EXTENDING LINEAR
SPACE-VALUED FUNCTIONS

by

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Section I: Introduction, basic definitions, and preliminary theorems

As the title states, we are concerned with the extension from a subspace to the whole space of continuous functions whose functional values lie in a linear topological vector space. We relate this concept to that of extending a continuous pseudometric from the subspace. The latter concept has been shown to be useful in many spaces, especially collectionwise normal spaces, normal spaces, and paracompact spaces. (For example, see [1], [10], and [14].) Recently it has been closely associated with the concept of a continuous uniformity (see [2]) and with the concept of a Z-embedded subspace (see [3]). By now relating it to the extension of linear space-valued functions we are able to give interesting characterizations of normal spaces and collectionwise normal spaces, as well as characterizations of the Hewitt realcompactification and the Stone-Čech compactification of a Tichonov space.

Also a negative reply can be given to a problem posed in [5]. A continuous function from a closed subset of a normal space into a bounded closed convex subset B of a Banach space cannot necessarily be extended continuously to the whole space with values still remaining in the subset B . For uniform subspaces S of a uniform space X , the situation is different. Every uniformly continuous function from S into a Fréchet space can be extended to a continuous function on X . This improves a result given in [10].

Let S be a nonempty subset of a topological space X and let γ be an infinite cardinal number. A pseudometric d is said to be γ-separable if there is a subset A of X having cardinality not greater than γ such that A is dense in $\mathfrak{J}_d$, the pseudometric topology generated by d. The pseudometric d is said to be totally bounded if for every $\epsilon > 0$, there is a finite subset F_ϵ of X such that X is contained in the union of the d-spheres with radius ϵ about the points of F_ϵ. The subset S is P-embedded (respectively P^γ-embedded, T-embedded) in X if every continuous (respectively γ-separable continuous, totally bounded continuous) pseudometric on S can be extended to a continuous (respectively γ-separable continuous, totally bounded continuous) pseudometric on X.

The following relationships concerning these embeddings are known. (α) The subspace S is P-embedded in X if and only if it is P^γ-embedded in X for all infinite cardinal numbers γ. (See Theorem 2.8 of [14].) (β) The subspace S is $P^{\aleph_0}$-embedded in X if and only if it is C-embedded in X, (that is, every continuous real-valued function on S extends to a continuous real-valued function on X). (For a proof of the necessity see Theorem 2.4 of [10] and for the proof of the sufficiency see Theorem 4.7 of [14]). (γ) The subspace S is T-embedded in X if and only if it is C^*-embedded in X, (that is, every bounded real-valued continuous function on S extends to a continuous function on X). In [1], Alo and Shapiro showed that if S is T-embedded in X, then it is C^*-embedded in X and that the converse is true if the space in question is a Tichonov space. In Section 2 we show that this requirement can be dropped.

A family $(S_\alpha)_{\alpha \in I}$ of subsets of X is discrete in X if every $x \in X$ has a neighborhood meeting at most one member of the family. A space X is collectionwise normal if for every discrete family $(F_\alpha)_{\alpha \in I}$ of closed subsets of X, there is a pairwise disjoint family $(G_\alpha)_{\alpha \in I}$ of open subsets of X such that $F_\alpha \subset G_\alpha$ for all $\alpha \in I$. Similarly, X is γ-collectionwise normal if the above condition holds for families $(F_\alpha)_{\alpha \in I}$, where $|I| \leq \gamma$. It is known that $\aleph_0$-collectionwise normality is equivalent to normality. C. H. Dowker in [8], (see also [12]) implicitly showed the relation between P-embedding and collectionwise normality. We give this here as follows: (δ) A space X is collectionwise normal if and only if every closed subset of X is P-embedded in X. An explicit proof of this is given as Theorem 5.2 of [14].

This result considered together with results (β) and (γ) mentioned above raises an interesting question. How can P-embedding itself be intrinsically related to the extension of continuous functions? Our results will show that linear space-valued functions are the key to answering this question.

In [14], the following result was shown. (ϵ) If S is a dense C-embedded subset of a Tichonov space X and if the cardinality of S is non-measurable, then S is P-embedded in X. Thus we will also obtain results concerning the Hewitt realcompactification of a completely regular T_1 space with non-measurable cardinality.

