ON THE BIFURCATION THEORY OF SEMI-LINEAR ELLIPTIC EIGENVALUE PROBLEMS

by

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1. This note is concerned with the bifurcation problem for the semi-linear elliptic eigenvalue problem

(1) $$-\Delta u = \lambda(P(x)u + g(x,u)), \quad x\in\Omega, \quad u|_{\partial\Omega} = 0.$$

Here Ω is a bounded region in R^N, $(N \geq 2)$ for which the Dirichlet problem is solvable, and g, defined for small u, is odd and monotone in u. For recent contributions to bifurcation theory for non-linear elliptic eigenvalue problems and for additional references see [6].

The object here is twofold, namely to show that in the variational treatment of the bifurcation problem for (1) a polynomial growth condition on g, as usually required, is unnecessary and using this fact and results of [3], to derive a bifurcation theorem for (1).

For simplicity we have restricted the linear operator on the left in (1) to be the Laplacian. Without difficulty the results obtained below can be extended to the larger class of real linear formally self-adjoint operators considered in [2]. The machinery for doing this is set up in [2].

2. The approach here to the eigenvalue problem (1) is through the study of the integral equation

(2) $$u(x) = \lambda\int_\Omega G(x,t)f(t,u(t))dt,$$

where $f(x,u) = P(x)u + g(x,u)$ and G is the Greens function for

(3) $$-\Delta w = \rho \text{ in } \Omega, \quad w|_{\partial\Omega} = 0.$$

To obtain certain required variational inequalities one must consider (2) in $L^p(\Omega)$, for an appropriate $p > 2$, and for this it is necessary to assume that f is defined on $\Omega \times R$ and satisfies a polynomial growth condition in u. It can be shown however, by a standard 'bootstrap argument', that for appropriate p and suitably restricted f, an L^p-eigenfunction of (2) agrees almost everywhere with a classical solution of (1), i.e. a function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and satisfying (1). This can be done, for example, as indicated in the proof of Theorem 4 in [1]. By similar arguments one can prove the following result which implies the superfluity of the polynomial growth condition in the bifurcation theory of (1).

Theorem 1. *Let* Ω *be a bounded region in* R^N $(N \geq 2)$ *and assume that the problem* (3) *has a Green's function* $G(x,t)$. *Let* $f(x,u)$ *be continuous on* $\overline{\Omega} \times R$ *and satisfy* $f(x,0) \equiv 0$, *and*

$$|f(x,u)| \leq K(1 + |u|^{\gamma}), \qquad (x,u) \in \Omega \times R, \tag{4}$$

for some $K > 0$, $\gamma \geq 0$.
If $p > \min(1, \frac{1}{2}N(\gamma - 1))$ *and* (λ_n, u_n) *is a sequence of* L^p-*solutions of* (2) *such that*

$$\|u_n\|_p = \left(\int_{\Omega} |u_n|^p \, dx\right)^{1/p} \to 0 \qquad \text{as} \quad n \to \infty, \tag{5}$$

and

$$\lambda_n \leq \Lambda < \infty, \qquad n = 1,2,\ldots, \tag{6}$$

then

$$\|u_n\|_{\infty} = \operatorname{ess\,sup}_{\Omega} |u_n(x)| \to 0 \quad \text{as} \quad n \to \infty. \tag{7}$$

The proof of Theorem 1 will require the following result.

Lemma 1. *The integral operator*

$$y \to \int_\Omega G(\cdot,t)y(t)\,dt$$

is completely continuous from $L^{r_1}(\Omega)$ to $L^s(\Omega)$, *for* $1 \le r_1, s \le \infty$, *provided*

$$s^{-1} > r_1^{-1} - 2N^{-1}$$

Proof. From the definition of the Green's function for (3) and the maximum principle for harmonic functions there follows, for $x,t\in\Omega$,

$$|G(x,t)| \le \begin{cases} \text{const. } (1 + |\log|x-t||), & N = 2, \\ \text{const. } |x - t|^{-N+2}, & N > 2, \end{cases}$$

and this implies, since Ω is bounded,

$$\sup_{x\in\Omega} \int_\Omega |G(x,t)|^a\,dt < \infty,$$

for $1 \le a < N/(N-2)$. This, together with the symmetry of G implies the assertion of the lemma, see Theorem 9.5.6 [5, p. 658].

