omega_{[1]}$ and $v \in \underset{\sim}{E}$ be given. Then $u = P(n-1)v$ is the unique solution of $(1.2)_{[n-1]}$ with $\Pi(n-1)u = v$ (Lemma 5.1). By (6.1), $A(n)v = -\Pi(n)u$. Therefore

$$(7.1) \quad (A(n)v)(s) = -u(n+s) = -u(n-1) + \int_{n-1}^{n+s} (M_{[n-1]}u)(\sigma)d\sigma \;,$$

$$-1 \leqq s \leqq 0 \;.$$

By (4.2), (4.3), and Lemma 5.2,

$$(7.2)\quad \|(M_{[n-1]}u)(\sigma)\| \leqq \|M_{\underset{\sim}{C}}\| [\![\Pi(\sigma)u]\!] \leqq \|M_{\underset{\sim}{C}}\| [\![\Pi(n-1)u]\!] \exp\|M_{\underset{\sim}{C}}\|$$

$$= \|M_{\underset{\sim}{C}}\| [\![v]\!] \exp\|M_{\underset{\sim}{C}}\| , \qquad n-1 \leqq \sigma \leqq n \ .$$

Combining (7.1) and (7.2), we find

$$(7.3)\quad \|(A(n)v)(s') - (A(n)v)(s)\| \leqq \int_{n+s}^{n+s'} \|(M_{[n-1]}u)(\sigma)\| d\sigma$$

$$\leqq (s'-s)\|M_{\underset{\sim}{C}}\| [\![v]\!] \exp\|M_{\underset{\sim}{C}}\| ,$$

$$-1 \leqq s \leqq s' \leqq 0 \ .$$

Formulae (6.2) and (7.3) show that $A(n)$ maps the unit ball of $\underset{\sim}{E}$ into a bounded equicontinuous set of continuous functions $[0,1] \rightarrow E$; when E is finite-dimensional, it follows from the Arzelà-Ascoli Theorem that the closure in $\underset{\sim}{E}$ of the image under $A(n)$ of that unit ball is compact. Hence $A(n)$ is a compact operator in that case.

We now state our main result, to the effect that admissibility of $(\underset{\sim}{C},\underset{\sim}{C})$ with respect to M implies a kind of "exponential dichotomy" of the solutions of $(1.2)_{[m]}$.

7.3. Theorem. Assume that E is finite-dimensional, and that $(\underset{\sim}{C},\underset{\sim}{C})$ is admissible with respect to M. Then there exist numbers $\nu, N > 0$ such that, for every real $m \geqq 0$, every bounded solution v of $(1.2)_{[m]}$ satisfies

$$\text{(i)} \quad \| \Pi(t)v \| \leqq Ne^{-\nu(t-t_o)} \| \Pi(t_o)v \| \quad \text{for all } t \geqq t_o \geqq m;$$

There further exist a finite-dimensional linear manifold $\underset{\sim}{W}$ of solutions of (1.2), and numbers $\nu', N' > 0$, $\lambda_o > 1$ such that, for every real $m \geqq 0$, every solution u of $(1.2)_{[m]}$ is of the form $u = v + w_{[m-1]}$, where v is a bounded solution and $w \in \underset{\sim}{W}$, and such that every solution $w \in \underset{\sim}{W}$ satisfies

$$\text{(ii):} \quad \| \Pi(t)w \| \geqq N'^{-1} e^{\nu'(t-t_o)} \| \Pi(t_o)w \| \quad \text{for all } t \geqq t_o \geqq 0,$$

$$\text{(iii):} \quad \| \Pi(t)w \| \leqq \lambda_o \| \Pi(t)w - \Pi(t)v \| \quad \text{for all } t \geqq m \geqq 0$$

and all bounded solutions v of $(1.2)_{[m]}$.

Proof. 1. By Theorem 7.1, $(\underset{\sim}{\ell}^{\infty},\underset{\sim}{\ell}^{\infty})$ is admissible with respect to A. We now refer to [1] and [6] in order to deal with equations (6.3), (6.4). Specifically, condition (c) of [6; Lemma 4.2] is satisfied with $\underset{\sim}{b} = \underset{\sim}{d} = \underset{\sim}{\ell}^{\infty}$; since every A(n) is compact by Lemma 7.2 (one would be enough), we conclude from [6; Theorem 4.3,(b)] that the covariant sequence $\underset{\sim}{E}_o$ is regular and that its terms (which are the sets of initial values of the bounded solutions of $(6.4)_{[n]}$; $n = 0,1,\ldots$) have constant finite co-dimension in $\underset{\sim}{E}$. We can therefore apply the fundamental "direct" result [1; Theorem 10.2] and find that this covariant sequence induces an exponential dichotomy for A.

