FUNCTIONS OF BOUNDED VARIATION
ON IDEMPOTENT SEMIGROUPS

by

Richard A. Alo and Andre de Korvin

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1. INTRODUCTION.

In [2], Dunford and Schwartz state a characterization of the dual space of the space of all absolutely continuous functions on an interval I. This latter space is represented by AC(I). In particular each element x^* of $AC(I)^*$ can be represented as $x^*(f) = a\, f(a) + \int_I gf'ds$ where g is in $L_\infty(I)$ -the space of essentially bounded Lebesgue measurable functions on I.

For the more general class BV(I) of functions of bounded variation on the interval $I = [0,1]$, T. Hildebrand in [4], has given a representation theorem for linear functionals on BV(I), continuous in the weak topology.

In [5], S. Newman defined a more general space of functions of bounded variation. Such functions and absolutely continuous functions are defined over an abelian idempotent semi-group and more particularly over a semi-group of semi-characters which themselves are defined over an abelian idempotent semi-group.

The main purpose of our paper is to study in more detail the space defined in [5]. In this respect let S denote a semigroup of semi-characters defined over an abelian idempotent semi-group A and let F denote a fixed function of bounded variation. Then AC(S,F) denotes all functions of bounded variation which are absolutely continuous with respect to F. The first main result of this paper deals with the representation of the dual space of AC(S,F). This representation is obtained via a fundamentally bounded convex set function K by the formula $T(G) = v\int G \bullet dK$ for all G in AC(S,F), where $v\int G \bullet dK$ is the "variational integral of G with respect to K".

Our second main result deals with the so called Lipschitz functions. It turns out that there exists a one-to-one and onto correspondence between Lipschitz functions and convex bounded set functions. Moreover the Lipschitz bound is equal to the bound of the corresponding convex function. The first two results are a generalization of some of the results contained in [3]. The techniques used in [3] depend strongly on the fact that the functions are defined on [0,1]. Thus our first main result would, for example, yield a representation for the dual of $AC(I \times I)$ while the techniques developed in [3] would not apply to this case.

Our third result is a generalization of the more classical results contained in, for example, [6]. It will be shown that if F and G are functions of bounded variation in the sense of [5] and if F satisfies some side conditions then $\frac{dG}{dF}$ in some sense exists almost everywhere.

Throughout the entire paper it will be seen that much is to be gained when functions of bounded variation are considered as bounded measures over some algebra.

2. BASIC DEFINITIONS AND RESULTS.

In this section we will define the notations and give some of the basic results of [5] which will be used throughout the manuscript. The latter are used considerably to obtain the first two main results. We will make frequent use of notation in [6].

To this end let A denote an abelian idempotent semi-group and let S be a semi-group of semi-characters on A containing the identity. Recall that a semi-character on a semi-group A is a non-zero, bounded, complex valued function on A which is a semi-group homomorphism. A semi-character on an idempotent semi-group is an idempotent function and hence it can assume only the values zero and one.

If f is in S, then A_f will denote the set $\{a \in A : f(a) = 1\}$ and J_f will denote the set $\{a \in A : f(a) = 0\}$. Now the sets $J_f (f \in S)$ generate a Boolean Algebra $\mathcal{A}$ of subsets of A. Let T_n be the Boolean algebra of all n-tuples of zeros and ones, let $X_n = \{f_1, \ldots, f_n\}$ be a finite subset of S, and let $\sigma \in T_n$. Then if $\sigma(i)$ denotes the i^{th} component of σ, let

$$(1) \qquad B(X_n, \sigma) = \Big(\bigcap_{\sigma(i)=1} A_{f_i} \Big) \cap \Big(\bigcap_{\sigma(i)=0} J_{f_i} \Big)$$

A set will be called *of B-type* if it is of the form $B(X_n, \sigma)$ (for some X_n and for some σ).

