October, 1970

GEOMETRIC PROGRAMS TREATED WITH SLACK VARIABLES*

by

R. J. Duffin+

and

Elmor L. Peterson++

Report 70-45

Abstract

Kochenberger and Woolsley have introduced slack variables into the constraints of a geometric program and have added their reciprocals to the objective function. They find this augmented program advantageous for numerical minimization. In this paper the augmented program is used to give a relatively simple proof of the "refined duality theory" of geometric programming. This proof also shows that the optimal solutions for the augmented program converge to the (desired) optimal solutions for the original program.

*This work was supported by Grant DA-AROD-31-124-71-817, Army Research Office, Durham.

+Carnegie-Mellon University

++Northwestern University

1. Introduction.

This paper gives a new and somewhat simpler proof of the "refined duality theory" of geometric programming. The first proof was given by Duffin and Peterson [1,2], and another proof was given by Duffin [3]. Moreover, Rockafellar [5] has related geometric programming to his "generalized convex programming". Those three proofs are all different, and each gives different insight into the structure of geometric programs.

The present proof does not employ Subsidiary programs [1,2], linear programming [3], or convexity [5]. Only basic principles of the calculus are needed. Geometric programming includes linear programming as a special case, so the present paper also furnishes a new proof of the duality theory of linear programming.

Given a geometric program with posynomial functions, the treatment here begins by adding a slack variable to each constraint function. Also, the reciprocal of each slack variable is added to the objective function. Clearly, the program so augmented is also defined in terms of posynomial functions. Moreover, it is obvious that the constraints of this augmented program are tight at the minimum. This property makes the augmented program easier to analyze because equalities replace inequalities. Finally, the properties of the original program are deduced by carrying out certain limit operations.

The concept of the augmented program is due to Kochenberger [6], and it has been employed by Woolsey [7]. They are mainly concerned with numerical calculations in geometric programming, which are known to encounter certain difficulties when slack constraints are present. Because the augmented program has no slack constraints those difficulties tend to be mollified.

Since the constraint inequalities of the augmented program may be assumed to be equalities it is possible to eliminate the slack variables. This results in an unconstrained program whose objective function is precisely the function introduced in the penalty methods of Carroll, Fiacco, and McCormick [4].

2. Basic concepts.

A posynomial $g(t)$ is a function of positive variables $t_1, t_2, \ldots, t_m$ expressed as a finite sum,

$$g(t) \triangleq \Sigma u_i(t),$$

where the terms $u_i(t)$ have the form

$$u_i(t) \triangleq c_i t_1^{a_{i1}} t_2^{a_{i2}} \ldots t_m^{a_{im}} .$$

The exponents a_{ij} are arbitrary real constants, but the coefficients c_i are positive constants. The primal geometric program to be considered is defined as follows.

Program A. Seek the minimum value of a posynomial $g_0(t)$ subject to the constraints

$$t_1 > 0, t_2 > 0, \ldots, t_m > 0,$$

and subject to the posynomial constraints

$$g_1(t) \leq 1, g_2(t) \leq 1, \ldots, g_p(t) \leq 1.$$

It is convenient to list all the terms as

$$u_1, u_2, \ldots, u_n$$

and then let

$$g_0 \triangleq u_1 + \dots + u_{n_0}, \quad m_0 \triangleq 1$$

$$g_1 \triangleq u_{m_1} + \dots + u_{n_1}, \quad m_1 \triangleq n_0 + 1$$

$$\vdots$$

$$g_p \triangleq u_m \,.$$

If there is a point t which satisfies the constraints then program A is said to be <u>consistent</u>. The infimum of $g_0(t)$ subject to the constraints of A is written as $M_A \triangleq \inf_A g_0(t)$. If $M_A > 0$ then program A is said to have a <u>finite infimum</u>.

Associated with the preceding minimization program is a maximization program termed the <u>geometric dual program</u>. This dual program B is defined as follows.

