PRABIR ROY'S SPACE A IS NOT

N-COMPACT

by

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Introduction* N-compact spaces were introduced by S. Mrowka in $[M_1]$, where the general concept of an E-compact space was defined: given a Hausdorff space E, a space X is E-compact if it is homeomorphic to a closed subspace of E^m for some cardinal number *Mi.* Thus the I-compact spaces (where I is the closed unit interval) are the compact Hausdorff spaces, the R-compact spaces are the realcompact spaces, and the 2-compact spaces (where 2 denotes the discrete two-point space) are the O-dimensional compact Hausdorff spaces. The N-compact spaces are those which can be embedded as closed subspaces in N^ where N is the set of natural numbers with the discrete topology.

The main properties of E-compact spaces were given in [EM], where it was asserted that the N-compact spaces are precisely the O-dimensional realcompact spaces. ('O-dimensional' there, as here, means 'having a base of clopen sets.') It is clear that every N-compact space is O-dimensional and realcompact, but the proof of the converse in [EM] was incomplete. The purpose of this paper is to show that the converse is in fact false, that Prabir Roy's space A is a counterexample. It is not N-compact, but it is metrizable of cardinality $2^{\aleph_0}$ and hence realcompact [cf. GJ, p. 232] and it is O-dimensional.

The space A was described by P. Roy in $[R_1]$ and many of its properties were proven in detail in $[R_2]$, including its metrizability and its zero-dimensionality. The fact that A is not N-compact is the new result, first established by the author using the proof given below. It is based on the following characterization of N-compactness, first discovered by H. Herriich

[H, Beispiele 6].

Theorem. A zero-dimensional Hausdorff space X is N-compact if, and only if, every clopen ultrafilter[1] on X with the countable intersection property[2] is fixed.

In what follows, we establish the existence of $2^{\aleph_0}$ distinct free clopen ultrafilters on A, any one of which is enough, by Herrlich's theorem, to establish that A is not N-compact.

Constructing the Ultrafilters. In order to facilitate comparison with [R_2], the numbers of the lemmas and theorems will begin with a 5, and if a lemma closely parallels a lemma or a definition in [R_2]* ^t w i ^ 1 ^e given a similar numbering. (Thus, the first two lemmas follow right from Definitions 5.3.3 and 5.3.4 in [$R_2$1 and are numbered 5.3.5 and 5.3.6; the next three lemmas are like Lemma 5.4 in [$R_2$1* and are numbered respectively 5.4.0, 5.4.1, and 5.4.2. And a later lemma is a generalization of Lemma 5.7 in [$R_2$1 and is numbered 5.7'.

All the notation used in this section, unless otherwise remarked, follows that of [R_2]. Not all the results in [R_2] will be used in proving the lemmas below; while familiarity with [R_2] up to the beginning of Section 5.3, p. 127 is certainly an asset, the reader will be able to get by with considerably less. Specifically, after the end of Section 2.1 only the following facts in [R_2J are used:

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[1] an ultrafilter on the Boolean algebra of clopen subsets of x is called simply a clopen ultrafilter.

[2] Given a cardinal number *4**\ , a filter 3 is said to have the ~~/*H-intersection property~~ if every collection of **n* or fewer sets in 3* has nonempty intersection. The countable intersection property is the case *AM* = N_0.

(1) Lemma 2.7, which shows that the regions R_x and R, * form a base for a topology, and which can be proven right after Lemma 2.1;

(2) Lemma 2.8, which shows that the regions R_x eT, form a local base at a point peP-^ and the regions $R_{(p,n)}$ x€l\ 2 form a local base at a point peP, and which can be proven right after Lemma 2.1;

(3) A modest version of Property II [p.126]: given a point peP-p let x_n be a member of X with $|x_n| = n$ and such that p extends x_n (this means $x_n(i) = p(i)$ for $i = 1,\ldots,|x_n|$); then fl R = [p]; similarly, given a point peP_0, Tl, R v = {p). These facts can be proven right after the definitions of Section 1.

(4) Lemma 5.1, which shows that every R_x and every R, is clopen. This can be proven after Section 2 and does require all the results in that section.

