SETS OF FORMULAS
VALID IN FINITE STRUCTURES

by

Alan L. Selman

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SETS OF FORMULAS VALID IN FINITE STRUCTURES

<u>Abstract</u>

A function Γ is defined on the set of all subsets of ω so that for each set K, the value, Γ_K, is the set of formulas valid in all structures of cardinality in K. An analysis is made of the dependence of Γ_K on K. It is easily seen that for all infinite sets K, $d(K) \vee 1 \le d(\Gamma_K) \le d(K)'$. On the other hand, we prove that $d(\Gamma_{K \vee \omega}) = d(\Gamma_K) = d(\Gamma_\omega)$, and use this to prove that for any two degrees a and b, $a \geq 1$, $a \le b \le a'$, and b r.e. a, there exists a set K so that $d(K) = a$ and $d(\Gamma_K) = b$. Various similar results are also included.

SETS OF FORMULAS VALID IN FINITE STRUCTURES[1]

by

Alan L. Selman[2]

B. A. Trachtenbrot [8] has shown that the set of formulas of first order logic valid in all finite structures is not recursively enumerable, although it is the complement of such a set. Let us define a function $\mathcal{V}$ on the set of all subsets of ω so that for each set K, the value, $\mathcal{V}_K$, is the set of formulas valid in all structures of cardinality in K. A. Mostowski has asked (in conversation, 1966) what can be said of $\mathcal{V}_K$, when K is known. In particular, if the Kleene-Post degrees of the two sets K and J are identical, are the degrees of $\mathcal{V}_K$ and $\mathcal{V}_J$ identical?

Let $\bar{K}$ denote the complement of the set K. (The universe of discourse is ω throughout.) It is shown that for all infinite sets K, $d(K) \vee 0' \leq d(\mathcal{V}_K) \leq d(K)''$. Nevertheless, in section 3 it is shown that there exist sets K for which $d(\mathcal{V}_K) \neq d(\mathcal{V}_{\bar{K}})$. This solves the above question in the negative. In section 4 we describe the extent to which $d(\mathcal{V}_K)$ is independent from $d(K)$. The principal result in this direction is Theorem 12. The techniques used to obtain our results involve both the writing of explicit algorithms and the application of standard theorems about the degrees of unsolvability.

It is assumed that we have at our disposal some first order language, $\mathcal{L}$, with equality whose grammar contains an infinite list of k-ary predicate letters M_i^k, F_i^k, ..., $i \geq 1$, for each k. $\varphi, \theta, \ldots$ shall denote formulas of this language. $\varphi(M_1^1, \ldots, M_n^1, F^1, \ldots)$, $n \geq 1$ is a formula containing among its predicate letters one or more occurrences of the one-place predicates $M_1^1, \ldots, M_n^1, F^1$.

Let $\mathfrak{A}$ be an interpretation of the formula φ. $\mathfrak{A}$ is a structure with domain A and k-ary relations $M_i^{\mathfrak{A}}$ corresponding to predicate letters M_i^k occurring in φ. We write $\|A\|$, for the cardinality of a set A. By the cardinality of a structure $\mathfrak{A}$ we mean the cardinality of its domain. A structure $\mathfrak{A}$ is finite, if its domain is. We write $\models_{\mathfrak{A}} \varphi$, if φ is valid in $\mathfrak{A}$.

Also, we will use the notation "A $\leq_T$ B" for "A recursive in B" and "A $\leq_m$ B" for "A is many-one reducible to B".

Definition 1. Let $K \subseteq \omega$.

(i) $\Lambda_K = \{\varphi : \forall \mathfrak{A}[\|A\| \in K \rightarrow \models_{\mathfrak{A}} \varphi]\}$;

(ii) $\Omega_K = \{\varphi : \exists \mathfrak{A}[\models_{\mathfrak{A}} \varphi \ \& \ \|A\| \in K]\}$;

(iii) $\omega_K = \{\varphi : \forall \mathfrak{A}[(\models_{\mathfrak{A}} \varphi \ \& \ \|A\| < \omega) \rightarrow \|A\| \in K]\}$.

Lemma 1. $\varphi \in \Lambda_K \Leftrightarrow \neg\varphi \notin \Omega_K$. $\varphi \in V_K \Leftrightarrow \neg\varphi \notin \Omega_K$.

ω_K and Ω_K defined above have conceptual interest, and, by Lemma 1, for each set K, $d(\Lambda_K) = d(\Omega_K) = d(\omega_K)$. In fact, we prefer to analyze the function ω, since as is easily seen, for all K, ω_K is r.e. in K.

1. Trachtenbrot's Theorem.

This section is concerned with certain generalizations of Theorem 1 of [8].

Throughout this paper we equate computable with recursive. As an instance, given a formula $\varphi(M_1^1,\ldots,M_n^1,F^1,\ldots)$, $n \geq 1$, define $\rho(m_1,\ldots,m_n,j,k)$ to be 0, if φ has a model $\mathfrak{A}$ of cardinality k so that $\|M_i^{1\mathfrak{A}}\| = m_i$, for $i \leq n$, and $\|F^{1\mathfrak{A}}\| = j$, and 1 otherwise. ρ is recursive.

Definition 2. Let K be a non-empty subset of ω. A formula $\varphi(M_1^1,\ldots,M_n^1,F^1,\ldots)$ is a K-representation of an n-place function f if

(i) $\forall m_1,\ldots,m_n\, \exists j \exists k[\rho(m_1,\ldots,m_n,j,k) = 0 \;\&\; k \in K]$,

and

(ii) $\forall k,m_1,\ldots,m_n,j\, [k \in K \;\&\; \rho(m_1,\ldots,m_n,j,k) = 0 \rightarrow f(m_1,\ldots,m_n) = j]$.

The proof of the following theorem is immediate.