2. To make this result manageable, we use the description of an exponential dichotomy given by [1; Theorem 7.1,(c)]. We observe from the proof of that theorem that we are free to choose the splitting q; and since $\underset{\sim}{E}_o(0)$ has finite co-dimension in $\underset{\sim}{E}$, we choose q to be a (linear) projection along $\underset{\sim}{E}_o$ onto a finite-dimensional complementary subspace, say $\underset{\sim}{Z}$. This choice of $\underset{\sim}{Z}$ and the regularity of $\underset{\sim}{E}_o$ imply that the values at any given n of the solutions of (6.4) starting on $\underset{\sim}{Z}$ constitute a complementary subspace to $\underset{\sim}{E}_o(n)$; in other words, if x is any solution of $(6.4)_{[n]}$, there exists a solution z of (6.4) with $z(0) \in \underset{\sim}{Z}$ such that $y = x - z_{[n]}$ --another solution of $(6.4)_{[n]}$--is bounded.

The combination of [1; Theorem 10.2] and [1; Theorem 7.1] as applied to our case then yields: there exist numbers $\nu, \nu', N_1, N_1' > 0$, $\lambda_1 > 1$, such that for any integers n_o, n with $n \geqq n_o \geqq 0$, any bounded solution y of $(6.4)_{[n_o]}$ and any solution z of (6.4) with $z(0) \in \underset{\sim}{Z}$

we have

$$(7.4) \qquad \| y(n) \| \leqq N_1 e^{-\nu(n-n_o)} \| y(n_o) \|$$

$$(7.5) \qquad \| z(n) \| \geqq N_1'^{-1} e^{\nu'(n-n_o)} \| z(n_o) \|$$

$$(7.6) \qquad \| z(n_o) \| \leqq \lambda_1 \| z(n_o) - y(n_o) \| .$$

3. It now remains to translate this information on (6.4) by means of Lemma 6.1 into the conclusion of the theorem. We define $\underset{\sim}{W}$ to be the finite-dimensional linear manifold of solutions w of (1.2) with $\Pi(0)w \in \underset{\sim}{Z}$. In the rest of this proof, whenever $w \in \underset{\sim}{W}$, we set $z = \varpi w$ and observe that, by Lemma 6.1, z is a solution of (6.4) with $z(0) = \Pi(0)w \in \underset{\sim}{Z}$, and that all solutions of (6.4) starting from $\underset{\sim}{Z}$ are of this form.

Let $w \in \underset{\sim}{W}$ and t_o, t be given, with $t \geqq t_o \geqq 0$. Let n_o be the greatest integer $\leqq t_o$ and n the least integer $\geqq t$. Combining (7.5) and Lemma 5.2 we find

$$\begin{aligned} \| \Pi(t)w \| \exp\|\underset{\sim}{M_C}\| &\geqq \| \Pi(n)w \| = \| z(n) \| \geqq \| z(n_o) \| N_1'^{-1} e^{\nu'(n-n_o)} \\ &= \| \Pi(n_o)w \| N_1'^{-1} e^{\nu'(n-n_o)} \\ &\geqq \| \Pi(t_o)w \| N_1'^{-1} e^{\nu'(t-t_o)} \exp(-\|\underset{\sim}{M_C}\|) , \end{aligned}$$

i.e., (ii) with $N' = N_1' \exp(2\|\underset{\sim}{M}_C\|)$.

Let now $m \geq 0$ be given and fixed in the sequel. If u is a solution of $(1.2)_{[m]}$, we choose an arbitrary integer $n \geq m$ and find that $u_{[n-1]}$ is a solution of $(1.2)_{[n]}$. By Lemma 6.1, $\varpi u_{[n-1]}$ is a solution of $(6.4)_{[n]}$; by Part 2 of this proof, there exists a solution z of (6.4) with $z(0) \in \underset{\sim}{Z}$, i.e., $z = \varpi w$ with $w \in \underset{\sim}{W}$, such that $\varpi u_{[n-1]} - z_{[n]} = \varpi(u_{[n-1]} - z_{[n-1]}) = \varpi(u - z_{[m-1]})_{[n-1]}$ is bounded. Then $(u - w_{[m-1]})_{[n-1]}$ is bounded, and $v = u - w_{[m-1]}$ is a bounded solution of $(1.2)_{[m]}$.