<u>Program B.</u> <u>Seek the maximum value of the product function</u>

$$v(\delta) \triangleq \prod_{i=1}^{n} \left(\frac{c_i}{\delta_i}\right)^{\delta_i} \prod_{k=0}^{p} \lambda_k^{\lambda_k},$$

<u>where</u>

$$\lambda_0 \triangleq \delta_1 + \dots + \delta_{n_0}, \quad m_0 \triangleq 1$$

$$\lambda_1 \triangleq \delta_{m_1} + \dots + \delta_{n_1}, \quad m_1 \triangleq n_0 + 1$$

$$\vdots$$

$$\lambda_p \triangleq \delta_{m_p} + \dots + \delta_{n_p}, \quad m_p \triangleq n_{p-1} + 1, \quad n_p \triangleq n.$$

<u>The variables</u> δ_i <u>are subject to the linear constraints</u>:

$$\delta_i \geq 0, \quad i = 1, \dots, n \quad \text{(positivity)}$$

$$\sum_{i=1}^{n_0} \delta_i = 1, \quad \text{(normality)}$$

$$\sum_{i=1}^{n} \delta_i a_{ij} = 0, \quad j = 1, 2, \dots, m. \quad \text{(orthogonality)}$$

<u>Here, the constants</u> c_i <u>are the posynomial coefficients in program</u> A, <u>and the constants</u> a_{ij} <u>are the posynomial exponents in program</u> A.

In evaluating the product function v(6) it is understood that $x^x = x\sim^x = 1$ for x = 0. This makes v(6) a continuous function in the otthant $6_i . \wedge 0$. Program B is said to be <u>consistent</u> if there is a point 6 which satisfies its constraints. The supremum of v(6) subject to the constraints of Program B is written as M_B A sup v(6). If M_* < oo then program B is said to have a <u>finite supremum.</u>

The main goal of duality theory is to show that M_A = 1VL.
Toward that end, the following lemma is needed.

Lemma 1. Let u. and 6. be real numbers such that u. > 0 and 6. J> 0 for i = 1,•••,N; then

$$(\sum_{i=1}^{N} u_i)^{A} \wedge \prod_{i=1}^{N} (\frac{u_i}{6_i})^{x} *\backslash$$

<u>where</u>

$$A \; A \sum_{i=1}^{N} 6_i \; .$$

<u>Moreover, this inecruality becomes an equality if, and only if</u>,

$$6_J \sum_i^N u_i = Au_j, \quad j = 1,2, \ldots ,N.$$

<u>Proofo</u> If all the 6_i are positive, let ^ 4 6^/A and U_i A $u_i/6_i$.
Then the e_i are "weights"[11], and the classical inequality stating that the weighted arithmetic mean of positive numbers U_1^U_2, ..₀,U_N is not less than the corresponding weighted geometric mean can be written as

$$\sum_1^N \; x \; x \qquad \prod_1^N \; e_x$$

This is equivalent to the inequality of the lemma. Moreover, the classical inequality is an equality if, and only if,

$$U_1 = U_2 = \cdots = U_N \; '$$

It is easy to see that this condition is equivalent to the condition stated in the lemma. The case where not all 6_i are positive is easily reduced to the case just treated, so the proof of Lemma 1 is complete.

Our first theorem shows that $M_A J \geq M_B$, and it also gives conditions that will ultimately help to prove that $ML_A = M_B$.

<u>Theorem 1_o If</u> t <u>satisfies the constraints of program</u> A <u>and if</u> 6 <u>satisfies the constraints of program B, then</u>

$$g_o(t) \; 2 \; v(6)$$

<u>Moreover, this inequality is an equality if, and only if,</u>

$$g_Q(t)6_i = u_i(t), \quad i = 1, \ldots, n_0,$$

<u>and</u>

1 K 1 K

<u>Proofo</u> By virtue of Lemma 1 we know that

$$(g)^{\lambda k} \wedge n_{m_k}^{n_k} (f)^{u_i 6.} {}^{x} X_k^{k} , \quad k = 0,1,\ldots,p.$$

Multiplying these p + 1 inequalities together gives the inequality

$$I \%^{P K} > 1<T?^{n u 6} i \ll^{p}_{O} V^{A} v(6>t l^{D_1} *2^{D_2} - * ».^{D_m}$$

where $D_j \wedge 2^n 6\wedge a\wedge_j$. Since 6 satisfies the orthogonality conditions $D_j = 0\wedge$ it follows that

$$g_0 \wedge \overset{P}{n_O} \wedge_k \overset{V}{\;} 2 \; v(6),$$

because $A_Q = 1$ and $g_k \wedge 1$ for $k = 1, \ldots, p$. This proves the inequality of the theorem.

