THE ZASSENHAUS LEMMA FOR CATEGORIES

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1. Introduction* The two Noether isomorphism theorems and the Zassenhaus Lemma for groups or rings can be considered as asserting that certain diagrams involving kernels and cokemels are commutative. In this form, the three results are clearly valid for every abelian category. However, groups and rings do not form abelian categories, and thus the question arises to find a larger class of categories for which the three theorems, and hence also the theorems of Jordan-Hölder and of Schreier, remain valid* Abellanas [l] has recently taken up this question, but he proves the Zassenhaus Lemma only for quotient objects of normal subobjects of a fixed object* In this note (which does not depend on [l]), we modify the method of Abellanas [l] to obtain the desired theorems in full generality, under assumptions which are actually weaker than those of [l]«

The categories to be considered must be pointed*, i.e. have zero morphisms*, and we avoid unnecessary complications by assuming that they have kernels and cokemels* This is not enough; we must require some vreak exactness conditions in the sense of [4]* The conditions used in this note are strictly weaker than those of [4], but not self-dual*

We use in general the notations and terminology of [3] for categories; exceptions to this rule will be duly noted*

2. Assumptions and examples. We assume throughout this note that & is a non-empty category which satisfies the following four conditions•

Z 0. $\mathcal{h}$ is pointed, i.e. *& has zero morphisms.

Z 1, ^ has inverse images, cokeraels, and cointersections of cokemels.

Z2, If em is defined in % for a cokernel e and a kernel m , then $e\,m * m^{f}\,e^{1}$ for a cokernel e^{f} and a kernel m^{f} .

Z *3.* If em is defined in *& for a cokernel e and a monomorphism m_{f} then $e\,m = m^{f}\,e^{f}$ for a cokernel e^{1} and a monomorphism m^{f} •

These are all the assumptions which we shall need. We note that Z 0 and Z 2 are self-dual; Z 1 and Z 3 are not. If k : A –> Z is a cokernel of an identity morphism id A of ^ , then Z is clearly a zero object of If . Thus \f has a zero object as well as zero morphisms, and the zero morphisms are those which factor through a zero object Z . By Z l_{f} the inverse image of 0 : Z –> B exists for every morphism f : A –> B of If . Thus if has kernels as well as cokemels. We shall denote by tor f the class of all kernels, and by coker f the class of all cokemels_{f} of a morphism f of if .

For images and coimages, we shall use the definitions of [2]. Thus an image of f £ ? , with respect to a class £ of morphisms of £ , is a pair $(e_{f}m)$ such that (i) f « m e in if and m e^ , and (ii) whenever $v\,f = m^{f}\,u$ in ^ with m^{1} $£^_{f}$ then u « x e and $v\,m = x\,m^{1}$ in ^ for a unique x 6 ^ , Coimages are defined dually. If ^ consists of monomorphisms of ^, then it suffices for (ii) to find x 6 ^ such that $v\,m$ « $m^{f}\,x$; the remainder of (ii) follows. If $\hat{y}$> is the class of all monomorphisms or the class of all kernels of % , then inverse f-images in the sense of [2] are inverse images in the usual sense, i.e. pullbacks of monomorphisins. The following lemma shows that Z 3 furnishes all the images and coimages which we shall need.

Lemma 2.1. If $f = me$ ^ ^ for a cokernel e and a monomorphism m, then (e, m) is an image of f for the class of all monomorphisms of £*, and also a coimage of f for the class of all cokernels of *)f •

We shall call (e, m) a normal coimage« or a conormal image, or shortly a conim. of f in this situation. The terms normal image« conormal coimage« and shortly ninu describe the dual situation.

Proof* If k £ ker f, then also k €. ker e, and e £ coker f. If we have $m^f u = v f$, then $m^1 u k * 0$, and $uk = 0$ if m^f is monomorphic. Thus $u = x e$ for a unique x £ ^, and $m^1 x * v m$ follows easily•

Let now $f u = v e^f$, where e^f is a cokernel of g in ^ • We have $f u g = v e^f g = 0$, and hence $e u g$ « 0 • Thus $e u = x e^1$ for a unique x €. *l?, and $v = m x$ follows easily!