In addition to the notations used in [R^l* we adopt the following notation: if XGX and x'€X [resp. pePñ] are such that I x? I J> I xl and $x'(i) = x(i)$ for $i = 1,\ldots,|x|$ [resp. $p(i) = x(i)$ for $i = 1,\ldots,|x|$] then we write $x < x'$ [resp. $x < p$]. Similarly, if Trell and TT' en [resp. xeX] [resp. p€P] are such that ITT'I 2 kl and 7r'(i) = ir(i) for i = 1,...,|TT| [resp. |x| ^ |TT| and x(i) = T(±) for i = 1, ...,|TT|] [resp. p(i) = 7r(i) for i = 1,...,|TT|] then we write IT < TT' [resp. IT < x] [resp. ir < p]. Also, we adopt the convention that if xeX is the (unique) member of X with $|x| = 0$, then $R_x = A$.

For the convenience of the reader, the following definitions in [R~] are repeated, with the above notation used where possible.

5.3.1. 7r = the set of all finite sequences of positive real numbers, defined on initial segments of the set of positive integers. If *ireH,* then |TT| = the greatest integer for which *IT* is defined.

5.3.2. K is an <u>indicator</u> means that K is a subset of *IT* with

(1) if 7r,7r^{f}fK, then |ir| = |TT^f| ^ and that integer is denoted by |K|

(2) {r:r = 7r(l) for some *veK)* is an infinite set, and

(3) if TreK and j is a positive integer with j < |TT|, then [r:r = TT^T(J+1) for some TT'GK with 7r^T(i) = 7r(i) for i = l,...,j} is an an infinite set.

5.3.3. If K is an indicator and xeX then

E(K,x$\pm$) = [R.,,:|x^T| = |x| + |K|,X < x^T, and for some TreK, x^f (|x|+i) = $\pm$7r(i) for i = 1,..., |K| }.

(Remark: this is a composite definition; the things actually defined are £(K,x+) and £(K,x-).)

5.3.4. If K is an indicator and R, (p^n) $_N$€F_O 2 then

E(K,(p,n)) = {R(q,$_m$^:m = n + |K|,q€R?p,n), and for some 7T€K,q_z(n-l+i) =7r(i) for i=l, . . ., | K|) .

5.3.5. <u>Lemma</u>. Let K be an indicator.

(E(K,(p,n)))* c: R_x « $R_{(p^n)}$e ^

<u>Proof</u>. ^: (E(K, (p,n)))* is a union of subregions of R, $_N$.

⇒:(Σ(K, (p,n)))* c R_X =, R_X n $R_{(p*n)}$?> ^ θ =» R_X^ 3 $R_{(p,n)}$#

5.3.6. Lemma. Let K be an indicator

(D CC(K,X+))* CR(pjn) - RxCR(p^n)

(2) (E(K,x-))* <=R(p,n) - R x ^ R ^ n ,

Proof. <*:(2(K,x+))* and (E(K,x-))* are both unions of subregions of R_x.

=*: Proof will be given of (1) in the case n > 1, the only case needed here.

Suppose (£(K5x+))* c R(p,n), then for each Rx»e£(K5x+)y we have Rxr^c R(p^n) and so Rxf n Ry(p^±)j for some j (2.1.3); furthermore, since Rv,x c: R7+(p*n) . (2.1.3), we have |x^T| J≥ |px |+n+l because Rx contains qeP2 of length |x^f| while all P2· points in R+(p,n) must have x-coordinates of length at least |px|+n+l. This implies Rγ(p,n,±)j => R? for exactly one j (2.1.1).

If Rx $ R(p^n),l*l £ IPx'+n (2-1-1)- pick xft such x"(i) = x'(i) for i= 0,... Jpx|+n,x^lf(|px|4.n+l) £ X'(|pxl+n+l), and Rvfl€S(K,x+) . If n > 1 we have q(|pv|+n+l) = p (|pv|+n-l) for all qePj^ 11 R(p, ^. Now let q^lf€P1 fff Rx,t and q^T€P]L fl Rx? . q and q^f cannot both be in R,_vp*n; w contradicting L(K,x+) c R(p,n).

5.4.0. Lemma. Let [rα}α∈Q be an infinite set of positive real numbers. For each ra let Ka be an indicator, with |KαI = n for all a. Now let K = f7r^f 517r^T | = n+1, TT^1 (1) = r^ for some a€G and, for the same a, 7r^f (1+i) = *ir(i)* for some TTGK }. K is an indicator.