The terminology and basic results we need concerning open covers are given in [14]. One of our techniques will be to map the subspace S into the space $C^*(S)$ of all bounded real-valued continuous functions on S. To this end, for a given bounded continuous pseudometric d on S we define the partial functions $(d_x)_{x \in S}$ associated with d as follows: $d_x(y) = d(x,y)$ for all $y \in S$.

We first state the major theorem of [14], as it will be frequently used in what follows.

Theorem 1.1: Let S be a nonempty subspace of a topological space X and let γ be an infinite cardinal number. The following statements are equivalent:

(1) The subspace S is P^{γ}-embedded in X.

(2) Every γ-separable bounded continuous pseudometric on S can be extended to a continuous pseudometric on X.

(3) Every locally finite cozero set cover of S of power at most γ has a refinement that can be extended to a normal open cover of X.

A characterization of T-embedding given in [1] will be stated here for convenience.

Theorem 1.2: If S is a nonempty subspace of a topological space X, then the following statements are equivalent:

(1) The subspace S is T-embedded in X.

(2) Every totally bounded continuous pseudometric on S can be extended to a continuous pseudometric on X.

(3) Every finite cozero set cover of S has a refinement that can be extended to a normal open cover of X.

We also need the following result. A proof is given in [15]. For notation and results on uniform spaces, see [11].

Theorem 1.3: Let S be a nonempty uniform subspace of a uniform space X. Every uniformly continuous pseudometric on S has an extension to a continuous pseudometric on X.

Section II: Extension of linear space-valued functions

Definition 2.1: Let S be a subset of X and Y a subset of Z, where X and Z are topological spaces. If f is a continuous function from S into Z such that $f(S) \subset Y$, then f extends continuously to X relative to Y if there is a continuous function f^* from X into Z such that $f^*|S = f$ and $f^*(X) \subset Y$. We emphasize the phrase "relative to Y" to stress that in the extension process the space Y must not be enlarged.

For the sake of precision in our theorems we introduce the following definitions and notation.

Definition 2.2: Let X be a topological space, let γ be an infinite cardinal, and let f be a function from X to a locally convex topological vector space (abbreviated LCTV space). We call f a M-valued function if the image of X under f is contained in a complete convex metrizable subset M of L (abbreviated CCM subspace). The function f is a (γ,M)-valued function if it is an M-valued function and if the image of X under f is a γ-separable subset of M. Recall that a

Fréchet space is a complete, metrizable LCTV space. The set of all bounded real-valued continuous functions on X is a Banach space under the sup norm, i.e., $\|f\| = \sup_{x \in X} |f(x)|$, and will be denoted by $C^*(X)$.

The following theorems will characterize the extendability of M-valued functions and (γ,M)-valued functions. This topic was first studied by R. Arens in [6]. There he showed the equivalence of (1) and (2) in our next theorem for closed subsets of a topological space.

Theorem 2.3: Let S be a nonempty subspace of X , let γ be an infinite cardinal, and let A be a discrete space such that $|A| \geq |S|$. The following statements are equivalent:

(1) The subspace S is P^γ-embedded in X .

(2) Given a CCM subspace M of a LCTV space L , every continuous (γ,M)-valued function on S extends continuously to X relative to M .

(3) Given a CCM subspace M of a LCTV space L, every continuous (γ,M)-valued function on S extends to a continuous function from X to L .

(4) Every continuous function from S to a Fréchet space, such that the image of S is γ-separable, extends to a continuous function on X .

(5) Every continuous function from S into $C^*(S)$, such that the image of S is γ-separable, extends to a continuous function on X .

(6) <u>Every continuous function from S into $C^*(A)$, such that the image of S is γ-separable, extends to a continuous function on X.</u>

Furthermore, the above conditions are also equivalent to the conditions obtained from (2) through (6) by requiring the image of S to be a bounded subset of the locally convex space in question.

<u>Proof</u>: We first show that (1) <u>implies</u> (2). Let f be a continuous (γ,M)-valued function from S to the LCTV space L, and let m be a complete metric for M. One easily verifies that $d = m \circ (f \times f)$ is a γ-separable continuous pseudometric on S. Hence by (1) it has an extension to a continuous pseudometric d^* on X. The space (S,d) is a subspace of the pseudometric space (X,d^*). The function f is uniformly continuous as a mapping from (S,d) into (M,m), since $d(x,y) \leq \epsilon$ implies $m(f(x),f(y)) \leq \epsilon$ for all x,y in S and for all $\epsilon > 0$. The space (M,m) is a complete Hausdorff uniform space. Hence f extends to a uniformly continuous function f^* from the closure of (S,d) in (X,d^*) to (M,m). Now we have a continuous function from a closed subset of a pseudometric space into a LCTV space L.