Clearly, $g_Q = v$ if, and only if, each of the p + 1 applications of Lemma 1 gives an equality. But the p + 1 conditions of the theorem are simply transcriptions of the equality condition of Lemma 1, so the proof of Theorem 1 is complete.

3. The augmented program. Given program A, its augmented program A^+ is defined as follows.

Program A^+. Seek the minimum value of the posynomial

$$G_0(t,T) \triangleq g_0(t) + \sum_{k=1}^{p} bT_k^{-1},$$

subject to the constraints

$$t_j > 0, \quad j = 1,\ldots,m,$$

$$T_k > 0, \quad k = 1,\ldots,p,$$

and subject to the posynomial constraints

$$G_k(t) \triangleq \theta g_k(t) + bT_k \le 1, \quad k = 1,\ldots,p.$$

The constants b and θ are positive, and $\theta < 1$.

Clearly, program A^+ is in the standard form of a geometric program. Moreover, program A^+ reduces to program A if $\theta = 1$ and $b = 0$.

To form the program B^+ which is dual to program A^+, it is necessary to add p additional dual variables Δ_k corresponding to the new terms bT_k^{-1} in the objective posynomial. Also, p more dual variables Δ_k' are needed to correspond to the new terms bT_k in the constraint posynomials. The corresponding factors in the dual objective function are of the form $(b/\Delta_k)^{\Delta_k}(b/\Delta_k')^{\Delta_k'}$. However, we might as well write this as $(b/\Delta_k)^{2\Delta_k}$ because the orthogonality condition on the new variables is $-\Delta_k + \Delta_k' = 0$. Thus, the augmented dual program B^+ can be defined as follows.

Program B^+. Seek the maximum value of the product function

$$v(\delta,\Delta) \triangleq \theta^{\sigma}\left(\prod_{i=1}^{n}(c_i/\delta_i)^{\delta_i}\prod_{k=1}^{p}\lambda_k^{\lambda_k}\right)\left(\prod_{k=1}^{p}(b/\Delta_k)^{2\Delta_k}\right)$$

where

$$\sigma \triangleq \sum_{i=1}^{n} \delta_i ,$$

$$\lambda_k \triangleq \bar{\lambda}_k + \sum_{m_k}^{n_k} \delta_i , \quad k = 1, \ldots, p.$$

The variables δ_i and λ_k are subject to the linear constraints

$$\delta_i \geq 0, \quad i = 1, \ldots, n$$

$$\lambda_k \geq 0, \quad k = 1, \ldots, p$$

$$\sum_{1}^{n} \delta_i + \sum_{1}^{p} \lambda_k = 1.$$

$$\sum_{1}^{n} \delta_i a_{ij} = 0, \quad j = 1, \ldots, m.$$

Here, the constants a_{ij}, b, c_i, and θ are as given in program A^+. Note that program B^+ reduces to program B if we set $\theta = 1$ and $\lambda_k = 0$.

A geometric program is said to be degenerate if a term u_i can be made to vanish without causing other terms to approach plus infinity. Otherwise, a program is said to be canonical. In what follows attention is restricted to canonical programs. The treatment of degenerate programs can then be reduced to that of canonical programs by deleting vanishing terms u_i (see Section VI.5 of [1]).

Without loss of generality it may be assumed that the matrix a_{ij} is of rank m (see Section III.3 of [1]). Then, the equations

$$\log(u_i / c_i) = \sum_{j=1}^{m} a_{ij} \log t_j , \quad i = 1, \ldots, n$$

show that the variables t_j are uniquely determined by the terms u_i. Consequently these equations show that if the terms u_i of a canonical program are bounded away from plus infinity, say $u_i \leq K$, then the variables t_j are confined to a compact set in the interior of the first orthant. Thus, if program A is canonical and consistent there is a point t^* such that $\inf_A g_0(t) = g_0(t^*)$.