If F is a real valued function on S and if X_n and σ are as above we can define an operator L by

(2) $$L(X_n, \sigma)F = \sum_{\tau \in T_n} m(\sigma, \tau) F\left(\prod_{i=1}^{n} f_i^{\tau(i)}\right)$$

where

$$m(\sigma, \tau) = (-1)^{|\tau| - |\sigma|} \qquad \tau \geqq \sigma$$

$$= 0 \qquad \text{otherwise}$$

is the Mobius function for T_n (see [7]). We note that in this notation $|\sigma|$ denotes the number of ones in the n-tuple σ. Now we call F a *function of bounded variation on S if*

(3) $$\sup_{X_n} \sum_{\sigma \in T_n} |L(X_n, \sigma)F| < +\infty$$

where the supremum is taken over finite subsets X_n of S. The norm of F, written $|F|_{BV}$ is the variation norm of F

defined by (3). Let BV(S) denote the set of all functions of bounded variation on S. Then any $F \in BV(S)$ is called <u>positive definite</u> if

$$L(X,\sigma)F \geq 0$$

for all finite subsets X of S and for all corresponding σ. For a fixed $F \in BV(S)$, by AC(S,F) we will mean all functions G' in BV(S) such that for any finite set $X_n = \{f_1, f_2, \ldots, f_n\}$ of S and for any subset H of T_n if $\epsilon > 0$ is given, there exists a $\delta > 0$ such that

$$\sum_{\sigma \in H} |L(X_n,\sigma)F| < \delta \quad \text{implies} \quad \sum_{\sigma \in H} |L(X_n,\sigma)G| < \epsilon .$$

Such functions G in BV(S) which satisfy this condition are called <u>absolutely continuous with respect to</u> F.

Let $M(\mathfrak{a})$ denote all bounded, finitely additive measures defined on the Boolean algebra $\mathfrak{a}$ generated by the sets $J_f (f \in S)$. In [5], it was shown that BV(S) is isometric, isomorphic as an algebra to $M(\mathfrak{a})$. If for $F \in BV(S)$, μ_F is the corresponding element in $M(\mathfrak{a})$ we have

$$|F|_{BV} = \|\mu_F\|$$

where $\| \ \|$ denotes the variation of the measure. Moreover, if $G \in AC(S,G)$, then $\mu_G \ll \mu_F$ and positive definite functions correspond to positive finitely additive measures.

Now it is clear that these definitions do generalize the spaces $BV(I)$ and $AC(I)$ where I is the unit interval. To see this let A be the semi-group $[0,1]$ under maximum multiplication (that is $x \ y = \max(x,y)$ for $x,y \in [0,1]$); and let $S = \{\psi_{[0,x]} : x \in [0,1]\}$ be the given semi-group of semi-characters on A where $\psi_{[0,x]}$ denotes the characteristic function on $[0,x]$. Now S under pointwise multiplication is a semi-group isomorphic to $[0,1]$ with the minimum multiplication. If we let $F(\psi_{[0,x]}) = x$, then we see readily that $BV(S)$ and $AC(S,F)$ coincide with $BV(I)$ and $AC(I)$ respectively. For further details see [5].

3. MAIN RESULTS.

We now assume that $F \in BV(S)$ is a fixed positive definite function. We first verify that the subspace $AC(S,F)$ *of* $BV(S)$ *is a Banach space*.

Lemma 3.1. *The space* $AC(S,F)$ *of functions of bounded variation absolutely continuous with respect to a fixed positive definite* $F \in BV(S)$ *is a closed subspace of* $BV(S)$.

Proof. In [5] it was shown that $BV(S)$ is a Banach space. Consequently if $\{P_n\}$ is a Cauchy sequence in $AC(S,F)$ then $\{P_n\}$ converges to $P \in BV(S)$ in the variation norm. For each $n \in N$ (the positive integers) let μ_n be the corresponding elements in $M(\mathfrak{G})$ of P_n and μ_P the correspondent of P. Then $\{\mu_n\}$ converges to μ_P in the variation norm. Since μ_n is absolutely continuous with respect to μ_F, it follows that μ_P is also absolutely continuous with respect to μ_F. Thus $P \in AC(S,F)$.