Proof. (1) is clearly satisfied.

(2) {r:r = TT(1) for some TTGK) is an infinite

set, namely $\{r_\alpha\}_{\alpha\in G}$.

(3) is also satisfied: for $j = 1$ it is satisfied because of property (2) of each K_α; for $j > 1$ it is satisfied because of property (3). of each K_α.

5.4.1. Lemma, If it is a collection of regions and R-- is such that, for uncountably many x^f with $|x^f| = |x|+l$, $x < x', x'(|x|+l) > 0$ [resp. $x'(|x|+l) < 0$] there is an indicator K_x, with $\mathcal{E}(K_x, x'+) c M$ [resp. $\mathcal{E}(K_x, x^1-) c W$] then there is an indicator K such that $E(K,x+)$ *a* 3J [resp. $\mathcal{E}(K,x-)$ c if].

Proof. Since [M:M = $|K_f|$ for some $x^?$} is a countable set, let N be a positive integer such that $|K_{,}| = N$ for infinitely many x^1. Let $\{r_\alpha\}_{\alpha\in G} = \{x'(|x|+l):|K_x,| = N\}$ [resp. $\{-x'(|x|+l):|K_{,}| = N\}$] and let K be as defined in the previous lemma. Then $\mathcal{E}(K,x+)$ *a U* [resp. $\mathcal{E}(K,x-)$ c Jt] .

5.4.2. Lemma. If W is a collection of regions and $R^{(p,n)}$. is such that, for uncountably many $r\in R^+$ there exists $q^r\in R^?_{(p,n)}$ ^x with $q_z^\wedge(n) = r$ and an indicator K_r such that $E(K_r, (q^r,n+1)) <= H$^ then there is an indicator K such that $\Sigma(K,(p,n))$ c *U.*

Proof. Proof is as in 5.4.1.

5.6.1. Lemma. Let *(^us^R£tij.* ke an open cover for R_x, with |fc| < C (C = 2**o) . Then there exists jSeB and an indicator K such that $(\mathcal{E}(K,x+))^*$ c $\backslash j_\beta$, and there exists *yefo* and an indicator K^1 such that $(E(K,x-))$^ c U_γ.

Proof. Given jSeB let $\#_\beta = \{R_x, : R_{x?}$ c U^}. We give the proof for the existence of a /J and a K as defined above (the proof for *y* and K^T is analogous) by applying 5.4.1 to each V

Suppose no such pair j8,K exists. Then, for each j8efl there are at most countably many $x^f > x$ with $|x^!| = |x|+1, x^f(|x|+1) > 0$, for which there is an indicator K_x, with

$$(\Sigma(K_{x'}, x'+))^* \subset U_\beta.$$

Hence, altogether, there are fewer than C distinct $x^1 > x$ with $|x^!j = |x| + 1, x^T(|x|+1) > 0$, for which there is some U_β and an indicator K_{x^f} with

$$(\pounds(K_{xf}, x<+))^* \subset Up.$$

Now, pick any $x\wedge > x$ with $|x_1| = |x|+1, x\wedge|x|+1) > 0$ for which no such pair exists. The process repeats: there are fewer than C distinct x" with $|x"| = |x|+2, x^M > x\wedge x^{11}(|x|+2) > o$ for which there is some U_p and an indicator $K_{x''}$ with

$$(E(K..,,,x«+))^* \subset U_{ft}.$$

Pick any $x_2 > x_1$ with $|x_2| = |x|+2, x_2(|x|+2) > o$ for which no such indicator exists.

In this way we get a nested sequence

$$\bar{x} \quad \bar{x}_1 \quad \bar{x}_2$$

and the unique point $p \in P_1$ such that $x_n < p$ for all n is in their intersection. For no n is it true that

$$(L(K_{\bar{x}_n}, x_n+))^* \subset XJ_\beta \quad \text{for some } \pounds \text{ and some } K_{x_n}.$$

And, a fortiori, none of the R_{x_n} is contained in U_β for any *fJ*. But this violates the hypothesis that the U_β constitute as open cover of R_x, for $p \in R_x$ and the R_{x_n} form a local base at p (2.8).