Theorem 1. If $\varphi(M_1^1,\ldots,M_n^1,F^1,\ldots)$ is a K-representation of f, then

$$f(m_1,\ldots,m_n) = j \leftrightarrow \exists k[k \in K \;\&\; \rho(m_1,\ldots,m_n,j,k) = 0].$$

Theorem 2. If f has a K-representation, then f is recursive in K. If f has a K-representation and K is r.e. in a set B, then f is recursive in B.

Proof. If f has a K-representation, then, using Theorem 1, the graph of f is r.e. in any set B which K is r.e. in. So f is recursive in K, and if K is r.e. in B, then f is recursive in B.

Corollary 1. If f has an <D-representation, then f is recursive.

Corollary 1 is due to Trachtenbrot.

Theorem 3. If B is an infinite set and f is recursive, then f has a B-representation.

Proof. The proof is essentially a repetition of the proof of Theorem 1 in [8]. It is shown in [8] that for each recursive function f there is an <u-representation <p. To complete the proof, it suffices to observe for each GO-representation <p, that if 21 is a model of <p with domain A, and if 31 is extended to a structure 9I' simply by enlarging the domain A, then 2I' is a model of <p. Since B is an infinite set, each <D-representation <p has a model of cardinality in B. Thus, *cp* is a B-representation.

Definition 3. The spectrum of a first-order formula *<p,* S(<p), is the set of all natural numbers n for which *cp* has a model of cardinality n.

It is well-known [1] that each *S(cp)* is an elementary set.

Let rng f denote the range of a function f.

Definition 4. The class of spectral functions of n-arguments, $Spr_n = \{ f: f \in {}^{(\omega^{\omega})}\omega$ & f has an ω-representation $\phi(M_1^1, \ldots, M_n^1, F^1, \ldots)$ so that rng f = $S((\phi(M_1^1, \ldots, M_n^1, F^1, \ldots) \wedge \forall x F^1 x)\}$.

Lemma 2. (1) The functions 2x, 2x+1, and x^2 belong to Spr_1. The function u + x belongs to Spr_2.

(2) Spr_1 is closed under substitution. More generally, if $g \in Spr_m$ and $f_1, \ldots, f_m \in Spr_n$, n,m > 0, then the function h defined by $h(x_1,\ldots,x_n) = g(f_1(x_1,\ldots,x_n),\ldots,f_m(x_1,\ldots,x_n))$ is contained in Spr_n.

Proof, (1) We again cite [8]. By that paper, the functions listed in (1) all have ω-representations. It is easy to see that these representations have the required property.

(2) Let f and g belong to Spr_1. Define h(x) = g(f(x)), f has ω-representation $\phi(M^1, F^1, \ldots)$ and g has ω-representation $\theta(M^1, F^1, \ldots)$, both satisfying Definition 4. By [8], h has ω-representation $\phi(M^1, G^1, \ldots) \wedge \theta(G^1, F^1, \ldots)$. Suppose y = h(x), for some x. $\phi(M^1, G^1, \ldots)$ has a model $\mathfrak{A}$ of cardinality $\|G^{\mathfrak{A}}\|$ = f(x). Extend (enlarge the domain) and expand (add additional relations) $\mathfrak{A}$ to a model $\mathfrak{B}$ of $\theta(G^1, F^1, \ldots)$ of cardinality $\|F^{\mathfrak{B}}\|$ = g(f(x)) = y. As observed in the proof of Theorem 3, $\mathfrak{A}$ can be extended to $\mathfrak{B}$ so that $\mathfrak{B}$ is still a model of $\phi(M^1, G^1, \ldots)$. Thus, $y \in S(\phi(M^1, G^1, \ldots) \wedge \theta(G^1, F^1, \ldots) \wedge \forall x F^1 x)$. It is immediate that if $y \in S((\phi(M^1, G^1, \ldots) \wedge \theta(G^1, F^1, \ldots) \wedge \forall x F^1 x)$, then $y \in$ rng h. Thus $h \in Spr_1$.

The proof of the second statement in (2) is identical.

2. Elementary Properties of to.

Suppose a Gödel numbering is given for the set of formulas of £ so that each number is used exactly once. Throughout this chapter let R(x,k) be the number theoretic predicate R(x,k) ≡ formula with Gödel number x has a model of cardinality k. R is a recursive predicate. Let ⌜<p⌝ denote the Gödel number of <p, and let $_t$Xj denote the formula with Gödel number x. For each set K, to_K = {(p:3k(R(<*</r ,$_x$) & keK) }. But, in what follows we will instead denote {x|3k(R(x,k) & keK) } by to .

Theorem 4. (1) For each set K, to_K r.e. K. In fact, VB [K e E_i^B → $\omega_K \in \Sigma_1^B$].

(2) If K is finite, then to_K is recursive.

(3) K £ S_n - l » K e E_n.

(4) K e n_n - io R e E_{n+1}.

(5) For each set K, K $<_m$ to_K .

Proof. The proofs of the first four clauses are immediate. Let E^n be a first order formula asserting the existence of exactly n distinct elements. x e K -*E^n_x "• e to_K • Thus, K $\leq_m$ to_K .

If <p is a formula with one free variable, let 31_n x <p be the formula asserting that there are exactly n distinct elements which satisfy <p.

Theorem 5. If every function recursive in K has a K-representation, then to_K is a completion of K. Thus, $d(to_K) = d(K)^f$.

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Proof. By Theorem 4, to r.e. K. Suppose P(x) r.e. K. P(x) ≡
K
K
3k[f^K (k) = x], where f_K is some function recursive in K. By
1 1

assumption f_K has a K-representation, say *cp(M* ,F ,...).

Let g(n) be the number theoretic function defined by

$$g(n) = <p(M \$F ,...) \ A \ 31 \ y F_n (y)$$

Then

$$P(n) = 3k[f^K(k) = n] = 3k[R(g(n),k) \ \& \ keK].$$

That is, P(x) $\leq_{in}$ to_K. Thus, to_K is a completion of K.