We can now define the analogue of a "polygonal function". Let $\psi_{B(X,\sigma)}$ denote the characteristic function of $B(X,\sigma)$ (to simplify notation let us have $X = X_n$ unless we wish to emphasize the number of elements in X) and let $S = \sum_{\sigma \in T_n} \lambda_\sigma \psi_{B(X,\sigma)}$

where λ_σ denotes some scalars. Consider now the integral $V_S(E) = \int_E S \, d\mu_F$. Clearly V_S is absolutely continuous with respect to μ_F and thus V_S corresponds to some element $P_S \in AC(S,F)$.

Definition 3.2. A function of the form P_S as demonstrated above is called a polygonal function relative to F.

Lemma 3.3. *The collection* $P(S,F)$ *of polygonal functions relative to* F *are dense in* $AC(S,F)$ *where* S *is as defined above*.

Proof. This follows immediately from a result in [1] which states that measures of the type $V_S(E)$ are dense in the variation norm in the set of measures which are absolutely continuous relative to μ_F .

We are now interested in a special class of polygonal functions. Let $G \in AC(S,F)$ and let $X = X_n$ be a finite set in S . Let

$$W_{X,G} = \sum_{\sigma \in T_n} \frac{\mu_G(B(X,\sigma))}{\mu_F(B(X,\sigma))} \cdot \psi_{B(X,\sigma)}$$

If $\mu_F(B(X,\sigma)) = 0$, then $\mu_G(B(X,\sigma)) = 0$ and the ratio is defined to be 0. Let

$$V_{X,G}(E) = \int_E W_{X,G}\, d\mu_F$$

for all $E \in \mathcal{Q}$. Clearly $V_{X,G}$ is absolutely continuous with respect to μ_F. Let pG_X be the function in $AC(S,F)$ corresponding to $V_{X,G}$.

Definition 3.4. The polygonal function pG_X defined above is called *a polygonal function associated with* G.

Lemma 3.5. *If* $G \in AC(S,F)$ *then* G *can be approximated, in the variation norm by polygonal functions associated with* G, *that is for* $\epsilon > 0$ *there exists a finite set* X_n *such that for all finite sets* Y_m *of* S *such that* $X_n \subset Y_n = Y$, *the difference* $|G - pG_Y|_{BV} < \epsilon$.

Proof. In [5] it is shown that μ_G is approximated in the variation norm by $V_{X,G}$. This establishes the lemma.

Continuing in an analogous fashion we may define the concept of *convex set functions*.

Definition 3.6. Let K be a function from all subsets of A of B-type to the reals. The function K is called *convex* (*relative to* F) if for every $B(X_n,\sigma)$ $(\sigma \in T_n)$ which is a disjoint union of sets $B(Y_m,\tau_j)$ $(X_n \subset Y_m$ and $\tau_j \in T_m)$.

we have

$$K(B(X_n,\sigma)) = \Sigma \lambda_j K(B(Y_m,\tau_j))$$

where

$$\lambda_j = \frac{\mu_F(B(Y_m,\tau_j))}{\mu_F(B(X_n,\sigma))} .$$

Thus $K(B(X_n,\sigma))$ is a convex combination of $K(B(Y_m,\tau_j))$. The function K is called bounded if for some constant C, $|K(B)| < C$ for all sets B of B-type. The least value of C is called the bound of K and is denoted by $\|K\|$.

Definition 3.7. Let H be a function from $BV(S)$. By the variational integral of H relative to the convex bounded set function K we mean the limit if it exists of $\sum_{\sigma\in T_n} L(X_n,\sigma)H \ K(B(X_n,\sigma))$. This limit is denoted by $v\int H \cdot dK$.

Thus if the limit exists in 3.7 then for every $\epsilon > 0$ there is a finite subset X_n of S such that for all finite subsets Y_m of S, $X_n \subset Y_m$, we have

$$|v\int H \ dK - \sum_{\tau\in T_m} L(Y_m,\tau)H \ K(B(Y_m,\tau))| < \epsilon$$

We can now show that all polygonal functions are integrable relative to the convex bounded set function.