~~Theorem 2. If program A is canonical and consistent, then program~~ B^+ ~~is consistent and~~

$$\min_{A^+} G_0(t,T) = \max_{B^+} V(\delta,\lambda) \quad \underline{\text{for}} \quad 0 < b \ \underline{\text{and}} \ 0 < \theta < 1.$$

<u>Proof.</u> Clearly, program A^+ is also a canonical consistent program. It follows that G_0 has a minimum for some values of the variables, say $t = t^*$ and $T = T^*$. For a fixed t the function G_0 is minimized by choosing the variables $T_1, \ldots, T_p$ to take the slack out of the p constraints; so we can eliminate these slack variables to obtain the relation

$$G_0(t,T) = g_0(t) + \sum_1^p \frac{b^2}{1-\theta g_k(t)} = \Gamma(t).$$

Thus, the function $\Gamma(t)$ has a minimum at $t = t^*$, which implies that the point t^* satisfies the necessary optimality conditions

$$t_j \frac{\partial \Gamma}{\partial t_j} = t_j \frac{\partial g_0}{\partial t_j} + \sum_1^p \frac{b^2\theta}{(1-\theta g_k)^2} t_j \frac{\partial g_k}{\partial t_j} = 0, \ j = 1,\ldots,m.$$

Carrying out the differentiations, we see that at t^*

$$\sum_1^{n_0} u_i a_{ij} + \sum_1^p \frac{b^2\theta}{(1-\theta g_k)^2} \sum_{m_k}^{n_k} u_i a_{ij} = 0, \ j = 1,\ldots,m.$$

Now, divide these equations by G_0 and use the tightness property $\theta g_k(t) + bT_k = 1$ to obtain the equations

$$\sum_1^n \delta_i a_{ij} = 0, \ j = 1,\ldots,m,$$

where we have let

$$\delta_i \triangleq u_i/G_0, \qquad i = 1,\ldots,n_0,$$

and

$$\delta_i \triangleq \frac{8u_i}{?}, \qquad i = m_k,\ldots,n_k, \quad k = 1,\ldots,p.$$

Also, let

then we see that

$$\sum_{i=1}^{n_0} \delta_i + \sum_{1}^{p} \lambda_k = \left(G_0 + \sum_{1}^{p} \frac{b}{T_k G_0}\right) - = 1.$$

Thus δ_i and λ_k so defined satisfy the positivity, orthogonality, and normality constraints of program B .

Now, Theorem 1 is to be employed to show that $G_0 = V(\delta,\lambda)$ with δ_i and λ_k defined as above. From those definitions we have

$$\lambda_k \triangleq \lambda_k + \lambda_k \delta_i = \frac{b}{T_k G_0} + \frac{b}{T_k^2 G_0} = \frac{2}{T_k^2 G_0}.$$

Using this result, we may redefine δ_i and λ_k by the following four relations:

$$\delta_i = \frac{u_i}{G_0} \qquad i = 1,\ldots,n_0,$$

$$\delta_i = \lambda_k \sigma u_i \qquad i = m_k,\ldots,n_k, \quad k = 1,\ldots,p,$$

$$\Delta_k = \frac{b}{T_k G_0} \qquad k = 1,\ldots,p,$$

$$\Delta_k' = \lambda_k b T_k \qquad k = 1,\ldots,p.$$

These are the equality conditions of Theorem 1 relative to programs A^+ and B^+, so the proof of Theorem 2 now follows from Theorem 1.

Let program A^8 be defined by putting $b = 0$ in program A^+. Its dual program B^8 is clearly obtained by putting $\lambda_k = 0$ in B^+.

Theorem 3. If program A is canonical and consistent, then program B and programs A^6 and B^8 are consistent, and

$$\min g\sim(t) = \max 0^{CT} v(6) \quad \text{for} \quad 0 < 6 < 1.$$

Proofo The proof of Theorem 2 shows that there exist $6_i > 0$ which satisfy the orthogonality conditions. Hence, the

$$\delta_i' \triangleq \delta_i / \Sigma_1^n \delta_j$$

satisfy both the orthogonality and the normality conditions. In other words program B is consistent.