Corollary 2. If f has a K-representation, then rng f $\leq_m$ to_K.

If K is an infinite set, then, by Theorem 3, every recursive function has a K-representation. Hence, the following Corollary 3 follows from Corollary 2.

Corollary 3. (1) $d(to_{(A)}) = 1$. In fact, $to_{(I)}$ is a complete E_X set (Theorem 2, [8]).

(2) If K is infinite, then $d(to) \geq 1$.

Suppose *K e TJ* $^{n+1}$. Then k = rng f, where f is recursive in Z_n. That is K is r.e. in a H_n-set. Thus, by Theorem 2,

if g is a function with a K-representation, then g is recursive in a E_n-set. Thus, if K e S_{n+1} and g has a K-representation, then g e A_{n+1}. Hence, not every function recursive in K has a K-representation.

This same conclusion follows from Theorem 5, since K *e* $£_n$ implies ID_K € 25_n.

Theorem 6. 3B [A = B[1]] -• d(U)$_A$) = d(A) .

Proof, A G E_1^B. Thus, to_A e S_X^B, by Theorem 4(1). Hence, $to_A \leq_r A_5$,
since A = B[1] . On the other hand, by Theorem 4(5) , $A \leq_m to_A$•

The following corollary follows from Theorem 4 and Corollary 3.

Corollary 4. For all infinite sets K, d(K) V 1 £ d(to $_K$) £ d(K)^f .

Corollary 3 and the following examples show that Corollary 4 gives the best possible upper and lower bounds to d(to$_K$). By example 3, d(K) and d(K)' are not the only possible values for d(to$_K$) .

Examples, 1. By Theorem 6 and Friedberg[1]s characterization of the degrees greater than 0^T [2],

$$Vd \geq 0' \ 3K[d(K) = d \ \& \ d(to_K) = d(K)].$$

2. Also by the result in [2], given a > 0' , choose b so that a = b' = b V o[1] & b10[1] . Choose K so that d(K) = b. Then, d(K) V 0' = *<5(^$_K$)* = £(K) ' -

3. By Theorem 4(3) and Corollary 3, if K *e* $£_1$ and K is infinite, then d(to$_K$) = 1. By a theorem of Sack's [16, p. 107],

$$3K[K \ e \ £_x \ \& \ 0 < d(K) < 1 \ \& \ d(K) » = 2] .$$

Thus, aKTd(K) < d(u$_K$) < d(K)'].

3. Relative Recursiveness

In the introduction to this paper we asked whether d(to$_{A}$) is a function of d(K). In this section we show that A <£$_r$ B does not imply to$_A$ <$_r$ to$_B$ and, more strongly, show that d(to$_K$) is not a function of d(K). We then show (see Corollary 8 and Theorem 10) that for each degree d there exist sets A and B so that d(A) = d(B), d(to^A) = d(A)' and d(to^B) = d(B) V 1.

Define p(x,y) = $(x+y)^2$ + y. Define t(x) = n, where n^2 is the largest square less than x. Define s(x) = x - t(x) and u(x) = t(x) - s(x), x ^ 3. Then, u(p(x,y)) = x and s(p(x,y)) = y.

It follows from Definition 4 and Lemma 2 that p(x,y) e Spr^{1}.

Thus, p(x,y) has an a)-representation <p(M^1,N^1,F^1,...) so that

rng p(x,y) = S(<p(M ,N ,F ,...) A VxF x) .

Let CJ(X) be the number theoretic function defined by

$$c>(n) = {}^r 3 1_n x |4^i x \text{ A } cp(M^1 j N^1, F^1, \ldots) \text{ A } V x F^1 x^{\sim i}.$$

We have now the following lemma,

Lemma 3. R(a(n),k) = 3y[k = $(n+y)^2$ + y].

Theorem 7. VA3B[B £$_r$ A & to$_3$ = A^T].

Proof, Let a set A be given. Choose 3yS^A(x,y) to be a complete A-generable predicate. By Lemma 3 and the definitions preceding Lemma 3,

$$3yS^A(x,y) = 3y[R(a(x),y) \ \& \ S^A(u(y),s(y))].$$

Define B = {y:S^A(u(y) , s(y)) } . B ^ A. 3yS^A(x,y) ≡ 3y[R(a(x),y) & yeB]. Thus, 3yS^A(x,y) $\le_m$ to_B . to_B r.e. A follows from Theorem 4(1), since B £ A. Thus, to is complete for A.

r B

Corollary 5. **3A,B[B $\overset{r}{£}$ A & to_A $\overset{\neq r}{<}$ ^.**

Proof. Choose A so that d(A) = d(to). Then apply Theorem 7.

Corollary 6. 3K[d(to) = d(K)' & d(to_) = d(K)].

Proof. Choose 3yVzP(x,y,z) to be a complete S^2 predicate. 3yVzP(x,y,z) = 3y[R(a(x),y) & VzP(u(y),s(y),z)]. Let K= [y:VzP(u(y),s(y),z)}. Then, 3yVzP(x,y,z) $\le_m$ to_K. But, to_K e $£^2$. Thus, d(to_K) = 0'' . By Theorem 4(3) and Corollary 3(2), since K is the complement of an r.e. set, d(to–) = 0' .

Thus, to_K does not induce a function on degrees and to_K does not preserve relative recursiveness.

Corollary 7. (1) 3K[d(tn$_K$) = d(K) & d(lh^) = d(K) »] .

(2) 3K[d(U --) = d(K)t & d(U--) = d(K)].

Proof. Corollary 6 and Lemma 1.

Thus, the functions ft\ and \s also do not induce functions on the degrees, and therefore do not preserve relative recursiveness.