<u>Lemma</u> 3.8. <u>If</u> P_S <u>is a polygonal function, then</u> $v\int P_S \, dK$ <u>exists. In fact for some finite set</u> Z <u>of</u> S,

$$v\int P_S \, dK = \sum_{\sigma \in T_n} L(X_n,\sigma) P_S \quad K(B(X,\sigma))$$

<u>for all finite subsets</u> X_n <u>of</u> S <u>with</u> $Z \subset X_n$.

<u>Proof</u>. For $Z \subset X = X_n$ let $S = \sum_{\sigma \in T_n} \lambda_\sigma \, \psi_{B(Z,\sigma))}$ then $V_S(B(X,\sigma)) = \lambda_\sigma \, \mu_F(B(X,\sigma))$. So $L(X,\sigma) P_S = \lambda_\sigma \, \mu_F(B(X,\sigma))$. Now let $Y = Y_m$ be a finite subset of S such that $Z \subset X \subset Y$ and such that $B(X,\sigma)$ is a disjoint union of sets $B(Y,\tau_j)$, $\tau_j \in T_m$. By the convexity of K we have

$$K(B(X,\sigma)) = \Sigma \lambda_j \; B(Y,\tau_j) \qquad \text{where} \qquad \lambda_j = \frac{\mu_F(B(Y,\tau))}{\mu_F(B(X,\sigma))}$$

Thus we have

$$\sum_{\sigma \in T_n} L(X,\sigma) P_S \; K(B(X,\sigma)) = \sum_{\tau_j \in T_m} L(Y,\tau_j) P_S \; K(B(Y,\tau_j))$$

for all X and Y such that $Z \subset X \subset Y$. This completes the proof of the lemma.

We now have everything necessary to demonstrate the first main result.

<u>Theorem</u> 3.9. <u>Let</u> T <u>be an element in the dual space</u> AC(S,F)* <u>of</u> AC(S,F). <u>There exists a unique convex bounded function</u> K <u>such that</u> $T(G) = v\int G \cdot dK$ <u>for all</u> G <u>in</u> AC(S,F). <u>Moreover</u> $\|T\| = \|K\|$.

<u>Proof</u>. Let $X = \{f_1, \dots f_n\}$ be a finite subset of S and let $\sigma \in T_n$. Define $W_{X,\sigma}$, a measure on $\mathcal{C}$, by $W_{X,\sigma}(E) = \frac{\mu_F[B(X,\sigma) \cap E]}{\mu_F(B(X,\sigma))}$. Since $W_{X,\sigma}(E) \leq \frac{\mu_F(E)}{\mu_F(B(X,\sigma))}$ for all $E \in \mathcal{C}$, $W_{X,\sigma} << \mu_F$. Let $\psi_{X,\sigma}$ be the corresponding function in AC(S,F). Define $K(B(X,\sigma)) = T(\psi_{X,\sigma})$. Since

$$V_{X,G}(E) = \int_E \sum_{\sigma \in T_n} \frac{\mu_G(B(X,\sigma))}{\mu_F(B(X,\sigma))} \psi_{B(X,\sigma)} \, d\mu_F = \sum_{\sigma \in T_n} \mu_G(B(X,\sigma)) W_{X,\sigma}(E)$$

it follows that $p\,G_X = \sum_{\sigma \in T_n} L(X,\sigma) G \; \psi_{X,\sigma}$. By Lemma 3.5,

$$T(G) = \lim_X T(p\,G_X) = \lim_X \Big(\sum_{\sigma \in T_n} L(X,\sigma) G \; K(B(X,\sigma)) \Big) .$$

Thus by Lemma 3.8, $T(G) = v\int G \cdot dK$, since Lemma 3.8 says that polygonal functions are integrable and Lemma 3.3 says that G is a limit of polygonal functions. Now $\|T\| \geq \sup |T(\psi_{X,\sigma})| = \|K\|$ since $|\psi_{X,\sigma}|_{BV} = \|V_{X,\sigma}\| \leq 1$. Also

$$|v \int G \cdot dK| \leq |G|_{BV}\|K\| ,$$

so

$$\|T\| = \|K\| .$$

Proposition 3.10. Let K be a convex and bounded set function. The map which maps $G \in AC(S,F)$ to $v\int G \cdot dK$ is a bounded linear functional on $AC(S,F)$.