Let v be a given bounded solution of $(1.2)_{[m]}$. Let t_o, t be given, with $t \geq t_o \geq m$. If there is no integer in the interval $[t_o, t]$, Lemma 5.2 implies

$$(7.7) \quad \|\Pi(t)v\| \leq \|\Pi(t_o)v\| \exp\|\underset{\sim}{M}_C\| \leq \|\Pi(t_o)v\| e^{-\nu(t-t_o-1)} \exp\|\underset{\sim}{M}_C\|;$$

otherwise, let n_o be the least integer $\geq t_o$ and n the greatest integer $\leq t$, so that $n \geq n_o \geq m$. By Lemma 6.1, $y = \varpi v_{[n_o-1]}$ is a bounded solution of $(6.4)_{[n_o]}$. Combining (7.4) with Lemma 5.2, we find

$$(7.8)\quad |\Pi(t)v|\exp(-\|M_{\underset{\sim}{C}}\|) \leqq |\Pi(n)v| = |y(n)|$$

$$\leqq |y(n_o)|N_1 e^{-\nu(n-n_o)}$$

$$= |\Pi(n_o)v|N_1 e^{-\nu(n-n_o)}$$

$$\leqq |\Pi(t_o)v|N_1 e^{-\nu(t-t_o-2)}\exp\|M_{\underset{\sim}{C}}\| .$$

Since $N_1 \geqq 1$ (by (7.4) for $n = n_o$), we conclude from (7.7), (7.8) that (i) holds with $N = N_1 e^{2\nu}\exp(2\|M_{\underset{\sim}{C}}\|)$.

Finally, with v as before and with $w \in \underset{\sim}{W}$, $t \geqq m$ given, let n_o be the least integer $\geqq t$; again $y = \varpi v_{[n_o-1]}$ is a bounded solution of $(6.4)_{[n_o]}$. Applying (ii) to w, (7.6) to y and $z = \varpi w$, and Lemma 5.2 to $w_{[m-1]} - v$, we find

$$|\Pi(t)w| \leqq N'|\Pi(n_o)w| = N'|z(n_o)| \leqq \lambda_1 N'|z(n_o) - y(n_o)|$$

$$= \lambda_1 N'|\Pi(n_o)w - \Pi(n_o)v| = \lambda_1 N'|\Pi(n_o)(w_{[m-1]}-v)|$$

$$\leqq \lambda_1 N'|\Pi(t)(w_{[m-1]} - v)|\exp\|M_{\underset{\sim}{C}}\|$$

$$= \lambda_1 N'|\Pi(t)w - \Pi(t)v|\exp\|M_{\underset{\sim}{C}}\| ,$$

i.e., (iii) with $\lambda_o = \lambda_1 N' \exp\|M_{\underset{\sim}{C}}\|$.

Remark. Conversely, the conclusion of Theorem 7.3 implies, via Lemma 6.1, that E_0 is a regular covariant sequence and induces an exponential dichotomy for A. From the "converse" theorem for difference equations, i.e., [1; Theorem 10.3], it follows that $(\ell^\infty, \ell^\infty)$ is admissible with respect to A; and hence, by Theorem 7.1, that (C, C) is admissible with respect to M. Thus the converse of Theorem 7.3 is valid. Since the proof, as outlined, is straightforward and requires no fresh insight, we omit it here.

References

1. C. V. Coffman and J. J. Schäffer, Dichotomies for linear difference equations. Math. Ann. 172, 139-166, (1967).

2. C. V. Coffman and J. J. Schäffer, Linear differential equations with delays: admissibility and conditional stability. Department of Mathematics, Carnegie-Mellon University, Report 70-2, Pittsburgh, Pennsylvania, (1970).

3. C. V. Coffman and J. J. Schäffer, Linear differential equations with delays: admissibility and conditional exponential stability. J. Differential Equations. (To appear).

4. J. L. Massera and J. J. Schäffer, Linear differential equations and function spaces. New York, Academic Press, (1966).

5. G. Pecelli, Dichotomies for linear functional-differential equations, (To appear).

6. J. J. Schäffer, Linear difference equations: closedness of covariant sequences. Math. Ann. 187, 69-76, (1970).

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