Definition 5. Let <p be a formula in prenex normal form and M^1 a one place predicate letter, not occurring in (p. Define tp_{M^1}, cp ~~relativized~~ to M^i , as follows:

(i) If φ is quantifier free and contains occurrences of the variables $x_0, \ldots, x_{n-1}$ and no others, then φ_{M^1} is $\varphi \wedge M^1(x_0) \wedge \ldots \wedge M^1(x_{n-1})$;

(ii) If φ is $\exists y \theta$, φ_{M^1} is $\exists y[M^1(y) \wedge \theta_{M^1}]$;

(iii) If φ is $\forall y \theta$, φ_{M^1} is $\forall y[M^1(y) \rightarrow \theta_{M^1}]$.

An easy argument proves the following lemma.

~~Lemma~~ 4. For every formula φ, φ has a model of finite cardinality y if and only if φ_{M^1} has a model $\mathfrak{A}$ so that $\|M^{1\mathfrak{A}}\| = y$.

Lemma 5. For every function f belonging to Spr^1 there is a recursive function g so that

$$\forall x \forall y[R(x,y) \leftrightarrow R(g(x),f(y))]$$

and

$$\forall x \forall z \exists y[R(g(x),z) \leftrightarrow (z = f(y) \ \& \ R(x,y))].$$

Proof. Assume $f \in Spr_1$. By Definition 4, f has an ω-representation $\varphi(M^1, F^1, \ldots)$ so that $[\varphi(M^1, F^1, \ldots) \ \& \ \forall x F^1(x)]$ has a model of cardinality z if and only if $\exists y[z = f(y)]$.

Let $g(x) = \ulcorner \theta_{M^1} \wedge \varphi \wedge \forall x F^1 x \urcorner$, where $\theta = \llcorner x \lrcorner$, and suppose $R(x,y)$. θ has a model of cardinality y. Thus, by Lemma 4, θ_{M^1} has a model $\mathfrak{A}$ so that $\|M^{1\mathfrak{A}}\| = y$. Since $f(y) \geq y$, $\mathfrak{A}$ can be extended and expanded to a model of $\theta_{M^1} \wedge \varphi(M^1, F^1, \ldots) \wedge \forall x F^1 x$ of cardinality $f(y)$. Thus $R(g(x), f(y))$.

Suppose $R(g(x),z)$. Then, $\theta_{M^1} \wedge \varphi(M^1, F^1, \ldots) \wedge \forall x F^1 x$ has a model $\mathfrak{A}$ of cardinality z. The restriction to $M^{1\mathfrak{A}}$ is a model of cardinality y of θ, where $f(y) = z$. Thus $R(x,y)$.

Definition 6. $A \le_{ms} B \iff A \le_m B$ by a function $f \in Spr_1$.

$\le_{ms}$ is a reducibility. That is, $\le_{ms}$ is a reflexive and transitive subrelation of $\le_T$. In fact, if $A \le_m B$ by $f \in Spr_1$, and $B \le_m C$ by $g \in Spr_1$, then $x \in A \iff g(f(x)) \in C$. Therefore, by Lemma 2(2), $A \le_{ms} C$. Hence $\le_{ms}$ is transitive. Since the identity function belongs to Spr_1, $\le_{ms}$ is reflexive.

Theorem 8. If $A \le_{ms} B$, then $\omega_A \le_m \omega_B$.

Proof. Suppose $A \le_m B$ by $f \in Spr_1$. By Lemma 5, there exists a recursive function g so that

$$R(x,y) \leftrightarrow R(g(x),f(y))$$

and

$$\forall x,z \exists y[R(g(x),z) \rightarrow (z = f(y) \;\&\; R(x,y))].$$

$$x \in \omega_A \leftrightarrow \exists y[R(x,y) \;\&\; y \in A]$$

$$\leftrightarrow \exists y[R(g(x),f(y)) \;\&\; y \in A]$$

$$\leftrightarrow \exists y[R(g(x),f(y)) \;\&\; f(y) \in B]$$

$$\rightarrow \exists y[R(g(x),y) \;\&\; y \in B]$$

$$\rightarrow g(x) \in \omega_B.$$

$$g(x) \in \omega_B \rightarrow \exists z[R(g(x),z) \;\&\; z \in B]$$

$$\rightarrow \exists y[R(g(x),f(y)) \;\&\; f(y) \in B]$$

$$\rightarrow \exists y[R(x,y) \;\&\; y \in A] \rightarrow x \in \omega_A.$$

Thus to$_A \le_m$ to$_B$ by g, concluding the proof of Theorem 8.

Define the recursive sup. of the two sets A and B by

$$2x \in A \vee B \leftrightarrow x \in A$$

$$2x + 1 \in A \vee B \leftrightarrow x \in B.$$

It is clear that $A \le_m A \vee B$, $B \le_m A \vee B$, and that d(AVB) is the least upper bound of d(A) and d(B) .

<u>Lemma</u> 6. For any two sets A and B, to$_A \le_m$ to$_{AVB}$ and to$_B \le_m$ ^$_{AVB}$.

<u>Proof</u>. By Lemma *2,* the functions 2x and 2x+l belong to Spr.^.

The proof follows then from Theorem 8.

Theorem 9. VA3C[d(C) = d(A) & to$_C$ is complete A-generable].

<u>Proof.</u> By Theorem *1,* 3B [B $\le_r$ A & to$_B$ is complete for A].

Let C = A V B. B < A, thus d(C) = d(A). to is r.e. in C and $C \le_r A$, thus to$_C$ is r.e. in A. By Lemma 7, to$_B \le_m$ to$_C$; also to$_B$ is complete for A. Thus, to$_C$ is complete for A.

Corollary 8. Vd3A[d(A) = d & d(to$_A$) = d(A) ' = d'] .

The following theorem (obtained by Thomas Grilliot, in personal communication) gives a positive solution to a question raised in [6].