5.6.2. Lemma. Let £u/?}/Jeft ^e an open cover for $R_{(p,n)}$ with $|H| < C$. Then there exists $\beta \in B$ and an indicator K such that $(E(K, (p,n)))^* \subset U_\beta$.

Proof. Proof is as in 5.6.1. What we obtain is, inductively, a sequence of $q^m \in R^\circ_{(q^{m-1}, n+m-1)}$ such that for no β and no indicator K_m is it true that

$$(\Sigma(K_m, (q^m, n+m)))^* \subset u_\beta.$$

and we thus get a nested sequence

$$R_{(p,n)} \supseteq R_{(q^1,n+1)} \supseteq \cdots \supseteq R_{(q^m,n+m)} \supseteq \cdots$$

and now define $q \in P_2$ with $q_x = p_x, q_y = p_y, q_z(i) = p_z(i)$ for $i = 1,\ldots,n-1, q_z(n+m-1) = q^m_z(n+m-1)$. It is easy to see that $R_{(q,n+m)} = R_{(q^m,n+m)}$ and so the above nested sequence forms a local base at q, and its intersection is $\{q\}$, and $q \in R^\circ_{(p,n)}$. But for no m is it true that $R_{(q,n+m)} \subset u_\beta$ for some β, violating the assumption that the U_β form an open cover for $R_{(p,n)}$.

5.7. ~~Lemma~~, if K is an indicator, and $x = y(p,n,\pm)j$ with $j \geq |K|+1$, then

$$(\Sigma(K,x))^* \subset (\Sigma(K,(p,n)))^*$$

while if $j < |K|+1$, $(\Sigma(K,x))^* \cap (\Sigma(K,(p,n)))^* = \emptyset$.

Proof. First, if $j \leq |K|$, it can be easily seen from the definitions that $R_x \cap R_{(q,m)}$ is empty, for all q in $S.S^j$ and hence a fortiori we get the result above. For the remainder of the proof, assume $j \geq |K|+1$.

For each $\pi \in K$, let $x_{\pi\mp} \in X$ such that

$$|x_{\pi\mp}| = |x|+|K|, x_{\pi\mp} > x, x_{\pi\mp}(|x|+i) = \pi(i) \quad \text{for } i = 1,\ldots,|\pi|,$$

and let $\phi \in P_2$ be such that $\psi_\pi \phi_{n} \ni (q\pi)_z(n+i-1) = \pi(i)$ for $i=1,\ldots,|\pi|$.

We have then by 5.3.3 and 5.3.4 that if $w \in K$ then

$R_{x_{\pi\mp}} \in \Sigma(K, x\mp)$ and R, ... h e£(K, (p,n)).

Furthermore, in presence of the definitions of $x_{\mp}$ and q^, an inspection of 1.3.7-1.3.11 of [R_2] will show that

$$x_{\pi\mp} = \gamma(q_\pi, n+|K|, \pm)_{j-|K|}.$$

Again by 5.3.3 and 5.3.4 we have

$$E(K,xT) = \{R_{x_{\pi\mp}} : \pi\in K\}$$

and

$$E(K,(p,n)) = (R_{(\%jn+|Kl)} : TT6K)$$

and so

$$(E(K,\overline{x+}))^* \subset \{R^T(a_\pi, n+|K|) : \pi\in K\}^* \subset (E(K, (p,n)))^*.$$

5.13.1. Lemma. Let {V } r be a partition of R into clopen sets, |G| < C. Then there is an aeG for which there is an indicator K^ such that $(L(K_1 \wedge x+))^\wedge \subset V_\alpha$ and there is a 6eG for which there is an indicator K_2 such that $(£(K_2,x-))^* \subset V_\beta$. Furthermore a = 6 and a is unique.