Proof of Theorem 1. In view of the conditions on f, Theorem 19.1 of [9] implies that $u \to f(\cdot,u(\cdot))$ is continuous from $L^r(\Omega)$ to $L^{r_1}(\Omega)$ for finite r_1 with $1 \le r_1 \le r/\gamma$. Combined with Lemma 1, this implies that the operator

$$u \to \int_\Omega K(\cdot,t)f(t,u(t))\,dt$$

is continuous from $L^r(\Omega)$ to $L^s(\Omega)$ for $1 \le r,s \le \infty$ and $s^{-1} > \gamma r^{-1} - 2N^{-1}$. In particular this operator is continuous from $L^r(\Omega)$ to $L^\infty(\Omega)$ for $r > \frac{1}{2}\gamma N$, and from $L^r(\Omega)$ to $L^{r+\delta}(\Omega)$ for $r \ge p > \frac{1}{2} N(\gamma-1)$, where $\delta = p(2p - (\gamma-1)N)/(\gamma N - 2p) > 0$, if $p < r < \frac{1}{2}\gamma N$. From (5), (6) and

$$u_n(x) = \lambda_n \int_\Omega G(x,t) f(t,u_n(t))dt$$

it follows by induction that

$$\lim_{n\to\infty} \int_\Omega |u_n|^r dr = 0$$

for $r \geq p$, and hence that (7) holds. This completes the proof of Theorem 1.

3. Theorem 1, combined with the results of [3], yields the following.

Theorem 2. _Suppose that_ $P(x)$ _is bounded, positive, and locally Hölder continuous in_ Ω _and that_ $g(x,u)$ _is bounded and locally Hölder continuous in_ (x,u) _on_ $\{(x,u) : x \in \Omega, |u| < c\}$, _where_ Ω _is as in Theorem_ 1. _Suppose in addition that_ $g(x,u)$ _is odd_

$$g(x,u) = -g(x,-u), \qquad x \in \Omega, \quad |u| < c,$$

and monotone

$$g(x,u_1) \geq g(x,u_2), \qquad x \in \Omega, \; c > u_1 > u_2 > -c$$

and finally that

$$g(x,u) = o(|u|), \qquad \text{as} \quad u \to 0,$$

uniformly with respect to $x \in \Omega$.

Then every eigenvalue μ_k _of the linear eigenvalue problem_

$$-\Delta u = \mu P(x) u, \qquad x \in \Omega, \qquad u|_{\partial\Omega} = 0, \tag{8}$$

is a bifurcation point of (1).

A more precise formulation of the conclusion of Theorem 2 is contained in the assertion (*) below.

In order to deduce Theorem 2 from the results of [3] we require the following.

Lemma 2. _Let_ $g_1(x,u)$ _be bounded and continuous in_ (x,u) _on_ $\Omega \times R$, _and satisfy_

$$g_1(x,u) = o(|u|), \quad \text{as} \quad u \to 0,$$

uniformly with respect to $x \in \Omega$. _Then for_ $1 \leq r < s < \infty$,

$$u \to g_1(x,u)$$

is a continuous operator from $L^s(\Omega)$ _to_ $L^r(\Omega)$, _and is Fréchet differentiable at zero; its Fréchet derivative at zero is zero._

Proof. Since $g_1(x,u)$ is bounded and continuous the continuity follows at once from Theorem 19.1, [9]. If we set

$$h(x,u) = \begin{cases} 0, & u = 0 \\ u^{-1} g_1(x,u), & u \neq 0, \end{cases}$$

then $h(x,u)$ is also bounded and continuous on $\Omega \times R$, thus, by the result just quoted, $u \to h(x,u)$ is continuous from $L^s(\Omega)$ to $L^{r'}(\Omega)$, where $\frac{1}{r'} + \frac{1}{s} = \frac{1}{r}$. Since multiplication $(v,w) \to v \cdot w$, in continuous from $L^s(\Omega) \times L^{r'}(\Omega)$ to $L^r(\Omega)$, the result follows.