Theorem 10. Vd3A[d(A) = d & d(\n$_A$) = d V 1] .

Proof. By Corollary 3, we already have this result for the case d = 0. Thereforej assume that d > 0, and choose K so that d(K) = d. Let $Ch_K(n)$ denote the characteristic function of K, and let $\tilde{Ch}_K(n)$ (= $\prod_{i<n} p_i^{Ch_K(i)}$, see [3, p. 231]) be the course-of-values

function for $Ch_K(n)$. Then, define A to be the complement of $\{\tilde{C}h_K(n):n\in\omega\}$. $K \le_r A$, and $A \le_r K$. Also, it is easy to see that 7? is recursive in every infinite subset of 75.

By Corollary 4, it suffices to show that £(1^) £ d̃(A) V 1̃. Let <p be any formula of £. Since d(A) > 0, A is not recursive in S(<£>). Therefore S(<p) cannot be an infinite subset of 75. Hence, either S(<p) is finite, or ⌜cp^ e to. That is, either 3yVz > y R(⌜cp~',y) or 3yfR(⌜<PSy) & Y^A]. The function f(x) = fly[[R(x,y) & y_GA] V Vz > y R(x,z)] is recursive in A and 0^T, and

$$V^e \ 1B_A \circ {}^a y \ 1 \ fC^) \quad [RCo^y) \ \& \ yeA].$$

Hence, d(to) ^_ d(A) V 1.

4. <u>Values of</u> $d(to_K)$, <u>for</u> K <u>of a given degree</u>

Are d(K) V o^T or d(K)' the only possible values for d(to) for any K? In this section we describe the extent to which $d(to_K)$ is independent from d̃(K), within the bounds given by Theorem 4 and Corollary 4.

Lemma 7. There is a recursive function f so that R(x,2y) •* R(f(x),y).

Proof, Let $x^0,x^1,\ldots,$ be a complete list of the individual variables in <£. Let S be a binary predicate letter and let a and b be individual constant letters. Given a formula <p in £, let x^K

be the highest index variable which occurs in <p. Let u^ denote the variable x_{k+1+i}. Then, none of the variables $u_0, u^, \ldots,$ occurs in (p. Also, we may suppose without loss of generality that *(p* contains no occurrences of S, a and b. (Otherwise, *<p^* can be found uniformly, where $(p_1$ contains no occurrences of S, a and b, and R(r<pn,2y) «• R(r(£0_1, 2y) .) We define a new formula *<p* as follows:

(1) (x^x^)* is $x_\pm = x_j$ A $u_\pm = u_j$;

(2) $P^n(x_{i_1}, \ldots, x_{i_n})^*$ is $P^{2n}(x_{i_1}, u_{i_1}, \ldots, x_{i_n}, u_{i_n})$;

(3) $(ip_1$ A $lb_2)^*$ is ib_1^* A 0_2^*•

(4) (-.0)* is -r(<M*;

(5) (3x$_i$0)'f is 3x$_i$au$_i$[S(x$_i$,u$_i$) A 0*].

Define

T(<p) = <p* A a ^ b A [VxS(x,a) A VxS(x,b)

A VxVy(S(x,y) -• y=a V y=b) A $s(x_{i_1}, u_{i_1})$

A ... A $S(x_{i_n}, U_{i_n})$],

where $x_{i_1}, \ldots, x_{i_n}$ is a list of the free variables occurring in (p.

<u>Claim</u>, <p is satisfiable in a structure of cardinality 2y if and only if T(<p) is satisfiable in a structure of cardinality y.

<u>Proof</u>. We first show that if <p is satisfiable in a structure of cardinality 2y, then T(<p) is satisfiable in a structure of cardinality y.

If a formula <p holds in a structure of cardinality 2y, then let

A = «l,l>,<2,l>,...,<y,l>J<1.2>,<2,2>,...Xy,2>}

be the domain of such a structure, 21. Define a structure 39 with domain B = {l,2,...,y} as follows:

(1) If R^ is a k-ary relation on A, then R^ is a 2k-ary relation on B defined by

$$R_{\mathfrak{B}}(i_1,j_1,\ldots,i_k,j_k) \leftrightarrow R_{\mathfrak{A}}(\langle i_1,j_1\rangle,\ldots,\langle i_k,j_k\rangle),$$

for $i_{l5}...,i_k$ € {l,2,...,y), and J^---*^ e t^{1*2}}-

(2) S_m = {<i,j>:i = l,...,y & (j=l or j=2)}.

(3) a is 1, b is 2.

It is clear that [VxS(x,a) A VxS(x,b) A VxVy(S(x,y) - y = a v y = b)] holds in ».

We prove by induction that *cp* is satisfiable in 31 (by an assignment a) if and only if T(<p) is satisfiable in 99 (by an assingment p). Moreover, $a(x_i) = \langle p(x_i),p(u_i)\rangle$.

Case <p is $x_i = x_j$. If a satisfies <p in 81, then for some <s,t> *e h,* $a(x_i) = a(x_j)$ = <s,t>. Define p by $p(x_i) = p(x_j) = s$ and $P(u_i) = p(u_j) = t$. Then p satisfies $<p^*$ in a. Since t = 1 or t = 2, p satisfies T((p) in 5.

<p* is $x_1 = x_3$ A $u_1 = u_3$. If p satisfies T(<p) in SB, then P(xi) = P(xj) = s and p(u±) = p(uj.) = t, for s,t e B. Also, s^ls^t). Thus, <s,t> *e* A. Define a by a(x.) = a(x) = <s,t>. a satisfies <p in 31.

~~Case~~ <p ~~is~~ P(xn,_____,x). If there is an assignment a so that PqT(a(x) . , . . . ,<x(x)), define p(x.) and p(u,), i = l,...,n, so that $a(x_1) = <p(x_1), p(u_1)>$. Then, by definition of P, $P_{gj}O(x_1),P(u_1),...,p(x_n),e(u_n))$. That is, p satisfies <p* in 39. Therefore, p satisfies T(cp) in S3.

