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HYBRID ADDITION OF MATRICES - A NETWORK THEORY CONCEPT*

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ABSTRACT

The parallel connection of networks suggested the concept of parallel addition of matrices to Anderson and Duffin. The hybrid connection of networks also suggests a matrix operation. Using the Kirchhoff current and voltage equations, a new operation, hybrid addition, is defined for the set of Hermitian semidefinite matrices. This operation is an Hermitian semidefinite order preserving semigroup operation. Hybrid addition is closely related to the work of Anderson on 'shorted operators', and to the gyration operation of linear programming and network synthesis.

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Hybrid Addition of Matrices-
A Network Theory Concept

1.0 <u>INTRODUCTION</u> Let A and B be the impedance matrices of n-port passive resistive networks. It is well known that A and B are Hermitian semidefinite. Figure 1 shows the series connection of two 2-port networks. More generally, in the series connection of n-ports, the corresponding ports of each network are connected in series. The impedance matrix of the series connection is A + B.

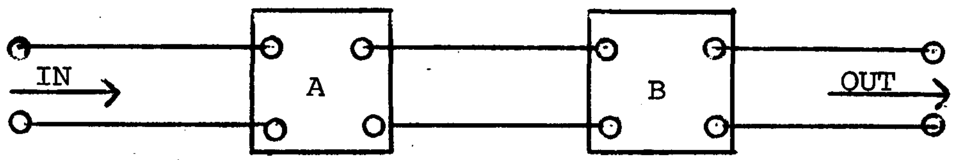

Figure 1: Series connection of two 2-ports

Figure 2 shows the parallel connection of two 2-ports; the n-port case is similar. Anderson and Duffin have shown that the impedance matrix of the parallel connection is given by $A(A+B)^{+}B$; where A^{+} is the Moore-Penrose generalized inverse of A. Defining $A : B = A(A+B)^{+}B$, the parallel sum of A and B, Anderson and Duffin have shown that parallel addition is a commutative semigroup operation on the set of Hermitian semidefinite matrices. Parallel addition also preserves the Hermitian semidefinite partial ordering, where $A \geq B$ means that $A - B$ is

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Hermitian semidefinite.

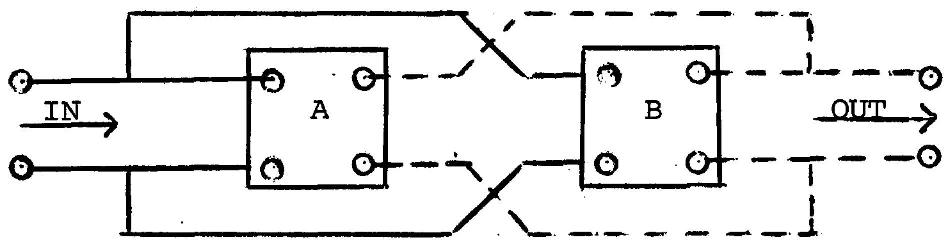

Figure 2: Parallel connection on two 2-ports

Another connection of n-ports used in electrical network theory is the hybrid connection [5,7]. This connection is a combination of the series and parallel connections, and contains each as special cases. Using the Kirchhoff current and voltage equations associated with the hybrid connection, we define hybrid addition of Hermitian semidefinite matrices. This new operation is commutative, associative and preserves the Hermitian semidefinite partial order. The 'gyration operator' of Duffin, Hazony, Morrison [7] and Tucker [11,12] is extended to obtain a matrix expression for the hybrid sum of two Hermitian semidefinite matrices. Anderson has defined the 'shorted operator' of a matrix and it is shown that hybrid addition is closely related to the shorted operator.

2.0 <u>NOTATION AND PRELIMINARY LEMMAS</u> This paper considers operators on a finite dimensional complex vector space. Matrix representations are used whenever possible. An operator A is said to be Hermitian semidefinite when $A = A^*$ and $(Ax,x) \geq 0$

for all x. Hermitian semidefinite is abbreviated HSD. If A is HSD then $(Ax,x) = 0$ if and only if $Ax = 0$. The range and null space of A are denoted $R(A)$ and $N(A)$ respectively. If A and B are HSD then $R(A+B) = R(A) + R(B)$. The Moore-Penrose generalized inverse [6,9] of A is denoted A^+. If A is Hermitian, then A restricted to $R(A)$ is one to one and therefore invertible. The generalized inverse is then defined by: $A^+ = A^{-1}$ on $R(A)$ and $A^+ = 0$ on $N(A)$.

Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ with A_{11} and A_{22} square submatrices.

It is clear that if A is HSD then A_{11} and A_{22} are HSD and $A_{12}^* = A_{21}$. The following lemma relates properties of submatrices to the partitioned matrix; $[x,y]$ denotes vectors partitioned corresponding to the matrix partition.