For $0 < b$, Theorem 2 shows that the augmented program A^+ has a minimum value

$$\mathbf{M(6,b)} = V(6,A),$$

where 6 and A denote an optimal solution to program B^+. Suppose that 9 > 9, and let V denote the corresponding dual function. By Theorem 1, we know that

$$M(9^*,b) \geq V^*(6,A),$$

so

$$\frac{M(8^*,b)}{M(9,b)} \geq \frac{V^*(6,A)}{V(6,A)} = \left[\frac{81}{9}\right]^{\circ} .$$

It is obvious from the form of program A^+ that M(8,b) decreases as b decreases. Thus, we infer the existence of lim M(9,b) $\leq$ K and lim M(9 ,b) ^ K as $b \to 0^+$. Moreover, the canonicality of program A implies that K > 0 and K* > 0, so the preceding inequality on *a* shows that a is bounded as $b \to 0^+$.

Since $a \sum_{m_1}^{n} 6_i$ is bounded and since $\sum_1^{m_0} 6_i + \sum_1^{P} A_k = 1$, it follows that the 6_i and A_k have limits 61 and A_k^* as $b \to 0^+$

(through a suitable subsequence).

From the proof of Theorem 2, we know that $\Delta_k^2 = {}^{A}i/{}^{G}Q^*$ so $\Delta_k \to 0^+$ as $b \to 0^+$. Moreover, this identity shows that the factor $\Psi_k \triangleq (b/\Delta_k)^{2\Delta_k}$ occuring in the augmented dual function can be written as

$$\log \% \wedge \frac{2b}{G_0} \wedge^k \wedge^l K\chi, A\Lambda^{1/2} + A_k \log G_Q .$$

From this identity it follows that $\log \% \to 0$ as $b \to 0^+$.

As $b \to 0^+$ we now see that

$$V(6,A) \to e^{CT} v(6^*)$$

because the function V is continuous and because each $\%_n \to 1$. Also, the domain of the t_j variables is compact, so we can assume that $t_j \to t_j$ as $b \to 0'$. Then

$$c_0(t,T) \to g_0(t')$$

because the proof of Theorem 2 shows that the extra terms in G_Q are of the form $b/T_k = {}^{G}r^k$ which approach zero. Also,

$$8g_k(t) = 1 - bT_k \wedge 1,$$

so $6g_k(t') \leq 1$. We now see that $g_Q(t^T) = 8^- v(6')$ and this together with Theorem 1 completes the proof of Theorem 3.

4. <u>The main theorems</u>. We now have enough machinery to establish the main theorems of geometric programming.

<u>Theorem 4</u>. <u>If program</u> A <u>is canonical and consistent, then program</u> B <u>is consistent, and</u>

$$\min_A g_0(t) = \sup_B v(6).$$

<u>Proof</u>. Theorem 3 asserts that program B is consistent. If t' and $6'$ denote optimal solutions whose existence is guaranteed by

Theorem 3, then have

$$g_0(t') \le \theta^{a'} v(\delta') \le v(\delta') \quad \text{for } 0 < \theta < 1.$$

Letting $\theta \to 1^-$, we infer from compactness and continuity of g_k that t' has a limit point t'' which satisfies the constraints of program A. Moreover, the preceding displayed relations imply that $g_0(t'') \le \sup_B v(\delta)$. But Theorem 1 shows that $g_0(t'') \ge \sup_B v(\delta)$, so the proof of Theorem 4 is complete.

<u>Theorem 5, Let program A be canonical, and suppose that program B is consistent and has a finite supremum M_B. Then program A is consistent, and</u>

$$\min_A g_0(t) = \sup_B v(\delta).$$

<u>Proof</u>. Since the dual objective function is

$$v(\delta) = \prod_{1}^{n} \left(\frac{c_i}{\delta_i}\right)^{\delta_i} \prod_{0}^{p} \lambda_k^{\lambda_k},$$

and since

$$\prod_{1}^{n} {}^{x} \qquad \prod_{0}^{p} {}^{k}$$

it is clear that v satisfies the identity

$$v(\delta) \equiv [v(\delta/a)]^a$$

for any $a > 0$.

