A CATEGORICAL CHARACTERIZATION OF CH

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An obviously appropriate concern of Categorical Topology is to characterize categorically various subcategories[1] of TOP (the category of topological spaces and continuous functions). For example, the sequential spaces can be characterized extrinsically as the mono-coreflective hull of the subcategory of TOP whose objects are just the homeomorphs of $u)_o + 1$ with its order topology. [F]

An example of a somewhat different nature is Hatcher[1]'s recent characterization of non-empty, complete subcategories of CH (the compact Hausdorff spaces and continuous functions) as precisely those which can be defined by a class of identities on some space in CH, i.e. given a subcategory of CH there is a compact Hausdorff space X and a class of pairs of maps

$$* \underset{b}{\overset{a,}{=\!=\!=\!=\!\Rightarrow}} x$$

(where * is the singleton space) such that a compact Hausdorff space Y belongs to the subcategory just in case

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1 Only full and replete subcategories will be considered.

$$\begin{array}{ccc} * & \xrightarrow{\ \ a\ \ } & X \\ {\scriptstyle b}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ X & \xrightarrow{\ \ f\ \ } & Y \end{array}$$

commutes for each f:X—>Y   [Ha].

Another remarkable example is the recent result of Herrlich
and Strecker [H-S] which roughly asserts that CH is the only
epi-reflective subcategory of T2 (the Hausdorff spaces and con-
tinuous maps) which is varietal [He].

The purpose of this note is to point out that a very ele-
gant theorem of de Groot can be given a categorical formulation
which yields an intrinsic characterization of CH in TOP.

De Groot[1]s theorem asserts that <u>the</u> <u>compact Hausdorff spaces</u>
<u>are</u> <u>the</u> <u>only</u> <u>non-trivial, productive, and closed hereditary</u> class
<u>of spaces which are absolutely closed</u>[2] <u>and preserved under closed</u>
<u>images</u> [W].

For the categorical translation let S denote the Sierpinski
two point space with 0 the open point and 1 the closed point,
and c:*—> S the closed map. Three easy lemmas will now lead us
to the theorem.

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2 If a subspace of a space in the class also belongs, then it is
   a closed subspace.

LEMMA 1.   A subspace F oj: c* space X _i^ closed iff there

is some   f:X—»S   such that

$$F \overset{i}{\rightarrowtail} X,$$

with   i aii extremal mono, is the pullback of

$$* \overset{c}{\longrightarrow} S$$
$$\uparrow f$$
$$X.$$

The proof is routine since any such f must be the character-
istic function of F, and this function will serve.

This lemma allows us to use the notion of a closed subspace
of an object of TOP in categorical characterizations of other
concepts. The next lemma is part of the folklore ([F],p.33).

LEMMA 2.   JIn any category with the extremal epi - mono factor-
ization property

   i)   factorization is unique and extremal epimorphisms com-
        pose iff

   ii) for each commutative diagram
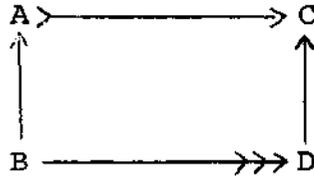
there <u>exists</u> <u>a</u> <u>unique</u> $D \xrightarrow{h} A$ <u>making</u> <u>everything</u>

<u>commute</u>.