Corollary 2.2, I£ $e m = m^f e^f$ in *£ for a kernel m_f a cokernel e_9 an eDimorphism e^1 and a monomorphism $m^f{}_9$ then m^f is a kernel and e^f a cokernel.

Proof. *By* Z 2_f $e m = m_1 e_1$ for a kernel nu_1 and a cokernel e^,_ . Then (e_1, m_1) is a nim and a conim by 2#1 and its dual. Thus $nu_1 = m^f x$, $e^f * x e^*_1$, and $e_1 = y e^f{}_t$ m^f « $m_1 y$ for morphisms $x_f y$ of ^€ • It follows easily that x and y are inverse isomorphismsB

Examples. In all examples which follow, Z 0 - Z 3 and the dual of Z 1 are valid. In all examples but one, the dual of Z 3 is also valid.

Groups, loops, and groups or loops with a fixed family of operators are familiar examples. In these examples, epimorphisms are surjective homomorphisms, and every epimorphism is a cokernel. Rings without unit element, with ideals as normal subobjects and surjective homomorphisms as cokernels, form another example.

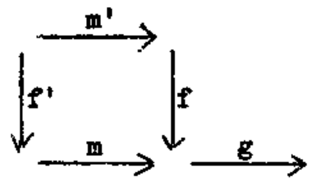
Pointed sets, i.e. sets with a basepoint and mappings which preserve basepoints, satisfy Z 0 - Z 3 and their duals. Monomorphisms are injective maps, and every monomorphism is a kernel. Epimorphisms are surjective maps, but not every epimorphism is a cokernel.

Pointed topological spaces and pointed Hausdorff spaces are also examples. In both cases, the kernel of a map $f : A \to B$ is the subspace inclusion map from $f^{-1}(e_B)$ to A, where e_B is the basepoint of B. Cokernels are the quotient maps which shrink a subspace to a point and are injective outside of that subspace. The dual of Z 3 is satisfied for topological spaces, where every subspace inclusion $A' \to A$ is a kernel, but not for Hausdorff spaces, where only closed subspaces A' of A, and not all closed subspaces if A is not regular, define kernels of maps f with source A.

2. Diagram lemmas. The results of this section are well known for abelian categories; we state and prove them under much weaker assumptions. We shall need two pullback lemmas, the Nine Lemma, and two forms of the Six Lemma.

As we pointed out in [4], exactness in non-abelian categories is not self-dual. For the present note, however, we need only a special, and self-dual, aspect of exactness. We call a pair (m, e) short exact if $m = \ker e$ and $e = \operatorname{coker} m$. It will also be convenient to call a commutative square $me = e'm'$ a conim square if two opposite sides e, e' are cokernels, and the other sides m, m' monomorphisms.

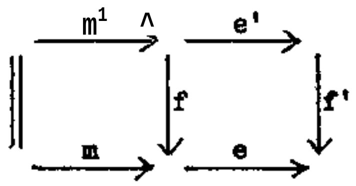
Lemma 3.1. If the square in



is a pullback and $m \in \ker g$, *then* m' *is a kernel of* gf.

This is well-known; we omit the straightforward proof.

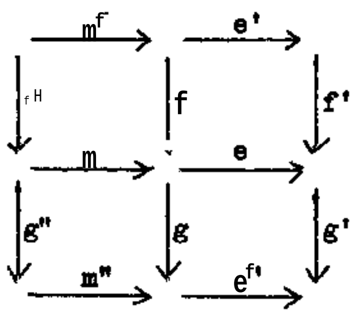
Lemma 3.2. If the righthand square in



is a pullback, with f *and* f' *monomorphic, and if* $m \in \ker e$, *then* m' *with* $m = f m'$ *exists to complete the diagram, and* $m' \in \ker e'$.

Proof. We have $e m = f' z$ for a zero map z, and thus m' exists, and $e' m' = 0$. If $e' x = 0$, then $e f x = f' e' x = 0$, and thus $f x = m x' = f m' x'$ for a unique $x' \in \mathcal{A}$, and $x = m' x'$ follows.