The consistency of program B clearly implies the consistency of program B^+. Let δ_i and λ_k satisfy the constraints of program B^+; then the $\delta_i' = \delta_i/\lambda_0$ satisfy the constraints of program B if program $\lambda_0 > 0$. Choosing $a = 1/\lambda_0$ in our identity, we see that

$$\prod_{1}^{n} \left(\frac{c_i}{\delta_i}\right)^{\delta_i} = v(\delta) \prod_{0}^{p} \lambda_k^{-\lambda_k} = [v(\delta')]^{\lambda_0} \prod_{0}^{p} \lambda_k^{-\lambda_k},$$

so the augmented dual objective function

$$v(\delta, \lambda) \triangleq \delta_0^{\delta_0} \prod_{1}^{n} \left(\frac{c_i}{\delta_i}\right)^{\delta_i} \prod_{1}^{p} \left(\frac{b}{\Delta_k}\right)^{2\Delta_k} \prod_{1}^{p} \lambda_k^{\lambda_k}$$

can be rewritten as

$$V(\delta, \lambda) = [v(\delta')^{\lambda_0}] \lambda_0^{-\lambda_0} \prod_1^p [(\frac{b}{\Delta_k})^{2\Delta_k}] [(1 + \frac{\Delta_k}{\lambda_k})^{\lambda_k}] [(\lambda_k + \Delta_k)^{\Delta_k} \theta^{\lambda_k}].$$

Since $\lambda_0 \leq 1$ and $\lambda_k \leq 1$, the factors in square brackets have bounds independent of δ_i and λ_k when $0 < \theta < 1$; in particular, $[v(\delta')]^{\lambda_0} \leq M_B^{\lambda_0} \leq \max\{1, M_B\}$. Also

$$\left(\frac{b}{\Delta}\right)^{2\Delta} \leq \max\{1, b^2\} \left(\frac{1}{\Delta}\right)^{2\Delta}$$

$$\left(1 + \frac{\Delta}{\lambda}\right)^{\lambda} \leq \left(1 + \frac{1}{\lambda}\right)^{\lambda},$$

$$(\lambda + \Delta)^{\Delta} \theta^{\lambda} \leq (\lambda + 1)^{\Delta} \theta^{\lambda} \leq (\lambda + 1)\theta^{\lambda}$$

Clearly the functions on the right sides of these inequalities are continuous on the positive real axis and have finite limits at 0 and ∞. Thus these functions are uniformly bounded. Likewise $[\lambda_0^{-\lambda_0}]$ is seen to be uniformly bounded. Consequently, if δ_i and λ_k satisfy the constraints of program B^+, and if $\lambda_0 > 0$, then

$$V(\delta,\lambda) \leq K(b,\theta),$$

where the function K is defined for $0 < b$ and $0 < \theta < 1$. However, V is a continuous function of δ, so the preceding bound remains true for $\lambda_0 = 0$.

For fixed $b > 0$, program A^+ is clearly consistent if θ is chosen very small. If program A^+ is not consistent for all $\theta < 1$ then there is a $\theta_1 < 1$ such that the constraints $\theta g_k + bT_k \leq 1$

can be satisfied for $6 < 6_{][}$ but not for $6 > e^{\wedge}$ If $0 \to 0_X$ from below it is obvious that some T_k -» 0 and hence min T(t) -> +oo. However, using the proof of Theorem 2 and the bound K, we have

$$\min T(t) = \max_{B^+} V(6,A) \; £ \; K(b,8).$$

But K(b,0) does not approach +oo as 0 -> 8^, so this is a contradiction.

The preceding contradiction shows that the constraints 0g^.(t) £ 1 can be satisfied for all 0 < 1. Then, the compactness property of canonical programs and the continuity of the g_k imply that the constraints can also be satisfied for 0 = 1. This shows that program A is consistent. But then Theorem 5 is a consequence of Theorem 4.

Theorem 6. Program A is canonical if, and only if, there is a vector 6^f with strictly positive components that satisfy the constraints of program B.

Proofo If program A is canonical it follows that it is consistent for 0 small. Then the proof of Theorem 2 shows that there are strictly positive 6_1 which satisfy the constraints of program B^+. Hence the $6_1' \; L \; 6_-/Tn^O \; 6.$ are strictly positive and satisfy the constraints of program B.

Conversely, suppose that $6^!$ is a positive vector satisfying the constraints of program Bo By orthogonality

$$\sum_1^n 6_i^f \; \log Cuj/Cj) = £_1^m(£_1^n \; 6_i^t \; a^\wedge \log t_j = 0_o$$

Thus, no term u_K can vanish without causing other terms u_1 to approach plus infinity, so program A is canonical and hence the proof of Theorem 6 is complete.