Lemma 1: Let A be HSD and partitioned as above, then $R(A_{11}) \supseteq R(A_{12})$ and $N(A_{21}) \supseteq N(A_{11})$.

Proof: Suppose the vector $x \in N(A_{11})$, then $A_{11}x = 0$. If $y = A_{21}x$, then for real a, $(A[ax,y],[ax,y]) = 2a\|y\|^2 + (A_{22}y,y)$. Since a is arbitrary, if $y \neq 0$, the right side can be made negative, contradicting the fact that A is HSD. This shows that $x \in N(A_{21})$. Since A_{11} is HSD and $A_{21}^* = A_{12}$, the other containment follows by taking complements.

The theory of parallel addition is used throughout this paper. The following summary of results is included for completeness. See Anderson and Duffin [2] for proofs.

<u>Lemma 2</u>: <u>Let</u> A, B, C, D <u>be</u> HSD <u>and</u> $A : B = A(A+B)^{+}B$ <u>then</u>

a) $A : B$ <u>is HSD</u>.

b) $A : B = B : A$.

c) $A : (B : C) = (A : B) : C$.

d) $A : B = (P(A^{+} + B^{+})P)^{+}$, <u>where</u> P <u>is the projection onto</u> $R(A) \cap R(B)$.

e) $(A : Bz, z) \leq (Ax, x) + (By, y)$ <u>for all</u> $x + y = z$. <u>Moreover, if</u> $x = (A+B)^{+}Bz$ <u>and</u> $y = (A+B)^{+}Az$ <u>then equality holds</u>.

f) $(A+B) : (C+D) \geq A : C + B : D$ <u>Series Parallel Inequality</u>.

g) $R(A : B) = R(A) \cap R(B)$

3.0 <u>ALGEBRA OF HYBRID ADDITION</u> Figure 3 illustrates the hybrid connection of two 2-ports. More generally in the hybrid connection of n-ports, the first r ports are connected in parallel and the remaining $n - r$ ports are connected in series. Ideal isolation transformers are assumed present at each port to insure proper current flow.

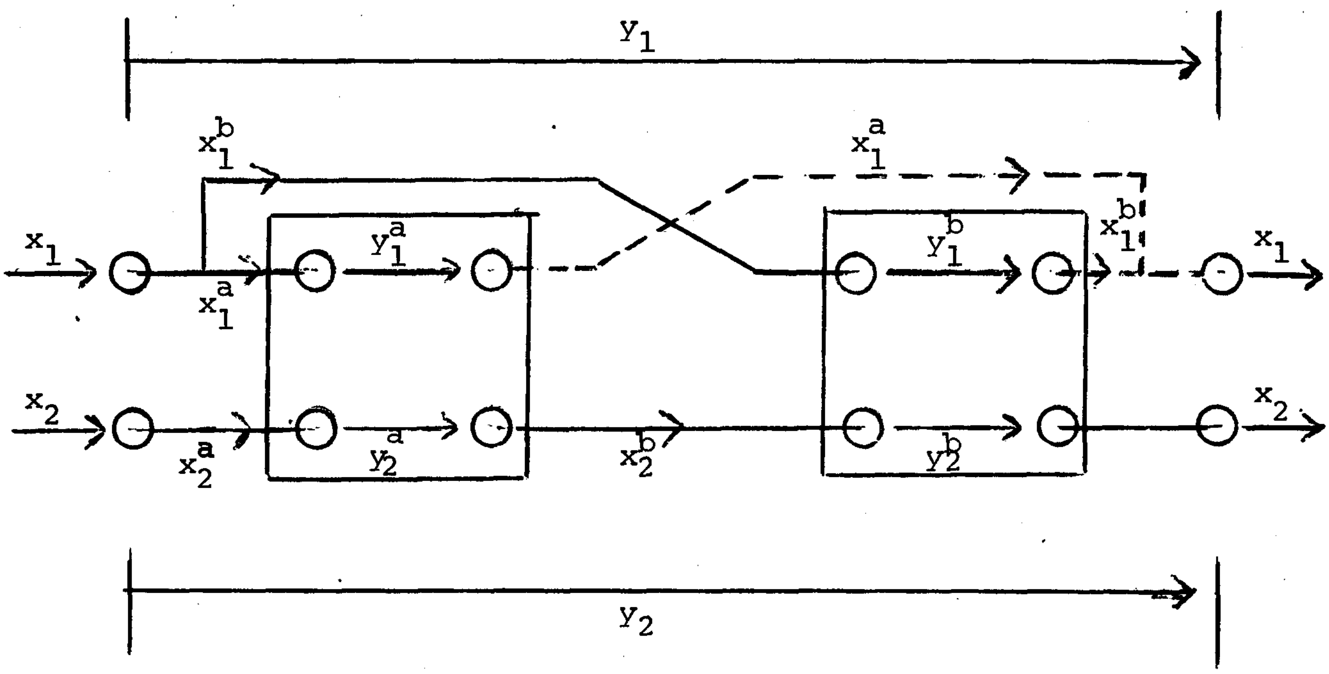

Figure 3: Hybrid Connection of Two 2-ports

Let $[x_1, x_2]$ represent current, $[y_1, y_2]$ voltage, and A and B the impedance matrices of the networks. Then the Kirchhoff current and voltage equations take the form:

$$A[x_1^a, x_2^a] = [y_1^a, y_2^a], \qquad B[x_1^b, x_2^b] = [y_1^b, y_2^b],$$

$$x_1 = x_1^a + x_1^b, \qquad x_2 = x_2^a = x_2^b,$$

$$y_1 = y_1^a = y_1^b, \qquad y_2 = y_2^a + y_2^b.$$

When written with the partitioned A and B, these equations become:

(1)

a) $A_{11}x_1^a + A_{12}x_2^a = y_1^a$ e) $x_1 = x_1^a + x_1^b$

b) $A_{21}x_1^a + A_{22}x_2^a = y_2^a$ f) $x_2 = x_2^a = x_2^b$

c) $B_{11}x_1^b + B_{12}x_2^b = y_1^b$ g) $y_1 = y_1^a = y_1^b$

d) $B_{21}x_1^b + B_{22}x_2^b = y_2^b$ h) $y_2 = y_2^a + y_2^b$

The next theorem shows that equations (1) can be solved for $[y_1, y_2]$ in the form $[y_1, y_2] = H[x_1, x_2]$. The matrix H is called the _hybrid_ _sum_ of A and B. If all the ports of the network are connected in series, then $H = A+B$, the ordinary series sum. If all ports are connected in parallel, then $H = A : B$ the parallel sum of A and B [1,2].

Theorem 3: _If_ A _and_ B _are_ HSD _then_ _equations_ (1) _can_ _be_ _solved_ _for_ _a_ _matrix_ H _so_ _that_ $[y_1, y_2] = H[x_1, x_2]$, _where_ H _is_ _given_ _by_:

$$H = \begin{bmatrix} D & D(A_{11}^{+}A_{12} + B_{11}^{+}B_{12}) \\ (A_{21}A_{11}^{+} + B_{21}B_{11}^{+})D & E \end{bmatrix}$$

with $D = A_{11}: B_{11}$, the parallel sum of A_{11} and B_{11} and

$$E = A_{22} + B_{22} - (A_{21} - B_{21})(A_{11} + B_{11})^{+}(A_{12} - B_{12}).$$

Proof: From b),d),f) and h) in (1) it follows that:

$$(2) \quad y_2 = A_{21}x_1^a + B_{21}x_1^b + (A_{22} + B_{22})x_2.$$

From a),c),e),f) and g) in (1) we have:

$$(3) \quad \begin{aligned} &a)\ y_1 - A_{11}x_1^a = A_{12}x_2 \\ &b)\ y_1 - B_{11}x_1^b = B_{12}x_2 \\ &c)\ x_1^a + x_1^b = x_1 \end{aligned}$$

Equations (2) and (3) are a linear system for y_1 and y_2 in terms of x_1 and x_2. To show uniqueness, suppose $x_1 = x_2 = 0$, then (3) can be rewritten:

$$y_1 = A_{11}x_1^a = B_{11}x_1^b, \quad x_1^a = -x_1^b.$$

Therefore $(A_{11} + B_{11})x_1^a = 0$. Since A_{11} and B_{11} are HSD, it follows that $y_1 = A_{11}x_1^a = B_{11}x_1^b = 0$. Since A and B are HSD, $N(A_{21}) \supseteq N(A_{11})$ and $N(B_{21} \supseteq N(B_{11})$. Therefore, $A_{21}x_1^a = B_{21}x_1^b = 0$, and then $y_2 = 0$ from (2). This completes the uniqueness argument.

To show existence and to solve for H, x_1 and x_2 are considered separately.

Let $x_2 = 0$. Then, since $x_1^a + x_1^b = x_1$, (3) can be written:

$$y_1 - A_{11}x_1^a = 0, \quad y_1 + B_{11}x_1^a = B_{11}x_1.$$

Therefore $(A_{11} + B_{11})x_1^a = B_{11}x_1$. Since $R(B_{11}) \subseteq R(A_{11} + B_{11})$, $x_1^a = (B_{11} + A_{11})^{+}B_{11}x_1$ is a solution. Substituting this solution for x_1^a into (3a) and (2), we have:

(4)

$$a)\quad y_1 = A_{11}(A_{11}+B_{11})^+B_{11}x_1$$

$$b)\quad y_2 = A_{21}(A_{11}+B_{11})^+B_{11}x_1 + B_{21}(I-(A_{11}+B_{11})^+B_{11})x_1.$$