Proof. It has been shown that $v \int G \cdot dK$ exists for all $G \in AC(S,F)$. In fact

$$|v \int p\, G_x\, dK - v \int p\, G_y\, dK| \leq |p\, G_x - p\, G_y|_{BV}\|K\| .$$

Thus by completeness $\lim_X v \int p\, G_x\, dK$ exists. So

$$v \int G \quad dK = \lim_X \quad v\int p\, G_x\, dK .$$

The inequality

$$|v \int G \quad dK| \leq |G|_{BV}\|K\|$$

finishes the proof.

The variational integral enables us to define absolutely continuous measures relative μ_F . If $\eta_G(E) = v\int_E G \cdot dK$ where $v\int_E G \cdot dK = v\int G_E \cdot dK$ (G_E corresponding to the measure $\mu_G(E \cap (\cdot))$), then it is clear that η_G is a finitely additive absolutely continuous measure relative to μ_F .

It is now possible to investigate the so-called Lipschitz functions of $BV(S)$. Let $M_c(F)$ denote all convex and bounded set functions (relative to F). Let $BV_L(S,F)$ denote all functions $G \in BV(S)$ satisfying the following: $|L(X,\sigma)G| < DL(X,\sigma)F$ for all finite subsets X of S, $\sigma \in T_n$ and where D is some positive constant and F is positive definite.

<u>Definition</u> 3.11. The functions $G \in BV_L(S,F)$ defined above are called <u>Lipschitz functions</u> (<u>relative to</u> F) or <u>F-Lipschitz</u>.

It is now possible to show that there is a one-to-one, onto linear map from $BV_L(S,F)$ onto $M_c(F)$. Hence as vector spaces the Lipschitz functions are isomorphic to the set of convex functions.

<u>Theorem</u> 3.12. <u>The spaces</u> $BV_L(S,F)$ <u>and</u> $M_c(F)$ <u>are isomorphic as vector spaces</u>. <u>Moreover the Lipschitz bound</u>

<u>of a function in</u> $BV_L(S,F)$ <u>is equal to the bound of the corresponding convex set function in</u> $M_c(F)$.

<u>Proof</u>. Let $K \in M_c(F)$ and let $\mu_K(B) = \mu_F(B)\ K(B)$ for all B-type sets B. Now for a finite disjoint family of sets B_i of B-type, $\mu_K(\cup B_i) = \mu_F(\cup B_i)\ K(\cup B_i)$. By convexity

$$K(\cup B_i) = \Sigma \frac{\mu_F(B_i)}{\mu_F(\cup B_i)} K(B_i) .$$

Thus

$$\mu_K(\cup B_i) = \Sigma\ \mu_F(B_i)\ K(B_i) = \Sigma\ \mu_K(B_i)$$

and $\mu_K << \mu_F$. Let H_K be the corresponding function to μ_K in $AC(S,F)$. Now

$$\begin{aligned} |L(X,\sigma)H_K| &= |\mu_K(B(X,\sigma))| \\ &\leq \|K\| \mu_F(B(X,\sigma)) = \|K\| L(X,\sigma)F . \end{aligned}$$

Thus H_K is a Lipschitz function and the Lipschitz bound is less or equal to $\|K\|$.

Now let H be a Lipschitz function, define

$$K_H(B(X,\sigma)) = \frac{\mu_H(B(X,\sigma))}{\mu_F(B(X,\sigma))} .$$

Then

$$\sum_\sigma \frac{\mu_F(B(X,\sigma))}{\mu_F(\cup B(X,\sigma))} K_H(B(X,\sigma)) = \sum_\sigma \frac{\mu_H(B(X,\sigma))}{\mu_F(\cup B(X,\sigma))}$$

$$= K_H(\cup B(X,\sigma))$$

where the union is taken over $\sigma \in T_n$. Then K_H is convex and $|\mu_H(B(X,\sigma))| \leq C\ \mu_F(B(X,\sigma))$ (where C is the Lipschitz bound for H) implies $C = \|K\|$.