Suppose p satisfies T(<p) in 35. *<p* is P(x 1,u 1. . . ,xn,un). $P_a(p(x_1),P(u_1),...,P(x_n),P(u_n))$, and S^(p(x^,p(u^), i=l,...,n. Thus <p(x1),P(u1)> € A, i = l,...,n. Define a(x.1) = <p(x.1), p(u1)>. a satisfies <p in 91.

~~Case~~ <p j^ *t1)* A 2^ . if a satisfies both *1b* and 2 in ^, then by induction hypothesis p satisfies *ij)*1 in SS and p satisfies 0_9^* in 35, where p is defined so that $oc(x_1)$ = $<p(x_1), p(u_1)>$, for each i. Thus p satisfies $<p^* = 0_1^*$ A i_2^* in 99. Thus, p satisfies T(<p) in ^8.

The other direction is identical. The case cp is •» 0 is straightforward.

<u>Case</u> cp <u>is</u> $3x_1.il)$. If $£x_1.0$ is satisfied in 31, then some assignment a satisfies $0(x_1)$ in 91. Thus p* defined by $a(x_j) = <p(x_j), p(u_j)>$, for all j, satisfies *ib* (x_1,u_1) in a.

Also, $S_{\infty}'(\sigma(x_i),\rho(u_i))$. Hence ρ satisfies $\exists x_i, u_i [S(x_i,u_i) \wedge \phi^*]$ in $\mathfrak{B}$. Thus, σ satisfies $T(\phi)$ in $\mathfrak{B}$.

The other direction is similar.

We have shown that if ϕ is satisfiable in a structure of cardinality 2γ, then $T(\phi)$ is satisfiable in a structure of cardinality γ. We show now that if $T(\phi)$ is satisfiable in a structure of cardinality γ, then ϕ is satisfiable in a structure of cardinality 2γ.

If $T(\phi)$ is satisfiable in a structure $\mathfrak{C}$ of cardinality γ, we may assume that $C = \{1,2,\ldots,\gamma\}$ is the domain, a is 1, b is 2, and $S_{\mathfrak{C}}(i,j) \Leftrightarrow j = 1$ or $j = 2$.

Define a structure $\mathfrak{B}$ from $\mathfrak{C}$ as follows:

(1) $B = \{1,\ldots,\gamma\}$, the domain of $\mathfrak{C}$;

(2) $S_{\mathfrak{B}} = S_{\mathfrak{C}}$;

(3) a is 1, b is 2;

(4) $P_{\mathfrak{B}}(i_1,j_1,\ldots,i_n,j_n) \Leftrightarrow$

$P_{\mathfrak{C}}(i_1,j_1,\ldots,i_n,j_n) \wedge S_{\mathfrak{C}}(i_1,j_1) \wedge \ldots \wedge S_{\mathfrak{C}}(i_n,j_n)$.

(Note that only 2k-ary relations appear in $T(\phi)$.)

We show by induction that every assignment γ which satisfies $T(\phi)$ in $\mathfrak{C}$ also satisfies $T(\phi)$ in $\mathfrak{B}$, and every assignment ρ which satisfies $T(\phi)$ in $\mathfrak{B}$ also satisfies $T(\phi)$ in $\mathfrak{C}$.

Our result follows easily from this, because $\mathfrak{B}$ is obtainable from a structure $\mathfrak{A}$ of cardinality 2γ as in the previous part of the proof, and we know that T((p) is satisfiable in 35 iff and only if <p is satisfiable in 91.

If cp is $x_i = x_j$, there is nothing to show, since 33 and E have the same domain.

Case (p is $P(x_1,...,x_n)$. Assume T(<p) is satisfiable in <£. Then there is a 2n-ary relation P~ on {l,...,y} and an assignment y to £ so that p_g(yix^), Y^),...,Y(x_n),Y(u_n))> and so that S(Y(x.),Y(u.)), i = l,...,n. Thus

$$P_{\mathfrak{B}}(\gamma(x_1),\gamma(u_1),\ldots,\gamma(x_n),\gamma(u_n)).$$

That is, y satisfies T(<p) in 33.

It is obvious that an assignment satisfying T(cp) in 33 also satisfies T(cp) in (S. This direction is clear in the following cases too.

Case <p i^ 0_1 A 0_2. Suppose an assignment Y satisfies T((p) in S. S^(Y(X.),Y(U.))$_9$ for all free variables x. occurring in <p. Thus Y satisfies ib_1 and 0_2 in 33, and Y satisfies T(cp) in 93.

Case <p j^ ^ 0. Suppose T(->0) is satisfied by an assignment Y in <£. S^f Y(x_i), Y(U_i)) for all free variables x_i occurring in 0. Thus, as above, Y is an assignment to S. Y satisfies ** (0) in E. Thus, Y does not satisfy 0 in E. By induction hypothesis, Y does not satisfy 0 in 53. Thus y satisfies "* (0) = ("7 0) in 33. That is, T(<p) is satisfied by y in ®*

Case <p is 3x.0. If T(<p) is satisfied in E, then 0 is satisfied by some y in <£. S(x_i, u_i). Thus y is an assign- ment to S3 and ^ (x_i^u_i) is satisfied by y in S3. Thus T(<p) is satisfied by y in 33.

The proof of the claim is complete. Let *d((p)* denote the universal closure of *cp. cp* is valid in a structure 91 of cardinality 2y if and only if *Cl{(p)* is satisfiable in 31 if and only if *T(d((p))* is satisfiable in a structure S3 of cardinality y if and only if *T(d{(p))* is valid in S3 (since *T(Cl(<p))* is closed). Define f(x) = ⌜T(Ot(<p)r, for x= ⌜*(p^*. Then, R(x,2y) - R(f(x),y).