A theorem of Dugundji's (see [9]) states that a continuous function from a closed subset of a metric space into a LCTV space can be extended to a continuous function on the metric space with values in the convex hull of the image of the subset. It is easily seen that his proof also applies to pseudometric spaces. Therefore, the function f^* extends to a continuous function g from (X,d^*) into M, since M is convex.

The function g is continuous with respect to the topology generated by d^*, and since this topology is contained in the original topology of X, the mapping g is the continuous extension of f relative to M that we seek.

The implications (2) implies (3) implies (4) implies (5) are immediate.

We now show that (5) implies (1). By Theorem 1.1 it is sufficient to show that every bounded γ-separable continuous pseudometric on S extends to a continuous pseudometric on X. Let d be a bounded γ-separable continuous pseudometric on S. Define a function f from S into $C^*(S)$ by $f(x)$ is the partial function d_x for all x in S. The function f is continuous since if $\epsilon > 0$ and $d(x,y) \leq \epsilon$, then for x,y in S $\|f(x) - f(y)\| = \sup_{z \in S} |f(x)(z) - f(y)(z)| = \sup_{z \in S} |d(x,z) - d(y,z)| = d(x,y) \leq \epsilon$. Let $(x_\alpha)_{\alpha \in I}$ be a dense subset of power at most γ of $\mathcal{J}_d$. It is easily verified that $(f(x_\alpha))_{\alpha \in I}$ is a dense subset of $f(S)$. Therefore by (5) the function f extends to a continuous function f^* mapping X into $C^*(S)$. Define a function d^* on $X \times X$ by $d^*(x,y) = \|f^*(x) - f^*(y)\|$ for all x,y in X. It is readily seen that d^* is a continuous pseudometric on X, and it extends the pseudometric d. In fact, if x and y are in S, then $d^*(x,y) = \|f(x) - f(y)\| = \sup_{z \in S} |f(x)(z) - f(y)(z)| = \sup_{z \in S} |d(x,z) - d(y,z)| = d(x,y)$. Thus S is P^γ-embedded in X.

The implication (1) implies (6) is now clear from the equivalence of (1) and (4).

To show that (6) implies (1) let d be a bounded continuous γ-separable pseudometric on S. Let G denote S with the discrete topology, and since the cardinality of the discrete space A is large enough, we may identify G with some copy of it in A. Therefore, from now on we will consider G as a subset of the space A. Define a function f from S into $C^*(A)$ by $(f(x))(a) = d(x,a)$ if a is in G and 0 if otherwise, for all x in S. Since A is discrete and d is bounded, $f(x)$ is an element of $C^*(A)$ for all x in F. The map f is continuous; if $\epsilon > 0$ and $d(x,y) \leq \epsilon$, then $\|f(x) - f(y)\| =$

$$\sup_{a\in A}|f(x)(a) - f(y)(a)| = \sup_{a\in G}|f(x)(a) - f(y)(a)| = \sup_{a\in S}|d(x,a) - d(y,a)| =$$

$d(x,y) \leq \epsilon$. Again it is easy to check that the γ-separability of d implies the γ-separability of $f(S)$. Hence by (6) the function f extends to a continuous function f^* on X. Defining d^* on $X \times X$ by $d^*(x,y) = \|f^*(x) - f^*(y)\|$ for all x,y in X, we see that d^* is a continuous pseudometric on X. By similar computations as those above, it is easily verified that d^* extends the pseudometric d. Therefore, S is P^γ-embedded in X.

To prove the last statement, call (2*) through (6*) the new conditions resulting from requiring the image of S to be bounded in (2) through (6), respectively. The implications (1) implies (2*) implies (3*) implies (4*) implies (5*) hold. The proof of (5*) implies (1) is as that of (5) implies (1) after one notes the following: By Theorem 1.1, it is sufficient to show that every bounded, γ-separable, continuous pseudometric on S extends to a continuous pseudometric on X . If d is bounded, then f(S) is a bounded subset of C*(S) . The implication (1) implies (6*) is clear; (6*) implies (1) is like (6) implies (1), noting that f(S) is a bounded subset of C*(A) .