<u>Proof</u>. 5.6.1 guarantees the existence of V_a and V_o. 5.8 of [R_2] can now be used to show that V^α and $V_{\bar{fi}}$ must have a nonempty intersection: assume on the contrary $V^\alpha \cap V_{\bar{g}} = j6,$ then apply 5.8 to $\bar{V}$ and V~ ; if $(E(K_9,x-))^* \subset v_m$ then a fortiori $(E(K_2,x-))^* \subset v^\wedge$ and now let $x_{]L} > x$ be such that $|x_1| = |xj + 1$ and there exist indicators K_3 and K_4 such that $(\Sigma(K_3, x_{j-}^\wedge))^* \subset v_{oc}$, $CE(K_4, x_{1-}))^\wedge \subset V^\wedge_{ex}$; by induction define $x_n > x_{n-1}$ with $|x_n| = |x_{n-1}| +1$ so that there exist indicators K_{x_n} and $(L(K_{x_n}, x_n+))^* \subset V_a$, $(S(K^\wedge_n, *_n-))^* \subset V\hat{}_7$ in this way we get a nested

sequence $[R_{x_i}\}_{i=l}^{\infty}$, and each set in the sequence has a nontrivial intersection with V_{OC} and V_{OC}^{C}. Now $\mathop{.£L}_{1-X} R_{X_{I}} = \{p\}$ for some $peP._{X},$, and since both V_{α} and V_{α}^{C} are closed, p is in both of them, which is absurd. So $V_{\alpha} D V_{\theta}$ ^ ^, and since these sets are members of a partition, $V^{\alpha} = V_{fi}$. Now suppose there exists $V\gamma$, yeG, and an indicator K such that (£(K,x+))* <= V^{γ} . Then V^{γ} D V_{fi} / θ (same proof as above with *y* substituted for a) and so $V^{\gamma} = V^{\alpha}$. Similarly, if we have V^{γ} ,yeG and an indicator K such that (£(K,x-.))* c V^{γ} then $V\gamma$ D V_a ^ *ft>₉* and $V^{\gamma} = V^{\alpha}$.

5.13.2. ~~Lemma~~o Let $\{V_{\alpha}\}_{\alpha\epsilon Q}$ be a partition of $R_{,(p,n)}$ into clopen sets, |G| < C. Then there is exactly one aeG for which there is an indicator K such that [£(K,(p,n))]* c V_{α}.

~~Proof~~o Existence of a is shown by using 5.6.2. To show uniqueness, suppose [£(1^,(p,n))]* c V_a, and [£(K_2,(p,n))]* c V_{β}. Let x = y(p,n,i)j with j J> max{ | K^J , | K_2 | }+l. Then

$$(LiK^xT))^* \ c \ [Et^, (p,n))]^* \ c \ V_a$$

and

$$(L(K_2,xT))^* \ c \ [£(K_2(p,n))]^* \ c \ v^{\wedge}_{\#}$$

Now the sets fv_{α} fl $R_x)_{a\epsilon}Q$ partition R_x into clopen sets and hence by 5«13.1, $V_{\alpha} = V_{\beta}$.

5.14₀1. Lemma o Let $U_{x} = \{U:U$ is clopen in A and there exists an indicator K with (E(K,x+))* c u}o U^{--} is a clopen filter.

~~Proof~~; Clearly if UeU^{--}_x and V is clopen in A with U c v, then VeU^{--}_x and clearly j^U . Now suppose U_{x_1} and U_{o} are both members of U^{x}, with associated indicators K_1 and K_2, respectively. Consider the four clopen sets U.$_1$ 0 U$_2$*U-i$_1$\$^U$2'^{U}2^^{U}1' and (U$_1$ U U$_2$) „ Their intersection with R$_x$ gives a partition of R$_X$ into clopen sets. Now (£(K,x+))* c R$_X$ for all • k. (U$_1$\u$_2$)eU$_x$ iff ((l^V^) n R$_x$)e^u$_x$- But

$$(E(K_2, x+))^* \subset (u_2 \cap R_x) \in U_x$$

and since ((U^t^) 0 R$_x$) n (U$_2$ 0 R^) = *f*>, we cannot have (U^U^el$_x$. In a similar way we eliminate ^{U}9\$^u$i an<* (^{u}i ^ ^2^°' so fc^at U, 0 U$_9$€U by 5.6.1.

5.14#2. Lemmao Given P€P$_2$^ let U$_{(p,n)}$= {u : U is clopen in A and there exists an indicator K with [£(K,(p,n))]* c u}. U . is a clopen filter.

Proof. Proof is as in 5.14.1.

5.15.1. Lemmao U$_x$ is a free clopen ultrafilter with the yffi-intersection property for all *M* < C, and U$_x$ ^U$_{x^f}$ are distinct whenever x ^ x^f.