Lemma 8. There is a recursive function g so that R($x_5$2y+1) H R(g(x),y).

Proof. As in the proof of Lemma 7, given *<p* in *SL,* let x^ be the highest index variable which occurs in <p and let u_i denote the variable XL_{k+1+i}, all i. Again as in the proof of Lemma 7, we can suppose without loss of generality that <p contains no occurrences of the binary predicate letter S and *(p* contains no occurrences of the individual constant letters a, b, and c. The formula <p is defined for *(p* as in the previous proof. Define

$$T(\varphi) = \varphi^* \wedge a \neq b \wedge a \neq c \wedge b \neq c$$

A [VxS(x,a) A VxS(x,b) A S(c,c) A VxVy(S(x,y) -> (y = a

Vy = bV (x = c A y = c)) A S(x_{i_1}, u_{i_1}) A,#.A S(x_{i_n}, u_{i_n})], where x. ,...,x. is a list of the distinct free variables occurring in <p.

Claim, <p is satisfiable in a structure of cardinality 2y+l if and only if T(<p) is satisfiable in a structure of cardinality y.

If <p holds in a structure of cardinality 2y+l, then let A = {<l,l>,...,<y,l>.<l,2>,...,<y,2>,<3,3» be the domain of such a structure, 91. Define a structure 99 with domain B = {l,...5y} as follows:

(1) If $R_{\mathfrak{A}}$ is a k-ary relation on A, then $R_{\mathfrak{B}}$ is a 2k-ary relation on B defined by

$$R_{\mathfrak{B}}(i_1,j_1,\ldots,i_k,j_k) \leftrightarrow R_{\mathfrak{A}}(\langle i_1,j_1\rangle,\ldots,\langle i_k,j_k\rangle),$$

for ii> — ->\ e {lj--->y)? 3₁^---0k € {1,2,3}.

(2) S_m is a binary relation defined by $S_m(i,l)$ and $S_M(i,2)$ for all i = l,2,...,y, and 8^(3,3).

(3) a is 1, b is 2, and c is 3.

An induction argument shows that <p is satisfied in 91 (by an assignment a) if and only if T(<p) is satisfied in SB (by an assignment p). Moreover, for all i, $a(x_i) = \langle p(x_i), P(u_i)\rangle$. It follows that if co is satisfiable in a structure of cardinality 2y+l, then T(<p) is satisfiable in a structure of cardinality y.

Conversely, if *T(cp)* is satisfiable in a structure ^ of cardinality y, we may assume that C = £l,...,y} is the domain, a is 1, b is 2, c is 3, and $S^{\mathfrak{C}}(i,j)$ ** (j=l or j=2 or (i=3 & j=3)). Define a structure $ from E as follows:

(1) B •=• {l,...,y);

(2)

(3) a is 1, b is *2,* c is 3;

(4) VV^i-'VV"

$P_{\mathfrak{C}}(i_1,j_1,...,i_n,j_n) \wedge S_{\mathfrak{C}}(i_1,j_1) \wedge ... \wedge S_{\mathfrak{C}}(i_n,j_n)$.

As in the proof of Lemma 7, an induction argument shows that if T(cp) is satisfied in E by an assignment *y,* then T(<p) is satisfied in SS by Y and conversely.

S is obtainable from a structure 91 of cardinality 2y+l as in the previous part of the proof; and we know that T(<p) is satisfiable in $8 if and only if *<p* is satisfiable in M. Thus, if T(<p) is satisfiable in a structure of cardinality y, then • <p is satisfiable in a structure of cardinality 2y+l. This completes the proof of the claim.

Define g(x) = ^{r}T(C^(cp))^n, for x = ^{r}*(p^n.* Then

R(x,2y+1) « R(g(x,y).

Theorem 11. d(to) = d(te) V d(to).

Proof. By Lemma *6,* d(to) v d('Jb) ^ d(to) .

~ A ~ B ~ AVB

3y[R(x,y) & y e AVB] •* 3y[R(x,2y) & 2y e AVB]

V 3y[R(x,2y+l) & 2y+l € AVB]

•• 3y[R(x,2y) *&* yeA] V 3y[R(x,2y+l) & yeB].

By Lemmas 7 and 8, let f satisfy $R(x,2y) \leftrightarrow R(f(x),y)$ and let g satisfy $R(x,2y+1) \leftrightarrow R(g(x),y)$. Then,

$$\exists y[R(x,2y) \;\&\; y \in A] \leftrightarrow \exists y[R(f(x),y) \;\&\; y \in A];$$

and

$$\exists y[R(x,2y+1) \;\&\; y \in B] \leftrightarrow \exists y[R(g(x),y) \;\&\; y \in B].$$

Thus, $x \in \omega_{A \vee B} \leftrightarrow f(x) \in \omega_A \vee g(x) \in \omega_B$. Thus, $\omega_{A \vee B} \leq_T \omega_A \vee \omega_B$.

We are now ready to prove our main results.

<u>Theorem</u> 12. $\forall \mathbf{a} \geq \mathbf{0}' \, \forall \mathbf{b} \, \exists K[(\mathbf{a} \leq \mathbf{b} \leq \mathbf{a}' \;\&\; \mathbf{b} \text{ r.e. } \mathbf{a}) \rightarrow (\mathbf{d}(K) = \mathbf{a} \;\&\; \mathbf{d}(\omega_K) = \mathbf{b})]$.