The preceding theorems constitute the major part of the duality theory of geometric programming. The theory may be completed by employing the concept of a subconsistent program. Program A is said to be <u>subconsistent</u> if program A^8 is consistent for all $6 < 1$. Using this terminology the preceding theorems may be restated so as to apply to degenerate programs. The details may be fourid in references [1],[2], and [3].

5. Convergence of the numerical methodo Although Kochenberger and Woolsey have been obtaining approximate numerical solutions to geometric programs by solving the appropriate augmented programs, they have not shown that the approximations can be made arbitrarily accurate by choosing b sufficiently close to zero. That such convergence is in fact the case can be readily established by examining the proof of Theorem 3. Such an examination should convince the reader that the following theorem is valid.

Theorem 7. Suppose that program A is canonical and consistent, and let 0 be fixed so that $0 < 8 < L$ Then the augmented program A^+ and its geometric dual program B^+ have optimal solutions t(b) and (6(b), A(b)) respectively when $0 < b$. Moreover, when b -» 0^+ ^through any sequence$_3$ the corresponding sequences (t(b)} and {6(b)} each have at least one limit point $t^!$ and $6^!$ respectively. Furthermore$_9$ each pair of limit points t^1 and $6^!$ generated in this manner are optimal solutions to programs A^6 and B^8 pectively.

Note that the consistency of program A implies the "super-consistency" of program A^g for $0 < 8 < 1$ (that is, there is a feasible solution t to program A such that $6g_k(t) < 1$, k = l,...,p). However, it is clear that Kochenberger's method can be applied only to superconsistent programs, because the augmented program A^+ for a consistent program A that is not superconsistent is obviously not consistent when $0 < b$ and $8 = 1$. Moreover, it is obvious that every superconsistent geometric program A^n can be formulated as a program A^Q corresponding to a consistent geometric program A by choosing each coefficient $c_i^{'} = c_i/8$ for some $8 < 1$ that is sufficiently close to 1. Needless to say, this restriction of the applicability of Kochenberger's method to superconsistent programs is a rather insignificant limitation.

Finally, we should mention that Kochenberger and Woolsey no longer use the augmented programs A^+ and B^+ for numerical minimization. Experimental investigations [7] indicate that it is numerically better to introduce an additional positive parameter r and add $bT_k^{r} + bT_k^{-r}$ (instead of just bT_k^{-1}) to the objective function g_0. In particular, they have obtained sufficiently accurate approximate optimal solutions to a number of programs of practical significance by choosing $b = r = .01$.

It is clear that there are many other posynomials in T_k that produce tight constraints when added to the objective function g_0; the only requirement on such posynomials is that they are not themselves minimized by any $T_k < 1$. There is little doubt that such methods converge in the sense of Theorem 7, but the proofs are probably more complicated than the proof given here and hence probably do not provide an even simpler proof of the refined duality theory of geometric programming. Of course, each such numerical method corresponds to the use of a different 'penalty function'. Moreover, each penalty function is known to produce a numerical method [4] for solving the primal program directly. Perhaps, a hybrid of the purely penalty function approach and Kochenberger's approach would be most effective. Such a hybrid method could conceivably exploit the fact that the primal constraint is slack when its corresponding dual positivity constraints are tight.

References

[1] Duffin, R. J., E. L. Peterson and C. Zener, Geometric Programming, John Wiley, New York, 1967.

[2] Duffin, R. J., and E. L. Peterson, Duality theory for geometric programming, SIAM J. Appl. Math. 14 (1966) pp. 1307-1349.

[3] Duffin, R. J., Linearizing geometric programs, SIAM Review 12 (1970), pp. 211-227.

[4] Fiacco, A. V., and G. P. McCormick, Sequential Unconstrained Minimization Techniques for Nonlinear Programming, John Wiley, New York, 1969.

[5] Rockafellar, R. T., Some convex programs whose duals are linearly constrained, Proceedings Symposium on Nonlinear Programming, Univ. of Wis., 1970.

[6] Kochenberger, Gary A., Geometric programming, extension to deal with degrees of difficulty and loose constraints, Ph.D. Thesis, University of Colorado, School of Business, 1969.

[7] Woolsey, R. G., private communication.