Proofo That U is an ultrafilter comes immediately from 5.6.1 and 5.14.1, and we can also use 5.6.1 to show that U has the /w-i.p. : Suppose there is a family of fU$_a$} ~ with U$_a$ eli$_x$ for all a and |G| £ *m* < c, then {U^c_a}$_{a€G}$ forms a clopen cover for R$_X$ and so one of them belongs to U$_X$ and so does its complement, a violation of 5.13.1. To show that U- is free, we show that for no R with |x^f| > |x| is it true that R€U$_x$* and that for no *Rf* (p,n) with n > 1 and R$_p$ c R$_x$ is it true that R$_{(p,n)}$ €U$_x$: if u$_x$ w©re fixed it would either be fixed on a point of P$_1$, in which case it would have as members R$_{x'}$ for arbitrarily large |x^f|, or else it would be fixed on a point $P\in {}^{p}9^{*}$ $\wedge^{n}$ which case it would have as members $R_{(p,n)}$ for arbitrarily high n, and since $p\in R_{x} \subset U_{..}$ we must also have $R_{px} \subset R_{x}$ by 2.1.1.

First, if $|x'| > |x|$, we have that $p(|x|+l)$ is the same real number for all $P\in R_{x}$» ^{n} ^{p}i while by (2) 5.3.2 and 5.3.3 it is possible to find infinitely many p in $(E(K,x+)^{*} \cap P$ with distinct values for $p(|x|+l)$. Hence R_{vt}<fck • Second, if $R_{x,x} \wedge U_{(p,n;}$ x then by 5.3.6 we have $R_{x} \subset R_{(p,n;}$. and by 2.1.2 and 2.1.3 we cannot have $R_{px} \subset R_{x}$. [Note: 5.3.6 was only proven in the case n > 1 but this is all we need.]

To conclude the proof of the Lemma, we note that if $x \neq x'$, then in the case $|x| \wedge |x'|$ we have either $R_{xx} \wedge U_{f}$ or $R_{xx} {}_{f}£U$ depending on which $|x|,|x'|$ is bigger, while we do have R_{x} eli x for all $x\in X$. if $|x| = |x'|$ but $x \wedge x'$, $R_{x} \cap R_{x'} = 0$ and so $R_{x'} \wedge U_{x}$.

5.15.2. Lemma. $U_{(p,n)}$, is a free clopen ultrafilter with the w~intersection property for all $M < C$, and $\wedge_{v} \wedge U.$. are always distinct, while $U_{(p,n)} = U_{(q,n)}$ iff $R_{(p,n)x} = R_{(q,m)y}$.

Proof. To show $U.$. is free, we show that no R with $|x| > |p_{x}|$ is in $U_{(p,n)}$, and that no $R_{(q,m)}$ with $|q_{x}|+m > |p_{x}|+n$ is in $U_{,}$. • freeness follows from the same considerations as above.

If $R_{x} \in U_{(p,n)}$ then $\exists K$ such that $(\Sigma(K,(p,n)))^{*} \subset R_{x}$ which implies **that** $R_{(p,n)} \subset R_{x}$. By **2.1.1** and 2.1.2 this implies $|x| \leq |p_{x}|$.

If $R_{(q,m)} \in U_{(p,n)}$ then for some $K, (\Sigma(K,(p,n)))^{*} \subset R_{(q,m)}$

which implies that $R^?_{(p,n)} \cap R_{(q,m)} \neq \emptyset$. Suppose first $R^?_{(p,n)} \cap R^\circ_{(q,m)} \neq \emptyset$ then we have $|q_x| = |p_x|$. Now all $q' \in P_2 \cap R^\circ_{(q,m)}$, have the same (k)th coordinate for $q \leq$ if $k<m$ while by 5.3.2 and 5.3.4 it is possible to find an infinite set of $q' \in P_9 \cap (\mathcal{E}(K, (p,n)))^*$ with $|q_x| = |p_x|$ and with distinct values for $a_{c^*}(n)$. Since the only $q' \in P_9 \cap R_{(q,m)}$ with $|q_x| = |q| = |p_x|$ are those in $R^{(q,m)}$, it follows that if $(\mathcal{E}(K(p,n)))^* \subset R^{(q,m)}$ then $m \leq n$, so that in this case $|q_x| + m \leq |p_x| + n$.