4. DERIVATIVE ANALOGUES.

It is now possible to develop results analogous to those found in the usual treatises for example as found in [6]. If F and G are two functions from BV(S) we would like to be able to say that the derivative $\frac{dG}{dF}$ exists μ_F a.e. It should be pointed out that even if μ_G is absolutely continuous with respect to μ_F the Radon-Nikodym derivative fails since the measures are only finitely additive. At best the derivative is represented by a net as is pointed out in [5].

The following assumptions should be made.

(i) *The measures* μ_G *and* μ_F *can be extended as countably additive set functions to* $\bar{\mathfrak{C}}$, *the* σ-*field generated by* $\mathfrak{C}$.

(ii) *For all* X *and* σ , $\mu_F(B(X,\sigma)) \neq 0$.

(iii) *For every* $\epsilon > 0$ *there exists a set of the form* $B(X,\sigma)$ *such that* $\mu_F(B(X,\sigma)) < \epsilon$.

Hence we are essentially assuming that the representative measures of F and G are extendable, that μ_F is faithful and that "there exists enough elements of S relative to the measure μ_F", or in a more intuitive manner, X , can be picked large enough so that $B(X,\sigma)$ has small measure. Of course such is the case if S can be identified with an interval.

Here we give a slightly modified version of Vitali's covering.

Definition 4.1. Let $\mathfrak{B}$ be all sets of B-type in A. The collection $\mathfrak{B}$ is expansive at a point x in A if whenever $C \in \mathfrak{B}$, $x \in C$ and $C \cap D \neq \emptyset$ where $D \in \mathfrak{B}$ and $\mu_F(C) < k\, \mu_F(D)$ for some real constant k then there is a $D' \in \mathfrak{B}$ such that $x \in D'$, $D \subset D'$ and $\mu_F(D') < k\, \mu_F(D)$. If $\mathfrak{F}$ is some sub-collection of $\mathfrak{B}$ and if $E \subset A$ then $\mathfrak{F}$ forms a V-covering of E if for every $x \in E$, $\epsilon > 0$ and $x \in B \in \mathfrak{B}$ there is a $C \in \mathfrak{F}$ such that $x \in C \subset B$ and $\mu_F(C) < \epsilon$. It will be said that A has the V-property if for every $E \subset A$ and for every V-covering $\mathfrak{F}$ of E and $\epsilon > 0$ there exists a sequence $\{B_1, \ldots, B_n\}$ of pair-wise disjoint sets in $\mathfrak{F}$ such that

$$\mu_{F*}(E - \bigcup_{i=1}^{n} B_i) < \epsilon .$$

We shall denote the outer measure generated by μ_F as μ_{F*}.

Lemma 4.2. If $\mathfrak{B}$ is expansive at each point x in A, then A has the V-property.

Proof. Choose disjoint sets from a V-covering $\mathfrak{F}$ of a subset E of A as follows:

Let B_1 be arbitrary and say $B_1, B_2, \ldots B_n$ have been chosen. Let $K_n = \sup \mu_F(B_\alpha)$ where the sup is taken over all sets B_α of $\mathfrak{F}$ which are disjoint from $B_1, B_2, \ldots B_n$. Clearly $0 \leq K_n < \infty$. If $E \not\subset \bigcup_{i=1}^{n} B_i$ obtain a set B_{n+1} such that $\mu_F(B_{n+1}) > \frac{K_n}{2}$ ($E - \bigcup_{i=1}^{n} B_i$ is a set in $\mathfrak{A}$ and thus contains a set of $\mathfrak{F}$).

Now for N large enough $\sum_{N+1}^{\infty} \mu_F(B_i) < \epsilon$. Let $R = E - \bigcup_{i=1}^{N} B_i$.