Since a subspace S of a topological space X is P-embedded in X iff it is P^{γ}-embedded in X for all infinite cardinal numbers γ , it is clear that from Theorem 2.3 we obtain characterizations of P-embedding by removing all mention of cardinality. In particular, we obtain the following.

Corollary 2.4: Let S be a subspace of a topological space X . Then S is P-embedded in X iff every continuous function from S into a bounded, closed, convex subset of a Banach space extends to a continuous function on X .

Thus Corollary 2.4 gives an interesting characterization of collectionwise normal spaces when considered in conjunction with statement (δ). In particular, we can now characterize collectionwise normal spaces in terms of the extension from closed subsets of a particular class of continuous functions just as in the case of the Tietze Extension Theorem for normal spaces.

Corollary 2.5: A topological space X is collectionwise normal if and only if for every closed subset F of X , every continuous function from F into a bounded closed convex subset B of a Banach space can be extended continuously to X relative to B .

Corollary 2.5 answers in the negative a question of Arens (see p. 19 of [5]). Arens asked whether a continuous function from a closed subset of a normal space into a bounded closed convex subset of a Banach space could be extended continuously to the whole space with values still in the subset. But Corollary 2.5 shows that this is equivalent to the collectionwise normality of the space. Bing has given an example of a space that is normal but not collectionwise normal.

As mentioned in result (β), $P^{\aleph_0}$-embedding and C-embedding are equivalent. Therefore, Theorem 2.3 with $\gamma = \aleph_0$ gives characterizations of C-embedding. In particular we can improve the Tietze Extension Theorem by stating as a corollary the following slightly stronger version of a result of Arens given in [5].

<u>Corollary 2.6</u>: <u>For any nonempty topological space X the following are equivalent:</u>

(1) <u>The space X is normal</u>

(2) <u>For every closed subset F of X, any continuous $(\aleph_0, M)$-valued function from F into a CCM subset M of a LCTV space L can be continuously extended to X relative to M</u>.

We also obtain the following result about the Hewitt realcompactification υX of a completely regular T_1 space X. (See Theorem 8.7 of [11]).

<u>Corollary 2.7</u>: <u>The Hewitt realcompactification υX of a Tichonov space X is that unique realcompactification of X for which every continuous function f from X to a Fréchet space, such that f(X) is separable, can be extended to a continuous function on υX</u>.

Moreover, if X has non-measurable cardinality, then υX is that unique realcompactification of X such that every continuous function from X into a Fréchet space extends to a continuous function on υX .

Proof: The result follows from characterizations of υX given in, for example [11], and from Theorem 2.3 and (β) and (ϵ) of Section 1 .

We now show, as promised in the introduction, that T-embedding and C*-embedding are equivalent with no hypothesis on the space in question.

Theorem 2.8: If S is a subspace of a topological space X , then S is T-embedded in X if and only if S is C*-embedded in X .

Proof: From the introduction we know that if S is T-embedded in X , then it is C*-embedded in X . It remains to prove the converse. Let $\mathcal{U} = (U_i)$ $i = 1, \ldots, n$ be a finite cozero set cover of S . By Theorem 1.2 it is sufficient to find a normal open cover $\mathcal{V}$ of X such that $\mathcal{V}|S$ refines $\mathcal{U}$.

Since $\mathcal{U}$ is normal, there exists a cozero set cover $\mathcal{W} = (W_i)$, $i = 1, \ldots, n$ of S such that W_i is completely separated from $S - U_i$ for $i = 1, \ldots, n$. (See [13]) . Hence there are continuous functions (f_i) on S , $i = 1, \ldots, n$, such that $f_i(W_i) \equiv 0$ and $f_i(CU_i) \equiv 1$, and $0 \leq f_i(x) \leq 1$ for all $x \in S$. By assumption these functions extend to continuous functions (f_i^*) $i = 1, \ldots, n$ on X . For each i , let $V_i = \{x \in X | f_i^*(x) < \frac{1}{2}\}$. Each V_i is a cozero set of X as is $V_0 = X - \bigcup_{i=1}^{m} Z(f_i^*)$.