Proof of Theorem 2. Clearly $g(x,u)$ has an extension $g_1(x,u)$ to $\Omega \times R$ which is odd and monotone and satisfies the hypothesis of Lemma 2. Since P is bounded and continuous, for any r,s with $1 \leq r < s < \infty$, the operator $u \to P(x)u + g_1(x,u)$ from $L^s(\Omega)$ to $L^r(\Omega)$ is continuous, and Fréchet differentiable at zero, with its Fréchet derivative at zero being the operator $u \to P(x)u$.

We consider the integral equation

(9) $$u(x) = \lambda\int_{\Omega} G(x,t)(P(t)u(t) + g_1(t,u(t)))dt,$$

in $L^p(\Omega)$ where

(10) $$2 \leq p < 2N/(N-2)$$

and regard the operator on the right in (9) as the composition of the Nemytsky operator $u \to P(t)u + g_1(t,u)$ from $L^p(\Omega)$ to $L^q(\Omega)$, and the integral operator $y \to \int_{\Omega} G(\cdot,t)y(t)dt$ from $L^q(\Omega)$ to $L^p(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$. By virtue of Lemma 1 the inequality (10) insures the continuity of the integral operator and, since $q < 2 < p$, Lemma 2 implies continuity and Fréchet differentiability at zero of the Nemytsky operator. It follows that the principle of linearization (†) of [3] is applicable to the integral equation (9) in $L^p(\Omega)$. We conclude from the result just quoted that every eigenvalue of

$$y(x) = \mu \int_{\Omega} G(x,t)P(t)y(t)dt,$$

or equivalently, every eigenvalue of (8), is a bifurcation point of (9), considered in $L^p(\Omega)$. More specifically, if μ is an eigenvalue of (8) then for every $\epsilon > 0$ there exist non-trivial L^p-solutions (λ,u) of (9) with $|\lambda-\mu| < \epsilon$, $\|u\|_p < \epsilon$. Because of Theorem 1 this assertion remains valid when the L^p-norm is replaced by the L^∞-norm (notice that in this case the constant γ in Theorem 1 can be taken to be 0). If (λ,u) is a solution of (9) with $\|u\|_\infty < c$, then, because of the Hölder continuity of P

and g, it readily follows that u is (up to difference on a set of measure zero) a solution of (1). Thus we have the following.

(*) <u>If μ is an eigenvalue of (8), then for every $\epsilon > 0$ there exist non-trivial solutions (λ, u) of (1) with</u> $|\lambda - \mu| < \epsilon$, $\|u\|_\infty < \epsilon$.

Furthermore, the results of [3], together with Theorem 1, imply the following assertion concerning the multiplicity of solutions of (1).

(**) <u>If μ is an eigenvalue of (8) of multiplicity m then for every $\epsilon > 0$ there exists $k_1 = k_1(\epsilon)$ such that for $0 < k < k_1$ the set of eigenfunctions u of (1) corresponding to eigenvalues in $(\mu-\epsilon, \mu+\epsilon)$ and satisfying</u>

$$\frac{1}{2}\int_\Omega P(x)u^2(x)\,dx + \int_\Omega\int_0^{u(x)} g(x,t)\,dt\,dx = k,$$

<u>is a compact symmetric set in $C(\Omega)$ of genus $\geq m$.</u>

For the definition of 'genus' see [3] or [7]; the genus of a symmetric set without zero in a Banach space is a special case of the co-index defined by Connor and Floyd, [4].

<u>Remark</u>. It is interesting to note that while Theorem 2 insures the existence of eigenfunctions of (1) when P is positive, regardless of the behavior of g for large u, (1) can fail to have any eigenfunctions when $P \equiv 0$. Such an example is due to Pohozaev, [8]; consider the problem

$$\Delta u + \lambda |u|^m \operatorname{sgn} u = 0, \qquad |x| < 1$$

$$u(x) = 0, \qquad |x| = 1.$$

An eigenfunction u of this problem, corresponding to the eigenvalue λ, must satisfy, [8],

$$\lambda(N/(m+1) - (N-2)/2) \int_{|x| \leq 1} |u|^{m+1} dx = \int_{|x|=1} u_\nu^2 ds,$$

where ds is the differential of surface area and u_ν is the outward normal derivative of u. One readily sees from this that the problem has eigenfunctions only if $m < (N+2)/(N-2)$, (by the maximum principle an eigenvalue of this problem must be positive).

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