<u>Proof</u>. (see Figure) Let $\mathbf{a}$ and $\mathbf{b}$ satisfy $\mathbf{a} \geq \mathbf{0}'$, $\mathbf{a} \leq \mathbf{b} \leq \mathbf{a}'$, and $\mathbf{b}$ r.e. $\mathbf{a}$. By Friedberg's characterization [2], $\exists \mathbf{c}\; \mathbf{a} = \mathbf{c}'$. $\mathbf{b}$ r.e. $\mathbf{c}'$, thus by Shoenfield [7], $\exists \mathbf{d}[\mathbf{b} = \mathbf{d}' \;\&\; \mathbf{c} \leq \mathbf{d} \leq \mathbf{c}']$.

By Theorem 6, choose A so that $\mathbf{d}(A) = \mathbf{d}(\omega_A) = \mathbf{a}$. By Corollary 8, choose B so that $\mathbf{d}(B) = \mathbf{d}$ and $\mathbf{d}(\omega_B) = \mathbf{d}' = \mathbf{b}$. Let $K = A \vee B$. $\mathbf{d}(K) = \mathbf{d}(A) = \mathbf{a}$. By Theorem 11, $\mathbf{d}(\omega_K) = \mathbf{d}(\omega_{A \vee B}) = \mathbf{d}(\omega_A) \vee \mathbf{d}(\omega_B) = \mathbf{a} \vee \mathbf{b} = \mathbf{b}$.

$\mathbf{a}'$

|

$\mathbf{d}(\omega_K) = \mathbf{b} = \mathbf{d}' = \mathbf{d}(\omega_B)$

|

$\mathbf{d}(K) = \mathbf{a} = \mathbf{c}' = \mathbf{d}(A) = \mathbf{d}(\omega_A)$

$K = A \vee B$

|

$\mathbf{d} = \mathbf{d}(B)$

|

$\mathbf{c}$

<u>Figure</u>

Theorem 13. $\forall \mathbf{a} \forall \mathbf{b} \exists K [\mathbf{a} \le \mathbf{b} \le \mathbf{a}' \rightarrow (\mathbf{d}(K) = \mathbf{b} \;\&\; \mathbf{d}(\omega_K) = \mathbf{a}')]$.

Proof. By Corollary 8 and Theorem 10, choose sets A and B so that $\mathbf{d}(A) = \mathbf{a}$, $\mathbf{d}(B) = \mathbf{b}$, $\mathbf{d}(\omega_A) = \mathbf{a}'$, and $\mathbf{d}(\omega_B) = \mathbf{b} \vee \mathbf{1}$. Let $K = A \vee B$. Then, $\mathbf{d}(K) = \mathbf{d}(A \vee B) = \mathbf{d}(B) = \mathbf{b}$. By Theorem 11, $\mathbf{d}(\omega_K) = \mathbf{d}(\omega_{A \vee B}) = \mathbf{d}(\omega_A) \vee \mathbf{d}(\omega_B) = \mathbf{a}' \vee \mathbf{b} \vee \mathbf{1} = \mathbf{a}'$.

Theorem 14. $\forall \mathbf{a} > \mathbf{0}' \forall \mathbf{b} \exists K[(\mathbf{a} \le \mathbf{b} \le \mathbf{a}' \;\&\; \mathbf{b} \text{ r.e. } \mathbf{a}) \rightarrow$

$$(\mathbf{d}(K) \le \mathbf{a} \;\&\; \mathbf{d}(\omega_K) = \mathbf{a} \;\&\; \mathbf{d}(K)' = \mathbf{b})].$$

Proof. Using [2] and [7], as in the first paragraph of the proof of Theorem 12, $\exists \mathbf{c},\mathbf{d}[\mathbf{c} \le \mathbf{d} \le \mathbf{c}' = \mathbf{a} \le \mathbf{d}' = \mathbf{b} \le \mathbf{a}']$. By Corollary 8 and Theorem 10, choose A and B so that $\mathbf{d}(A) = \mathbf{d}$, $\mathbf{d}(B) = \mathbf{c}$, $\mathbf{d}(\omega_A) = \mathbf{d} \vee \mathbf{1}$, and $\mathbf{d}(\omega_B) = \mathbf{c}' = \mathbf{a}$.

$$\mathbf{d}(A \vee B) = \mathbf{d} \vee \mathbf{c} = \mathbf{d} \le \mathbf{a}.$$

$$\mathbf{d}(A \vee B)' = \mathbf{d}' = \mathbf{b}.$$

$$\mathbf{d}(\omega_{A \vee B}) = \mathbf{d}(\omega_A) \vee \mathbf{d}(\omega_B) = \mathbf{d} \vee \mathbf{1} \vee \mathbf{a} = \mathbf{a}.$$

Take $K = A \vee B$.

Bibliography

[1] Asser, B. , "Das Repreśantantenproblem in Prädikatenkalkiil der ersten Stufe mit Identität," Zeitschr. f. Math. Logic und Grundlagen d. Math. 0,(1955) , 252-263.

[2] Friedberg, Richard, ftA Criterion for Cbmpleteness of Degrees of Unsolvability," J.S.L. 22(1957),159.

[3] Kleene,S., Introduction to Metamathematics, 5th ed., D. Van Nostrand, Princeton, New Jersey, 1950.

[4] Sacks, Gerald E., "Degrees of Unsolvability," Annals of Mathematics Studies,^55^ Princeton University Press, Princeton, New Jersey, 1963.

[5] Selman, A. L., "First Order Formulas True in Structures of Arbitrary Finite Cardinality," Abstract 672-202, Notices of the A.M.S.17(1970),140.

[6]________________, "Arithmetical Reducibilities and Sets of Formulas Valid in Finite Structures," Ph.D Thesis, Pennsylvania State University, 1970.

[7] Shoenfield, J. R., "On Degrees of Unsolvability," Ann. of Math.69(1959),644-653.

[8] Trachtenbrot, B. A., "Impossibility of An Algorithm for the Decision Problem in Finite Classes," Dokl. Akad. Nauk SSR 70(1950),569-572.

Footnotes

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2. The author is presently a postdoctoral research fellow at Carnegie-Mellon University.