Let $D = A_{11}(A_{11}+B_{11})^+B_{11}$; D is the parallel sum of A_{11} and B_{11}. Since $A_{21}A_{11}^+A_{11} = A_{21}$, $B_{21}B_{11}^+B_{11} = B_{21}$ and $B_{21}((A_{11}+B_{11})^+A_{11}+(A_{11}+B_{11})^+B_{11}) = B_{21}$, we may rewrite (4) as follows:

(5)

$$a)\quad y_1 = Dx_1$$

$$b)\quad y_2 = (A_{21}A_{11}^+ + B_{21}B_{11}^+)Dx_1.$$

Equation (5) gives the first column of the matrix H. For the second column suppose $x_1 = 0$.

If $x_1 = 0$, then since $x_1^a + x_1^b = 0$ equation (3) can be rewritten:

(6)

$$a)\quad y_1 - A_{11}x_1^a = A_{12}x_2,$$

$$b)\quad y_1 + B_{11}x_1^a = B_{12}x_2.$$

Therefore $(B_{11}+A_{11})x_1^a = (B_{12}-A_{12})x_2$. Since

$$R(B_{12}-A_{12}) \subseteq R(B_{12}) + R(A_{12}) \subseteq R(B_{11}) + R(A_{11}) = R(A_{11}+B_{11}),$$

$x_1^a = (B_{11}+A_{11})^+(B_{12}-A_{12})x_2$ is a solution. Substitution of this value for x_1^a into (6a) yields:

$$y_1 = [A_{12}+A_{11}(B_{11}+A_{11})^+(B_{12}-A_{12})]x_2.$$

Since $A_{11}(A_{11}+B_{11})^+A_{12} = A_{12}-B_{11}(B_{11}+A_{11})^+A_{12}$, we may write:

$$(7)\quad y_1 = D(A_{11}^+A_{12}+B_{11}^+B_{12})x_2.$$

Replace x_1^a given above in (2) and we have:

$$y_2 = [(A_{21} - B_{21})(A_{11} + B_{11})^+(B_{12} - A_{12}) + A_{22} + B_{22}]x_2.$$

Letting $E = (A_{21} - B_{21})(A_{11} + B_{11})^+(B_{12} - A_{12}) + A_{22} + B_{22}$, we can rewrite the above as:

$$(8) \qquad y_2 = Ex_2$$

Equations (7) and (8) give the second column of H and this completes the proof. QED

The matrix H, called the hybrid sum of A and B is denoted $A * B$. The generalized inverse is used in the definition of H. Since $R(A_{11}) \supseteq R(A_{12})$ and $R(B_{11}) \supseteq R(B_{12})$, the generalized inverse is only a convenient notation, and is really not needed.

The following notation is used throughout the remainder of the paper. Let

$$F = A_{11}^+A_{12} + B_{11}^+B_{12},$$

$$S_A = A_{22} - A_{21}A_{11}^+A_{12},$$

$$S_B = B_{22} - B_{21}B_{11}^+B_{12}.$$

S_A and S_B are related to the shorted operator of Anderson which is considered in Section 5.0.

From the matrix expression for $A * B$, we see that the upper right and lower left blocks are given by DF and F^*D respectively. The next Lemma simplifies the lower right block of $A * B$.

<u>Lemma 4</u>: <u>Let</u> D, F, S_A <u>and</u> S_B <u>be as above, then</u>:

$$A_{22} + B_{22} - (A_{21} - B_{21})(A_{11} + B_{11})^+(A_{12} - B_{12}) = S_A + S_B + F^*DF.$$

Proof: The terms S_A and S_B contain A_{22} and B_{22} respectively, therefore it is sufficient to show that:

$$-A_{21}A_{11}^{+}A_{12} - B_{21}B_{11}^{+}B_{12} + (A_{21}A_{11}^{+} + B_{21}B_{11}^{+})D(A_{11}^{+}A_{12} + B_{11}^{+}B_{12}) = -(A_{21} - B_{21})(A_{11} + B_{11})^{+}(A_{12} - B_{12}) \text{ .}$$

Expansion of the left hand side, using, $D = A_{11}(A_{11} + B_{11})^{+}B_{11}$, $A_{21}A_{11}^{+}A_{11} = A_{21}, A_{11}A_{11}^{+}A_{12} = A_{12}$ and similarly for B_{12} and B_{21}, yields:

$$-A_{21}A_{11}^{+}A_{12} - B_{21}B_{11}^{+}B_{12} + A_{21}(A_{11} + B_{11})^{+}B_{12} + A_{21}(A_{11} + B_{11})^{+}B_{11}A_{11}^{+}A_{12} + B_{21}(A_{11} + B_{11})^{+}A_{12} + B_{21}(A_{11} + B_{11})^{+}A_{11}B_{11}^{+}B_{12}.$$