Second, suppose $R^?_{(p,n)} \cap R_{(q,m)} = \emptyset$, then $R^?_{(p,n)} \cap R_{(q,m+j)} \neq \emptyset$ for some j which implies by 2.1.1 that $|q_x| + m + 1 \leq |p_x|$ and so here too $|q_x| + m \leq |p_x| + n$.

The /H-intersection property follows from 5.6.2 and an argument like that in the previous lemma. Similarly, $U_{(p,n)}$ is an ultrafilter.

To show that U_x and $U_{(p,n)}$ are always distinct, note that $R_x \in U_x$ and $R_{(p,n)} \in U_{(p,n)}$, so that if $R_x \cap R_{(p,n)} = \emptyset$ we are done. If $R_x \cap R_{(p,n)} \neq \emptyset$ then $R_x \ni R_{p_x}$, which rules out $R_x \subset R_{(p,n)}$. Hence by 5.3.6 $R_{(p,n)} \notin U_x$ and so the two clopen-set ultrafilters are distinct in this case. If $R_x \cap R^?_{(p,n)} = \emptyset$ but $R_x \cap R_{(p,n)} \neq \emptyset$ then $|x| > |p_x|$ and as shown above $U_x \neq U_{(p,n)}$.

Finally, suppose $U_{(p,n)} = U_{(q,m)}$. Then, as shown above, this implies $|q_x| + m \leq |p_x| + n$, and also $|p_x| + n \leq |q_x| + m$, so $|p_x| + n = |q_x| + n$. Also, $R^?_{(p,n)} \cap R^?_{(q,m)} \neq \emptyset$, otherwise $|q_x| + m + 1 \leq |p_x|$ as shown above. This implies $R_{(p,n)} = R_{(q,m)}$ by 2.1.4.

5.16. ~~Property VI~~; A is not N-compact.

The proof consists of either 5.15.1 or 5.15.2 together with H. Herrlich's theorem quoted in the Introduction.

Concluding Remarks> With the problem quoted in the Introduction thus solved in the negative, two other problems, also having their roots in [EM], become better defined.

First, as pointed out in [M_2], what [EM] really showed about N-compact spaces is that if X is realcompact and /?X is zero-dimensional (a strictly stronger condition the zero-dimensionality of X) then X is N-compact. (An explicit statement and proof of this may be found in [H, Beispiele 5,6].) The first unsolved problem before us is the converse of this statement. In other words, given a closed subset X of N ^{m}, /m any cardinal number, is it true that j8X is O-dimensional? (The other condition, realcompactness of X, does hold [cf« CJ, pp. 119-120, and p. 72]$_o$) A counter-example would still be zero-dimensional and realcompact, of course, and the author is unaware of any spaces other than A itself and spaces trivially obtainable from A which are zero-dimensional and realcompact and whose Stone-Čech compactification is not zero-dimensional. There are, however, spaces which may be of this sort and which are moreover known to be N-compact. One example is the Sorgenfrey plane: it is the product of two copies of the real line with intervals of the form [a,b) as a base for the topology. Each factor is Lindelof and thus is both realcompact [cf. GJ, p.115] and has zero-dimensional Stone-Čech compactification [cfo GJ, pp.245-7] and is thus N-compact. Hence the Sorgenfrey plane is N-compact. But is its Stone-Cech compactification O-dimensional? Another possible counterexample is the N-compactification of A. (For a definition and construction of the E-compactification of a space, cf$_o$ [EM] or [H, Kapitel I, §3,§9].)

For further discussion of spaces with zero-dimensional Stone-Čech compactification, see [GJ, ch.16] (where these spaces are called simply 'zero-dimensional', while the spaces which this paper calls zero-dimensional are simply designated as 'having a base of open-and-closed sets'), [E, ch,6^§2] (where these spaces are called 'strongly O-dimensional'), [H, Beispiele 5,6], and [N,§2].

The second problem is this: is there a single space E such that the class of O-dimensional realcompact spaces is the class of E-compact spaces? In this paper we have shown that if such a space exists it cannot be N, or any other N-compact space. Might it be A ?