It is easy to verify that $\mathcal{V} = (V_0) \cup (V_i)$ $i = 1, \ldots, n$ is a (locally) finite cozero set cover of X (and hence normal) and that $\mathcal{V}|S$ refines $\mathcal{U}$.

The following theorem characterizes T and C*-embedding in terms of the extension of totally bounded linear space-valued functions.

Theorem 2.9: Let S be a nonempty subspace of X, and let A be a discrete space such that $|A| \geq |S|$. The following statements are equivalent:

(1) The subspace S is C*-embedded in X.

(2) Given a CCM subspace M of a LCTV space L, every continuous M-valued function f on S, such that f(S) is totally bounded, extends continuously to X relative to M.

(3) Given a CCM subspace M of a LCTV space L, every continuous M-valued function f on S, such that f(S) is totally bounded, extends to a continuous function from X to L.

(4) Every continuous function f from S to a Fréchet space, such that f(S) is totally bounded, extends to a continuous function on X.

(5) Every continuous function f from S into C*(S), such that f(S) is totally bounded, extends to a continuous function on X.

(6) Every continuous function f from S into C*(A), such that f(S) is totally bounded, extends to a continuous function on X.

Proof: We first show that (1) implies (2). Let f be a continuous M-valued function on S such that f(S) is totally bounded. Let m be a complete metric for M. Then $d = m \circ (f \times f)$ is a continuous pseudometric on S which is also totally bounded. To see this, let $\epsilon > 0$ and let F_ϵ be a finite subset of S such that $f(S)$ is contained in the union of the m-spheres with radius ϵ about the points $f(x)$, where $x \in F_\epsilon$. Then S itself is the union of d-spheres of radius ϵ centered about the points of F_ϵ. By Theorem 2.8 and (1) there exists a continuous pseudometric d^* on X that extends d. The proof now proceeds exactly as the proof of (1) _implies_ (2) of Theorem 2.3.

The implications (2) _implies_ (3) _implies_ (4) _implies_ (5) are immediate. To show that (5) _implies_ (1) it is sufficient to show, by Theorems 2.8 and 1.2 that every totally bounded continuous pseudometric on S extends to a continuous pseudometric on X. Let d be a totally bounded continuous pseudometric on S and define a function f from S into $C^*(S)$ by sending x into the partial function d_x for all x in S. As we saw in the proof of (5) _implies_ (1) of Theorem 2.3, the function f is continuous. The image of S under f is totally bounded. In fact if S is the union of the d-spheres of radius ϵ centered about the points of F_ϵ, then $f(S) \subset \bigcup_{x \in F_\epsilon} S(f(x))$, where $S(f(x)) = \{z \in C^*(S): \|z - f(x)\| < \epsilon\}$. The proof now proceeds in a similar fashion as (5) _implies_ (1) of Theorem 2.3.

If S is C^*-embedded in X, it is now clear that (6) holds. Conversely, if (6) holds the proof of (6) _implies_ (1) of Theorem 2.3 and the remarks above pertaining to $C^*(S)$ applied to $C^*(A)$ will give the result that S is C^*-embedded in X .

Since any completely regular T_1 space is C^*-embedded in its Stone-Čech compactification we obtain the following corollary to Theorem 2.9. (See Theorem 6.5 of [11]).

Corollary 2.10: The Stone-Čech compactification of a completely regular T_1 space X is that unique compactification βX of X for which every continuous function f from X to a Fréchet space, such that $f(X)$ is totally bounded, can be extended to a continuous function on βX .

We now obtain easily the following result on the extension of uniformly continuous functions with values in a Fréchet space. This improves a result in [10].

Theorem 2.11: If S is a nonempty subspace of a uniform space X and if L is any Fréchet space, then every uniformly continuous function from S into L can be extended to a continuous function on X .

Proof: Let f be a uniformly continuous function from S to a Fréchet space L with complete metric m . Then $d = m \circ (f \times f)$ is a uniformly continuous pseudometric on S . Hence by Theorem 1.3, d extends to a continuous pseudometric d^* on X . The proof now proceeds as in (1) implies (2) of Theorem 2.3.

In [10], an example is given which shows that we cannot request that the extended function also be uniformly continuous. In fact for the case in which the range space is the real line, the boundedness of the function is needed to insure uniform continuity of the extended function.

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