Simplifying this we have:

$$A_{21}(-I + (A_{11} + B_{11})^{+}B_{11})A_{11}^{+}A_{12} + B_{21}(-I + (A_{11} + B_{11})^{+}A_{11})B_{11}^{+}B_{12} + A_{21}(A_{11} + B_{11})^{+}B_{12} + B_{21}(A_{11} + B_{11})^{+}A_{12}.$$

Since $(A_{11} + B_{11})^{+}B_{11} + (A_{11} + B_{11})^{+}A_{11} = P$, where P is the projection onto $R(A_{11}) \cap R(B_{11})$, the above becomes:

$$-A_{21}(A_{11} + B_{11})^{+}A_{12} - B_{21}(A_{11} + B_{11})^{+}B_{12} + A_{21}(A_{11} + B_{11})^{+}B_{12} + B_{21}(A_{11} + B_{11})^{+}A_{12}.$$

But this may be rewritten as:

$$-(A_{21} - B_{21})(A_{11} + B_{11})^{+}(A_{12} - B_{12}).$$

QED

Lemma 4 allows us to abbreviate the hybrid matrix as follows:

$$(9) \qquad A * B = \begin{bmatrix} D & DF \\ F^*D & S_A + S_B + F^*DF \end{bmatrix}.$$

If A and B are HSD, it is clear that S_A and S_B are Hermitian. Since D is also Hermitian by Lemma 2, equation (9) shows $A * B$ is Hermitian. Lemma 2 also states that $A_{11}(A_{11}+B_{11})^+B_{11} = B_{11}(B_{11}+A_{11})^+A_{11}$, therefore $A * B = B * A$. This proves the following theorem:

<u>Theorem 5</u>: <u>If</u> A <u>and</u> B <u>are</u> HSD <u>then</u> $A * B$ <u>is Hermitian</u> <u>and</u>

$$A * B = B * A.$$

Duffin, Hazony and Morrison have used the hybrid connection in network synthesis problems. In their synthesis they consider the gyrator, which is a non-reciprocal network element first used by Tellegen. Their study of the gyrator led them to the concept of matrix gyration. The gyration is a partial inverse defined for partitioned matrices. If A is partitioned as before, then the <u>gyration</u> of A, denoted by $\Gamma(A)$, is given by:

$$\Gamma(A) = \begin{bmatrix} A_{11}^+ & -A_{11}^+A_{12} \\ A_{21}A_{11}^+ & A_{22} - A_{21}A_{11}^+A_{12} \end{bmatrix}.$$

Tucker has also studied the gyration operator in connection with linear programming [11,12].

The submatrix A_{11} is termed the 'pivot', and these authors do not define gyration unless A_{11} is invertible. So actually the above definition is a generalized gyration. The following lemma justifies the term partial inverse.

Lemma 6: If A_{11} is invertible then $A[x_1,x_2] = [y_1,y_2]$ implies $\Gamma(A)[y_1,x_2] = [x_1,y_2]$.

Proof: In partitioned form $A[x_1,x_2] = [y_1,y_2]$ becomes:

$$A_{11}x_1 + A_{12}x_2 = y_1$$

$$A_{21}x_1 + A_{22}x_2 = y_2$$

Since A_{11} is invertible, we can solve these equations for x_1 and y_2 and we have:

$$x_1 = A_{11}^{-1} y_1 - A_{11}^{-1} A_{12}x_2$$

$$y_2 = A_{21}A_{11}^{-1} y_1 + (A_{22} - A_{21}A_{11}^{-1} A_{12})x_2 .$$

Rewritten these are:

$$[x_1,y_2] = \Gamma(A)[y_1,x_2]$$

QED

Using the notation $S_A = A_{22} - A_{21}A_{11}^{+}A_{12}$ introduced previously, we may write the gyration as:

$$\Gamma(A) = \begin{bmatrix} A_{11}^{+} & -A_{11}^{+}A_{12} \\ A_{21}A_{11}^{+} & S_A \end{bmatrix} .$$

The next lemma shows that gyration is an idempotent operation when restricted to Hermitian semidefinite matrices.

Lemma 7: If A is HSD then $\Gamma(\Gamma(A)) = A$.