For $x \in R$, there exists $C \in \mathfrak{F}$ such that $x \in C$, C is disjoint from $B_1, B_2, \ldots B_N$. Suppose that C is disjoint from $B_1, B_2, \ldots B_n$, then $\mu_F(C) \leq K_n < 2\,\mu_F(B_{n+1})$. Since $\sum_{N+1}^{\infty} \mu_F(B_i) < \epsilon$ it follows that $\mu_F(B_n) \to 0$. On the other hand $\mu_F(C) \neq 0$; thus C intersects some B_n. Let k be the first integer such that $C \cap B_k \neq \emptyset$ $(k > N)$. Since $\mathfrak{B}$ is expansive at each x in A, there is a $D^k \in \mathfrak{B}$ such that $\mu_F(D^k) < 2\,\mu_F(B_k)$. Therefore

$$\mu_{F*}(R) \leq 2 \sum_{k \geq N} \mu_F(B_k) < 2\,\epsilon .$$

In particular there exists $Q \in \overline{\mathfrak{A}}$, the σ-field generated by $\mathfrak{A}$ (Q is a countable disjoint union of sets of $\mathfrak{A}$) such that $\mu_{F*}(E - Q) = 0$.

Corrections to Research Report 70-39

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Page 21, between line 3 and line 4, insert:

"For the remainder of this report it is assumed that the collection B of all sets of B-type in A is expansive at each point x in A."

Page 22, line 8:

Change "Lemma 5" to "Lemma 4.2".

11-18-70

/ps

It should be remarked that if the fundamental sets are taken to be half open intervals of $A = I$, then A has the V-property.

<u>Definition</u> 4.3. Let $\{C_n\}$ be a sequence of sets of the B-type. The sequence $\{C_n\}$ converges to the point $p \in A$ if

(1) $\mu_F(C_n)$ converges to zero.

(2) For every set B of B-type such that $p \in B$, $C_n \subset B$ for all but a finite number of n.

(3) $p \in C_n$ for all n.

If C_n converges to p we shall write $C_n \to p$.

Lemma 4.4. *Assume that for every* $p \in H \subset A$, $\liminf \frac{L(C_n)G}{L(C_n)F} < r$ *for some sequence* $\{C_n\}$ *converging to* p. *Let* $H \subset B$ *where* B *is a countable union of disjoint sets of* B-*type*. *Then* B *contains a set* $Q \in \overline{\mathfrak{A}}$ *such that* $\mu_F^*(H - Q) = 0$ *and* $\mu_G(Q) < r\,\mu_F(Q)$.

Proof. Consider $H \subset B$ and let $\mathfrak{F}$ denote all sets C of B-type such that $L(C)G < r\,L(C)F$, $C \subset B$. Then $\mathfrak{F}$ forms a V-covering of H. By Lemma 5 there exists a set Q in $\mathfrak{F}$ which is a countable disjoint union of sets of B-type and $\mu_F^*(H - Q) = 0$. Let $Q = \cup B_n$. Then

$$\mu_G(Q) = \Sigma\,\mu_G(B_n) < r\,\Sigma\,\mu_F(B_n) = r\,\mu_F(Q) .$$

Lemma 4.5. *If for each* p *in* H, $\limsup \frac{L(C_n)G}{L(C_n)F} > r$ *for some sequence* $\{C_n\}$ *converging to* p *and if* $H \subset B$ *where* B *is a countable union of sets of* B-*type*, *then* B *contains a set* Q *in the* σ-*field* $\overline{\mathfrak{A}}$ *such that* $\mu_G(Q) > r\,\mu_F(Q)$ *and* $\mu_F^*(H - Q) = 0$.

<u>Proof</u>. The argument follows that in Lemma 4.4 once we let $\mathfrak{F}$ be all sets C of B-type such that $L(C)G > r\,L(C)F$ and $C \subset B$.