This problem is admittedly less attractive than the first one. If the answer is affirmative, a proof of this result might have to depend on the construction of a non-N-compact, O-dimensional realcompact space that is substantially easier to work with than A I

Bibliography

[E] R. Engelking, *Outline of General Topology*, Amsterdam, North Holland Publishing Co., 1968.

[EM] Engelking and Mrówka, On E-compact spaces, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astr. Phys. J5 (1958), 429-436.

[GJ] Gillman and Jerison, *Rings of Continuous Functions*, Princeton, N.J., Van Nostrand Co., 1960.

[H] H. Herriich, *(g-kompakte Räume* II Mathematisches Institut der Freien Universityt Berlin, 1965• Reprinted in Math. Zeitschr. .96 (1967), 228-255.

[M_1] S. Mrówka, On universal spaces, Bull. Acad. Polon. Sci. ± (1956) 479-481.

[M_2]___________, On E-compact spaces. II, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astr. Phys. 1± (1966), 597-605.

[N] P. Nyikos, Not every 0-dimensional realcompact space is N-compact, Report 70-30, Department of Mathematics, Carnegie-Mellon University.

[R_1] P. Roy, Failure of equivalence of dimension concepts for metric spaces, Bull. Amer. Math. Soc. j58 (1962), 609-613.

[R_2]________, Nonequality of dimensions for metric spaces, Trans. Amer. Math. Soc. jL3£ (1968), 117-132.

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CORRIGENDUM

The statement of the proof of 5^3₀6 is not quite precise* It. is fcrme that n > 1 is the only case needed to prove that £ is not N-eompactj, but I do eventually make *use* of the case n = 1 in order to **show** that an **ultrafilter of the form U** $_x$ **i&** alv;< y distinct f^fom one of the form li, .» **There** are also a few typographical errors,, Here is a corrected proof-.

£?% Suppose (S(K_9.xv)> c R-(p, n).. Than for each Rx,*:E{X,:-r>;- we 'have R , c *R,* . and so R $_f$ fl R -, - $_J$*4* ,0 for serase j (2..1,3); furthermore^ since R c R-(?^ n)' (2.1,3), we **have** '•**(J: »'R I ^ n 4- 1 because sx contains CrP^ **tfifch** 'c^j «« j^i: while all P^ points in R7 $_{K\,pp\,n}$^ must ha-^e x-coordxnates of Ie$_f$rç:h at least [p f + n •+ l> This implies R » • * 3 R , for exactly one j (2olol;

If R..<f: R,.. -t*ix| < 'p$_v$i + n (2..1.1) , Pick x^{J}' such that x^{rf}(i) « x''(i) for i » 0^o,o,-p^: 4 n, x$^{9)}${j p $_x$j * n ^ 1; x^{l}(!p$_x$l ^ n -^ 1) and Py?r{K;x+), If n > i we have q(;p$_x$l + n + 1) = P^^n - I) for all q-^ n *i<*$_{(p,n)}$, New let q^fp^ H R$_x$« and **q^r-$Pj** H R $_x$ - *q^n* and q cannot bcth *he ii<.* ^{K}/$_{(Pi-n)}$^^« contradicting £{K,X4-) c R$_{(p_9 n)}$

If n « 1 we distinguish three cases; . *x'* » jp j i.$_r$ ix[- |p-ic |xj < jp i* If ;-; « |p j 4^ 1 then we take q^{*1} and q in Cs(K^x4-))* n P$_1$ With q^{r9}C;pj - 2/ ^ q'dP i ^ 2/ Since the ClP$_x$^ ^ D-coordinates of q$^{c?}$ and q are the sam-c-. and since any P$_1$^points in R$_{(p,3)}$ with identical CiP$_x$! ^ li- coordinates also have icientxcal (ip^f-^- 2)-coordinates (1,3,.9-10)

q^{fl} and q^* cannot both be in $R^*{}_{1>0}$ If $jxf =* \{p_v j$ the same argument works since $I K| j£ 2$ and so we can find q^M and q^* that differ in the $(|p_V(+ 2)$-coordinate. The last case is disposed of by finding $q€(£(K^*xf))^* n Pj^\wedge$ such that $q(\{p_x() >= P_K(|p_x f)$.

Now the coitmient in brackets on page 12, lines 9-10 becomes superfluous* as does "with $n > l$" on page 11, four lines fron the bottom*

CS
12/15/70