Proof: From the definition of gyration we see that:

$$\Gamma(\Gamma(A) = \begin{bmatrix} A_{11}^{++} & A_{11}^{++} A_{11}^{+}A_{12} \\ A_{21}A_{11}^{+}A_{11}^{++} & S_A + A_{21}A_{11}^{+}A_{11}^{++} A_{11}^{+}A_{12} \end{bmatrix} .$$

Since A is HSD, $A_{11}^{++} = A_{11}$, $A_{11}^{++} A_{11}^{+} A_{12} = A_{12}$ and $A_{21} A_{11}^{+} A_{11}^{++} = A_{21}$, the terms of $\Gamma(\Gamma(A))$ simplify to those of A. QED

Whenever A_{11} and B_{11} are nonsingular, the hybrid connection $A * B$ is given by $\Gamma(\Gamma(A) + \Gamma(B))$, see Duffin, Hazony and Morrison [7] or Belevitch [5]. However if A_{11} or B_{11} is singular $\Gamma(\Gamma(A) + \Gamma(B))$ exists but is not $A * B$. The following theorem gives a formula for $A * B$ in terms of Γ and the projection onto $R(A_{11}) \cap R(B_{11})$.

Theorem 8: *Let* A *and* B *be* HSD, P *be the projection onto* $R(A_{11}) \cap R(B_{11})$, *and*

$$Q = \begin{bmatrix} P & O \\ O & I \end{bmatrix},$$

then $A * B = \Gamma(Q(\Gamma(A) + \Gamma(B))Q)$.

Proof: From the definition of gyration, we see $\Gamma(A) + \Gamma(B)$ can be written:

$$\begin{bmatrix} (A_{11}^{+} + B_{11}^{+}) & -F \\ F^{*} & S_A + S_B \end{bmatrix},$$

where, as before, $F = A_{11}^{+} A_{12} + B_{11}^{+} B_{12}$. Multiplication on the right and left by Q yields:

$$\begin{bmatrix} P(A_{11}^{+} + B_{11}^{+})P & -PF \\ F^{*}P & S_A + S_B \end{bmatrix}$$

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Since $(P(A_{11}^{+} + B_{11}^{+})P)^{+} = A_{11} : B_{11}$ by Lemma 2, taking the gyration of $Q(\Gamma(A) + \Gamma(B))Q$ gives $A * B$. QED

This new expression for the hybrid sum is used to show that hybrid addition is associative. First we need a preliminary lemma.

Lemma 9: Let A and B be HSD, and Q be as in Theorem 8, then $\Gamma(A * B) = Q(\Gamma(A) + \Gamma(B))Q$.

Proof: The hybrid sum is given by:

$$A * B = \begin{bmatrix} D & -DF \\ F * D & S_A + S_B + F*DF \end{bmatrix}.$$

Therefore $\Gamma(A * B)$ is:

$$\begin{bmatrix} D^{+} & -D^{+}DF \\ F*DD^{+} & S_A + S_B + F*DF - F*DD^{+}DF \end{bmatrix}.$$

Since $D^{+} = P(A_{11}^{+} + B_{11}^{+})P$, $D^{+}DF = F$ and $F*DD^{+} = F*$ the terms of $\Gamma(A * B)$ simplify to those of $Q(\Gamma(A) + \Gamma(B))Q$. QED

Theorem 10: If A,B and C are HSD then $A * (B * C) = (A * B) * C$, and if P is the projection onto $R(A_{11}) \cap R(B_{11}) \cap R(C_{11})$ and $Q = \begin{bmatrix} P & O \\ O & I \end{bmatrix}$, then $A * B * C = \Gamma(Q(\Gamma(A) + \Gamma(B) + \Gamma(C))Q)$.

Proof: Since $R(B_{11} : C_{11}) = R(B_{11}) \cap R(C_{11})$ by Lemma 2, we may write $P = R(A_{11}) \cap R(B_{11}) \cap R(C_{11}) = R(A_{11}) \cap R(B_{11} : C_{11})$. With Q as above, Theorem 8 yields:

$$A * (B * C) = \Gamma(Q(\Gamma(A) + \Gamma(B * C))Q).$$

By Lemma 9, $\Gamma(B * C) = Q_1(\Gamma(B) + \Gamma(C))Q_1$ where $Q_1 = \begin{bmatrix} P_1 & O \\ O & O \end{bmatrix}$ with $P_1 = R(B_{11}) \cap R(C_{11})$. Therefore $A * (B * C)$ can be written:

$$\Gamma(Q(\Gamma(A) + Q_1(\Gamma(B) + \Gamma(C))Q_1)Q).$$

From the definition of P and P_1, we see $PP_1 = P_1P$ and therefore $QQ_1 = Q_1Q = Q$. $A * (B * C)$ can then be written:

$$\Gamma(Q(\Gamma(A) + \Gamma(B) + \Gamma(C))Q).$$

A similar computation holds for $(A * B) * C$.

QED

Using equations (1),(2) and (3), we can complete the proof that hybrid addition is a commutative semigroup operation. The following lemma shows that $A * B$ is semidefinite.

<u>Lemma 11</u>. If $[x_1^a, x_2^a]$ and $[x_1^b, x_2^b]$ are determined in (2) and (3) from $[x_1, x_2]$ then $(A * B[x_1,x_2],[x_1,x_2]) = (A[x_1^a,x_2^a],[x_1^a,x_2^a]) + (B[x_1^b,x_2^b],[x_1^b,x_2^b])$.