<u>Definition</u> 4.6. Let F and G be two functions in BV(S) satisfying conditions (i), (ii), and (iii), let $p \in A$ and assume that there exists at least one sequence $\{C_n\}$ converging to p. By the <u>derivative</u> $\frac{dG}{dF}(p)$ we mean $\lim\limits_{C_n \to p} \frac{L(C_n)G}{L(C_n)F}$ where the limit is independent of the sequence $\{C_n\}$ converging to p.

<u>Theorem</u> 4.7. <u>Assume that every</u> $p \in A$ <u>is the limit of at least one sequence</u> $\{C_n\}$. <u>Then for some set</u> E <u>with</u> $\mu_F{}^*(E) = 0$, $\frac{dG}{dF}(p)$ <u>exists for all</u> $p \in A - E$.

<u>Proof</u>. Due to the Jordan decomposition we may assume that G is positive definite. Let

$$E = \{p \in A: \limsup_{C_n \to p} \frac{L(C_n)G}{L(C_n)F} > \liminf_{C_n' \to p} \frac{L(C_n')G}{L(C_n')F}\}$$

where $\{C_n\}$ and $\{C_n'\}$ denote two sequences converging to p. Let

$$E_{m,n} = \{p \in A: \limsup_{C_n \to p} \frac{L(C_n)G}{L(C_n)F} > \frac{m+1}{n} > \frac{m}{n} > \liminf_{C_n' \to p} \frac{L(C_n')G}{L(C_n')F}\}$$

If $\mu_F^*(E) \neq 0$ then for some positive integers i and j, $\mu_F^*(E_{i,j}) \neq 0$. Let $k = \inf_{B \supset E_{i,j}} \mu_F(B)$ where $B \in \mathfrak{a}$. Then $k > 0$ and there exists $B_1 \in \mathfrak{a}$ such that $\mu_F(B_1) < k+\epsilon$. By Lemma 4.4 there exists also a set Q_1 such that $Q_1 \subset B_1$, $\mu_F^*(E_{i,j} - Q_1) = 0$,

$$\mu_G(Q_1) < \frac{i}{j} \ \mu_F(Q_1) < \frac{i}{j} (k+\epsilon) .$$

Now consider $E_{i,j} \cap Q_1$. Since Q_1 is itself a countable union of disjoint sets of B-type, Lemma 4.5 provides a set $Q_2 \in \bar{\mathfrak{a}}$, $Q_2 \subset Q_1$, $\mu^*(E_{i,j} \cap Q_1 - Q_2) = 0$, and

$\mu_G(Q_2) > \frac{i+1}{j} \ \mu_F(Q_2)$. Now the set $E_{i,j}$ is contained in the disjoint union of the sets Q_2, $E_{i,j} - Q_1$, and $E_{i,j} \cap Q_1 - Q_2$. Consequently

$\mu_F{}^*(E_{i,j}) \leq \mu_F{}^*(Q_2)$ and $\mu_F{}^*(Q_2) \geq k$. Also for every $\epsilon > 0$,

$$\frac{i+1}{j} k \leq \frac{i+1}{j} \mu_F(Q_2) < \mu_G(Q_2) \leqq \mu_G(Q_1) < \frac{i}{j} (k+\epsilon)$$

which is a contradiction. Hence $\mu_F{}^*(E) = 0$. This completes the proof.

Corollary 4.8. *Let* F *and* G *be two functions in* BV(S) . *Suppose the measure* μ_F *is extendable to* $\mathfrak{a}$ *as a countably additive measure and assume that*

(1) $\mu_F(B(X,\sigma)) \neq 0$ *for all* X *and* σ

(2) *given* $\epsilon > 0$ *there exists a set* B *of* B-*type such that* $\mu_F(B) < \epsilon$.

(3) *the function* G *is Lipschitz relative to* F.

Then $\frac{dG}{dF}(p)$ *exists, except on a null set, if for every* $p \in A$ *there is at least one sequence* $\{C_n\}$ *converging to* p.

Proof. With the above hypothesis μ_G extends as a countably additive set function to $\bar{\mathfrak{a}}$, the σ-field generated by $\mathfrak{a}$.

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