Proposition 3.3. For a commutative diagram



in $\mathcal{A}$, *the following two statements are logically equivalent.*

(i) A?.l rows and columns are short exact pairs,

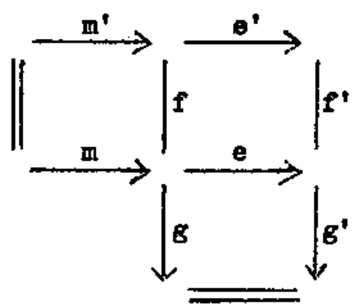
(ii) The middle row and the middle column are short exact, the tot) left SQuare is a pullback and the bottom right square a pushout« f^1 and m^M are monomorphisms« and e^1 and g^{fl} are epimorphisms,

groof. If (i) is satisfied, let $f x = m y$. Then $m^{fl} g^H y$ « g f x « 0 , and thus g" y « 0 f and y « f" z for a unique z £ ^ • It follows easily that also $x = m^f z$. The pushout part of (ii) is proved dually, and the other parts of (ii) follow immediately from (i)«

Conversely, if (ii) is satisfied, then m^f is a kernel of $e f = f^f e^f$ by 3»lt and hence also a kernel of e^1 • e^f is a cokernel by Z 2 and 2«2f and thus $(m^f.e^f)$ is short exact, (f'',g^{fl}) is short exact for the same reason, and $(f^f{}_f g^f)$ and (m^{fl},e^{lf}) are short exact for the dual reasonjf

We note that the diagram of 3»3 is determined up to isomorphisms in the four corners by the middle row (m,e) and the middle column (f,g) . If these are short exact, then the pullback square and the pushout square can be constructed by Z 1, and the other two squares exist by Z 2.

Proposition 3«4« In the commutative diagram



let the top right square be a conim square and g^f £ coker f^f , and let (m,e) be short exact> Then (m^f,e^f) is short exact, g £coker f , and the top right square ie a wtshout.

Proof. The top left square is a pullback. Thus m^f is a kernel of f^f e^f , Since f^f is monomorphic and e^f a cokernel$_f$ (m^e^1) is short exact*

g is epimorphic, and g f = g^1 $f^!$ e^1 « 0 , If xf = 0 , then x m « 0 $_f$ and x * y e for sane y £ £, with y f^f e^f = x *t* = 0 , and hence y f• » 0 . Now y s z g^1 for some z £ %, and x = z g • Thus g <£coker f • If u f = v e^1 , then u m = v e^f m^f « 0 , end u = z e for a unique z *Q* 𝒞. One sees easily that v = z f , and thus the top right square is a pushout|

Proposition 5*5• In the commutative diagram of 3»4, let (m,e) be short exact and g^f *&* coker f* , let, f be a kernel, and let e^1 be epimorphic and f^1 ponomorpit^L^• Then $(m^f{}_f e^f)$ ^nd both columna of %b^_^jflffram fire short exact$_f$ and the top right square is a pullback and a pushout*

Pfroof. Except for the pullback part, eveiything follows from 2.2 and *3A.* This is a self-dual situation, and the proof of the pullback part is dual to the proof of the pushout part of 3.4|

4. Operations on subobjects. We define a subob.iect of an object C of if as a monomorphism with target G . We put a^1 ^ a for subobjects a , a^1 of G if a^1 » a u for some u £ ^ , and we put a ' 4 a if this u is a kernel* We call a and a^1 equivalent, in signs a£^a^f , if a^1 ^ a and a ^ a^1 , i.e. if a^f « a u for an isomorphism u of ^ . The preordered class of subobjects of G has finite intersections, by Z 1. a f) b , for subobjects a , b of G , is obtained by the usual pullback construction. Like the operations constructed below, a ^ b is defined only up to equivalence.