<u>Proof</u>: $(A * B[x_1,x_2],[x_1,x_2]) = ([y_1,y_2],[x_1,x_2])$

$= (y_1,x_1) + (y_2,x_2)$

$= (y_1^a,x_1^a) + (y_1^b,x_1^b) + (y_2^a,x_2^a) + (y_2^b,x_2^b)$

using 1e,1f,1g,1h

$= ([y_1^a,y_2^a],[x_1^a,x_2^a]) + ([y_1^b,y_2^b],[x_1^b,x_2^b])$

$= (A[x_1^a,x_2^a],[x_1^a,x_2^a]) + (B[x_1^b,x_2^b],[x_1^b,x_2^b])$

QED

<u>Theorem 12</u>: <u>If</u> A <u>and</u> B <u>are</u> HSD <u>then</u> $A * B$ <u>is</u> HSD.

<u>Proof</u>: Theorem 5 guarantees that $A * B$ is Hermitian, and Lemma 11 assures that $A * B$ is semidefinite. QED

Theorems 5,10 and 12 complete the proof that hybrid addition is a commutative semigroup operation. The next section develops the matrix inequalities necessary to prove that hybrid addition preserves the Hermitian semidefinite partial order.

4.0 HYBRID INEQUALITIES Theorem 12 states that the network is 'passive'. If the currents x^a and x^b satisfy the constraints 1e) and 1f) but are not necessarily solutions of (2) then the power equality of Lemma 11 becomes the inequality of the following lemma.

Lemma 13: *Let* x^a, x^b *be solutions of* (2) *and* (3) *determined from* $x = [x_1, x_2]$. *Let* z^a, z^b *be vectors such that* $z_1^a + z_1^b = x_1$ *and* $z_2^a = z_2^b = x_2$ *then*

$$(A * Bx, x) \leq (Az^a, z^a) + (Bz^b, z^b).$$

Proof: Since $x_1^a + x_1^b = x_1 = z_1^a + z_1^b$, there is a t such that $z_1^a = x_1^a + t$, $z_1^b = x_1^b - t$, therefore we have:

$$(Az^a, z^a) + (Bz^b, z^b) =$$

$$(A[x_1^a + t, x_2^a], [x_1^a + t, x_2^a]) + (B[x_1^b - t, x_2^b], [x_1^b - t, x_2^b])$$

$$= (A * Bx, x) + (A_{11}t, t) + (B_{11}t, t) +$$

$$2 \text{ Re } \{(A_{11}x_1^a, t) + (A_{12}x_2^a, t) - (B_{11}x_1^b, t) - (B_{12}x_2^b, t)\}$$

$$= (A * Bx, x) + (A_{11}t, t) + (B_{11}t, t) \quad \text{by 1a), 1b), 1g)}$$

$$\geq (A * Bx, x) \qquad \text{Since } A_{11} \text{ and } B_{11} \text{ are HSD}$$

QED

When $(Ax, x) \leq (Bx, x)$ for all x, we will write $A \leq B$. It is clear that this is a partial order on the set of Hermitian semidefinite matrices.

Parallel addition is a special case of hybrid addition. The following theorem gives a generalization of the series-parallel inequality given by Anderson and Duffin [2]. We term this new result the <u>series-hybrid inequality</u>.

<u>Theorem 14</u>: <u>If</u> A,B,C, <u>and</u> D <u>are</u> HSD <u>then</u>

$$(A + B) * (C + D) \geq A * C + B * D$$

<u>Proof</u>: There are x_1^{ab}, x_1^{cd} such that $x_1 = x_1^{ab} + x_1^{cd}$ and

$$((A + B) * (C + D)[x_1,x_2],[x_1x_2]) =$$

$$((A + B)[x_1^{ab},x_2],[x_1^{ab},x_2]) + ((C + D)[x_1^{cd},x_2],[x_1^{cd},x_2])$$

$$= (A[x_1^{ab},x_2],[x_1^{ab},x_2]) + (C[x_1^{cd},x_2],[x_1^{cd},x_2]) +$$

$$(B[x_1^{ab},x_2],[x_1^{ab},x_2]) + (D[x_1^{cd},x_2],[x_1^{cd},x_2])$$

$$\geq (A * C [x_1,x_2],[x_1,x_2]) + (B * D [x_1,x_2],[x_1,x_2])$$

by Lemma 13.

QED

Letting $D = 0$ in Theorem 14, we have $(A + B) * C \geq A * C$ for all A,B,C that are HSD. Since $A + B \geq A$ for all B, we have the following corollary:

<u>Corollary 15</u>: <u>If</u> A,B <u>and</u> C <u>are</u> HSD, <u>and</u> $A \geq B$, <u>then</u> $A * C \geq B * C$.