If A—^-^0 and B *—b—> G are subobjects of G , then we may denote by A o B the source of a *f)* b , and we shall use similar notations for other operations on subobjects. If N—£-£ G is a kernel, then we denote by G/N the target of a cokernel of n • Objects like A *r*\ B and G/N are defined only up to an isomorphism of *%* , and they **are** constructed from morphisms of *$* and not from objects*

If G—^->H is a cokernel, and if A—=->G is a subobject of G and **B—£->H one of H , then we define subobjects f[a] of H and f"¹[b] of G , with sources f[A] and f"¹[B] , by commutative squares**

(4.1)

A——>f[A] f^O]——>B

la If[a] and |f^[b] Jb ,

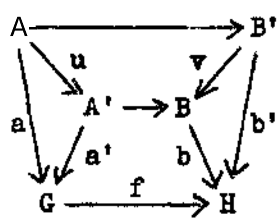
G——£—a> H G —-£—>H

where the square at left is a conim square and the one at right a pullback. *By* Z 1 and Z 3, these squares always exist in **€* .

<u>Proposition 4,2</u>. <u>The mappings</u> af—>f[a] <u>and</u> b|—^f⁻¹[b] <u>preserve order and satisfy</u> f[a]^'b <£=!> a ^ f~¹[b] .

In other words, we have a covariant Galois correspondence.

<u>Proof</u>. Consider a commutative diagram



If a¹ as f⁻¹ [b] , then the lower quadrangle is a pullback, and u exists if v exists• This shows that b h-> f"¹[b] preserves order, and that f[a] <b ⟹ a 4 f⁻¹[b] . If b^f « f[a] , then the outer quadrangle is a conim square, and it follows from 2.1 that v exists if u exists. This shows that a V-> f[a] preserves order, and that a ^ f"¹[b] =^> f[a] ≤! b |

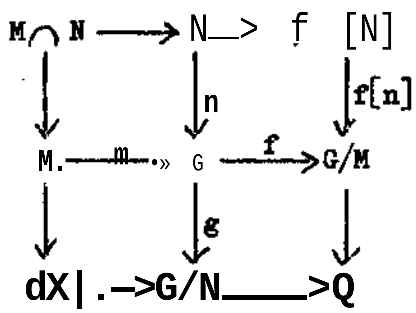
If N —a→'G is a kernel and A —a—5>G a subobject of G , then we put

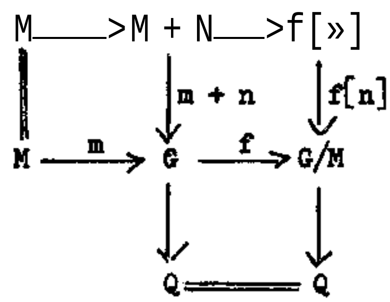
(4.3) n + a = f-HfU]] : N +G A —> G t

for a cokernel f of n . This defines a subobject n + a of G up to equivalence. We omit the subscript G in $N +_G A$ whenever the circumstances permit it. We note that a ^ n + a by 4.2, and that n ^ n + a by 3.2.

<u>Proposition 4.4</u>. Let M —m→ G and N —^D^ G be kernels, with cokernels G – i - ^ G/M <u>and</u> G –^-^ G/N . <u>Then</u> m -f nit n + m , <u>and</u> m + n <u>is a kernel of the cointersection of</u> f <u>and</u> g •

<u>Pjroof</u>. We need the following two commutative diagrams.



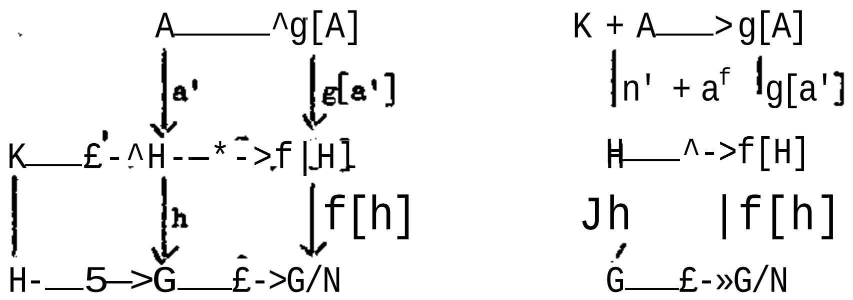


At left, G —*—^>Q is the cointersection of f and g , and all rows and columns are short exact by 3.3• At right, the middle row and the right column are from the diagram at left, and the top righthand square is a pullback. Thus m + n is a kernel by 3.1. f[m + n] rSftf^tfCnlllriffEn] by 4.2, and thus the top righthand square is also a conim square. Now m + n and G – > Q form a short exact pair. If we construct the diagram at right with m and n, and f and g, interchanged, then we see that $n + m$ and $G \to Q$ also form a short exact pair, $n + m \cong m + n$ follows.

<u>Proposition 4.5</u>. <u>Let</u> $N \xrightarrow{n} G$ <u>be a kernel, and let</u> $A \xrightarrow{a} G$ <u>and</u> $H \xrightarrow{h} G$ <u>be subobjects of</u> G <u>such that</u> $a \leq h$. <u>If</u> $n \cap h = h n'$ <u>and</u> $a = h a'$, <u>then</u> n' <u>is a kernel, and</u> $n + a \cong h\,(n' + a')$.

This means that the sources satisfy $N +_G A \cong N +_Q ((N \cap H) +_H A)$.

<u>Proof</u>. We put $K = N \cap H$ and consider the following two diagrams.



At left, the bottom squares are a pullback and a conim square, and the top square is a conim square. The middle row (n', g) is short exact by 3.1. The rectangle on the right is a conim square, and thus $g[A] \cong f[A]$. At right, the top square is a pullback, and $g[n' + a'] \cong g[g^{-1}[g[a']]] \cong g[a']$ by 4.2. Thus the top square also is a conim square, and the rectangle is a conim square, and

$$f[K + A] \cong g[K + A] \cong g[A] \cong f[A]$$

follows. Applying f^{-1} on both sides, we obtain $N +_G (K +_H A) \cong N +_G A$.

<u>Proposition 4.6</u>. <u>If</u> $N \xrightarrow{n} G$ <u>is a kernel and</u> $A \xrightarrow{a} G$ <u>a subobject of</u> G, <u>and if</u> $a = h a'$ <u>and</u> $n = h n'$ <u>in</u> $\mathcal{A}$ <u>for a kernel</u> $H \xrightarrow{h} G$, <u>then</u> n' <u>is a kernel</u>, $n + a \cong h\,(n' + a')$, <u>and</u> $N +_G A \cong N +_H A$.

Proof, We use the proof of 4.5 with K • N . Since h is a kernel, 3.5 applies, and the common lower square of the two diagrams is a pullback. Thus the rectangle at right is also a pullback, and N +G A£^ N +fi A followsj

5. Isomorphism theorems

Theorem 5.1. If N –n→ G is a kernel and A –a→ G a subobject of G, then (n n a) 4 a and n O (n + a) . I£ K ^ N o A and S ££ N + A , then A/K £? f[A] £ S/N for a cokernel G –^ G/N of n .

Proof. Consider two commutative diagrams

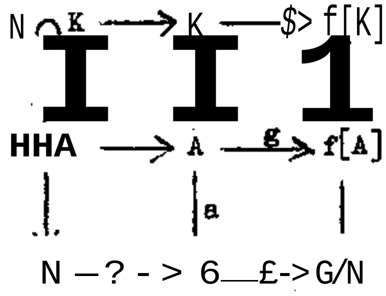
```
K_______>A______>f[A]          N ——> S ——> f[A]
|        |a       |f[a]    and  ||     |n + a  |f[a]   .
N____2 L^G____i-> G/N          N____2~> G --£->G/N
```

The squares at left are a pullback and a conim square, and the top row is short exact by 3*1. At right, the righthand square is a pullback, and by 4.2 also a conim square, and the top row is short exact by 3.2|

Theorem 5.1 is the first Noether isomorphism theorem. It is well known that the second Noether isomorphism theorem is equivalent to the Six Lemma 3.5. Thus both isomorphism theorems are valid under our assumptions. We turn now to the Zassenhaus Lemma.

Proposition 5.2, Let A –a→ G and K –k→ G be subob.iects of G such that k < 3 a , If H –n→ G –^ G/N is short exact, then f induces a kernel f [K] –^ f [A] , and f [A]/f [K] ^ A/K if n ^ a ^ k .