Corollary 15 shows that hybrid addition preserves the Hermitian semidefinite partial order. There is a duality between the series operation (+) and the parallel operation (:). Thus Theorem 14 has the following dual theorem.

<u>Theorem 16</u>: <u>If</u> A,B,C, <u>and</u> D <u>are</u> HSD <u>then</u>

$$(A : B) * (C : D) \leq (A * C) : (B * D).$$

We term this new result the <u>parallel-hybrid inequality</u>. The proof is omitted.

5.0 <u>SHORTED</u> <u>OPERATORS</u> The shorted matrix has been defined and studied by Anderson [3]. Let A be partitioned as before, then the <u>shorted</u> <u>matrix</u> of A, denote $S(A)$, is defined by:

$$S(A) = \begin{bmatrix} 0 & 0 \\ 0 & A_{22} - A_{21}A_{11}^{+}A_{12} \end{bmatrix}.$$

If A is the impedance matrix of a passive resistive network, then $S(A)$ is the impedance matrix resulting from shorting out the first r ports (A_{11} is $r \times r$).

From the definition of $A * B$, it is easily seen that:

$$A * B = S(A) + S(B) + \begin{bmatrix} D & DF \\ F^{*}D & F^{*}DF \end{bmatrix}.$$

Let P and P' be the orthogonal projections onto the complementary subspaces corresponding to the partitioning of A and B.

<u>Theorem 17</u>: $S(A * B) = S(A) + S(B)$.

<u>Proof</u>: It is sufficient to consider $P'S(A * B)P'$.

$$\begin{aligned} P'S(A * B)P' &= P'[S(A) + S(B)]P' + F^{*}DF - F^{*}DD^{+}DF \\ &= P'[S(A) + S(B)]P' \end{aligned}$$

QED

Using Theorem 17, Lemma 4 may be rewritten as:

$$P'[S(A) + S(B)]P' + F^{*}DF = A_{22} + B_{22} - (A_{21}-B_{21})(A_{11}+B_{11})^{+}(A_{12}-B_{12})$$

The right hand side of the above is $P'S(A+\overline{B})P'$. $\overline{B}$ is B with B_{12} and B_{21} replaced by $-B_{12}$ and $-B_{21}$. The above formula gives:

Theorem 18:

$$S(A+\overline{B}) = S(A) + S(B) + \begin{bmatrix} 0 & 0 \\ 0 & F^*DF \end{bmatrix} .$$

Since $S(\overline{B}) = S(B)$, and B is arbitrary, Theorem 18 gives an Hermitian semidefinite bound for $S(A+B) - S(A) - S(B)$. This bound is guaranteed by Anderson [3] but is not explicitly determined. It should also be noted that since $A * 0 = S(A)$, the inequality $S(A) + S(B) \leq S(A+B)$ follows directly from Theorem 14.

6.0 GENERALIZATIONS AND RELATED WORK The ideas used in developing hybrid addition can be used in more general situations. It would seem that any electrical interconnection of n-ports will give rise to a corresponding Hermitian semidefinite matrix operation. For example, the cascade connection of two n-ports is a very common connection in electrical engineering [4,5,10]. Figure 4 shows the cascade connection of two 2-ports. The basic ideas of hybrid addition can also be used to define an Hermitian semidefinite matrix operation, cascade addition. This will not be considered here.

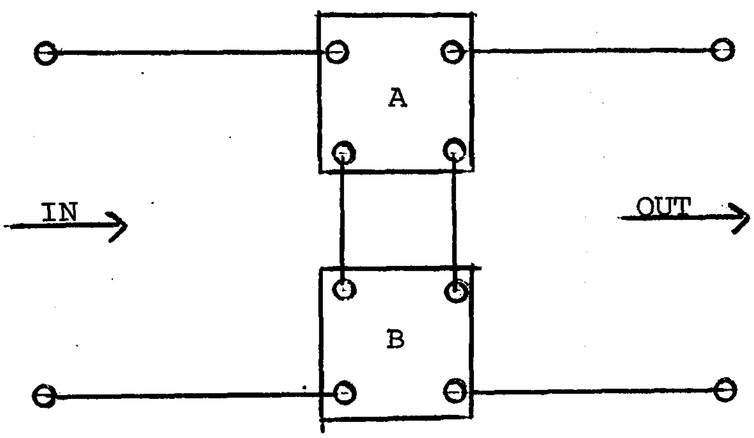

Figure 4: Cascade connection of two 2-ports

Anderson [1] and Lewis and Newman [8] have considered 'almost positive definite' matrices. Hybrid addition can be extended to almost positive definite matrices but the power inequality of Lemma 13 is not necessarily true and the succeeding matrix inequalities need not hold.

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