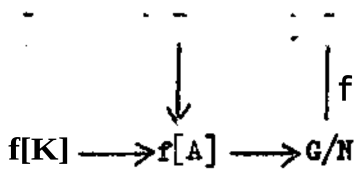
Proof. We consider the following commutative diagram.



We have pullback squares on the left, conim squares on the right, and f[K] •——> f[A] is a kernel by 2,2. If n n a ^ k , then n n k C f n 0 a f and 3.4 applies. Thus q g is a cokernel of K–3>k if q is a cokernel of f[K]——> f[A] , and A/K ^ f[A]/f[K] results(J

Proposition 5.3. <u>Let</u> N —n— G —f— G/N <u>be short exact, and let</u> A – 2 _ ^ G <u>and</u> K——k^> G <u>be subobjects of</u> G <u>such that</u> k 4 a . <u>If</u> n + a <u>fo</u> B –^– G <u>and</u> n + k £± L —^> G , <u>then</u> -/^J b , aM B/L ^ f[A]/f[K] .

<u>Proof</u>. We have a commutative diagram

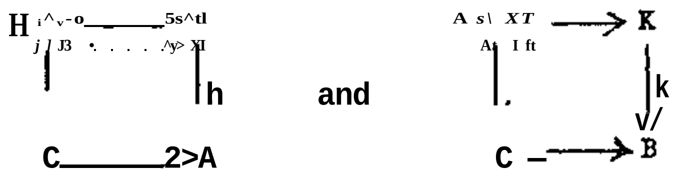


with pullback squares. f[K] ——> f[A] is a kernel by 5.2, and thus L – > B is a kernel by 3.1. We have t[b]£z f[f HfCa]]] ^ f[a] by 4.2, and similarly t[t] £z f[k] . Since b n n ^ n ≤ ℓ , we have B/L £ f[B]/f[L] by 5.2, and thus also B/L ^ f[A]/f[K] |J

We consider now subobjects A – a > G and B——b-> G of an object G , and kernels H——h> A and K ——k> B . We put C ^ A ∩ B , and

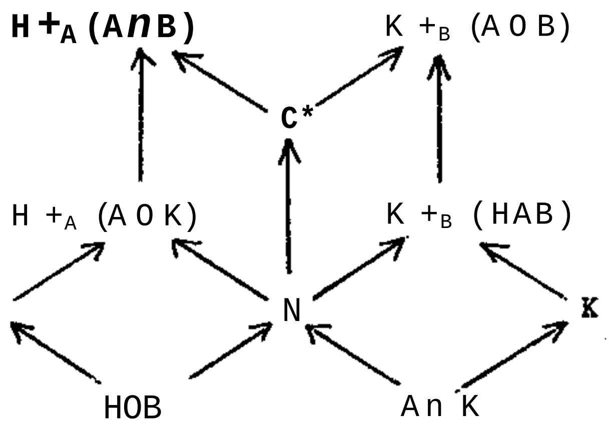
$$N \cong (H \cap B) +_C (A \cap K) \cong (A \cap K) +_C (H \cap B) ,$$

using 4.4* This is justified since the squares



are pullbacks, with kernels at right, and hence with kernels at left by 3#1« Thus the familiar "butterfly diagram" can be constructed from our data, in the following form.

(5.4)



Now we state the Zassenhaus Lemma as follows.

Theorem 5*5* If A —a^ G and B—b J> G are subobjects of an object G J2£ % t and H — h ^ A ^ffli K—k^B kernels in *£f then (5«4) is defined> and the three vertical lines in (5«4) represent kernels* with isomorphic quoti^flt objects.

Proof. N — ^ C is a kernel by 4.4, and E +A N —£>H +A C is a kernel by 5.3, with H + C/H + N £ f[c]/f[N] for a cokernel A —^ A/H of h . But H n C ^ H n B ^ H , and thus f[c]/f[N] fe C/N by 5.2. Now

H +A N £ H +A ((HnC)+c(AAK)).^ H +A (AnK) by 4*5. Thus the left vertical arrow in (5.4) is a kernel, and the corresponding quotient object is isomorphic to c/N . Sine? (5.4) is symmetric by 4.4, the same argument shows that the right vertical arrow in (5.4) is a kernel, with quotient object isomorphic to C/N §

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