It can be readily seen that TOP satisfies the dual of condi-
tion ii) and hence extremal monos compose in TOP. To show
that the intersection of extremal monos is again an extremal
mono consider the commutative diagram
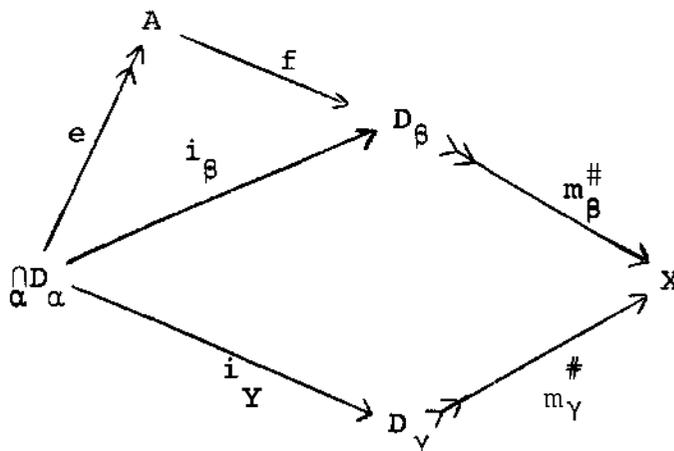
where each m_α is an extremal mono, and e is epi. The i_α
are all mono by a standard unique factorization argument. Con-
dition ii) of the lemma yields an $h_\gamma : A \twoheadrightarrow D_\gamma$ for each γ so
that the system $\{A, h_\gamma\}$ (with $h_\beta = f$) is a lower bound. Then the
unique $\hat{f} : A \to \cap D_\alpha$ obtained because $\cap D_\alpha$ is a limit is, in fact,
a left inverse for the epimorphism e, so each i_α is extremal.

It is now meaningful to discuss the "smallest" extremal

mono satisfying a given condition and Lemma 3 can be stated.

LEMMA 3. A morphism $f:X \twoheadrightarrow Y$ .in TOP .is closed iff for each

closed $F \rightarrowtail \xrightarrow{i} X$ the smallest $G \rightarrowtail \xrightarrow{j} \bullet Y$ through which fi fac-

tors is closed.

Proof: Consider the following diagram

$$
\begin{array}{c}
S \\
\nearrow^{g} \quad \uparrow c \quad \nwarrow^{h} \\
X \xrightarrow{\quad f \quad} Y \\
\uparrow i \quad\quad \uparrow \quad\quad \uparrow j \\
\quad\quad * \\
\nearrow \quad\quad \nwarrow \\
F \xrightarrow{\quad\quad} G
\end{array}
$$
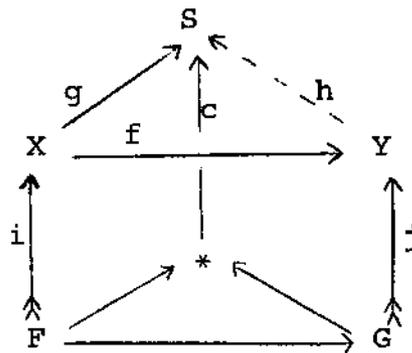
It is clear that G is in fact f(F) and that the lemma is

merely a restatement of the definition of a closed map.

Lemma 3 provides the needed categorical translation of

"closed epi"•   In terms of it we may now state the main result.

THEOREM.   CH JLS the only nontrivial productive subcategory K

of TOP which contains closed epimorphic images and satisfies

    (1) for any extremal mono $F \twoheadrightarrow \xrightarrow{m^{\#}} X$ in TOP with

        X an object of K, F jLs jan object of K

        iff $* <: \underline{\quad} F \xrightarrow{m^{\#}} -) X$ is a pullback for some

        $* \xrightarrow{\quad c \quad} S \longleftarrow X.$

The proof consists of verifying that our translation is "correct", i.e., that our conditions are equivalent to those of de Groot (only "imply" is needed). Thus one need only check that (1) is equivalent to saying that the class of objects of K is hereditarily closed and absolutely closed, since the rest is obvious.

Note that (1) can be subdivided into two categorical conditions, the first a translation of "hereditarily closed" and the second of "absolutely closed". This makes the proof completely trivial but results in an ungainly statement of the Theorem.

Finally, each of the hypotheses of the Theorem is necessary and independent. There follows a list giving for each condition of the Theorem a subcategory which satisfies the hypotheses except for that condition.

Non-trivial: The category of singleton spaces

Productive: The category of compact and sequentially compact Hausdorff spaces

Contains closed epimorphic images: The category of zero-dimensional compact Hausdorff spaces

(1): The category of continua.

(De Groot's conditions are also independent: continua work for closed hereditary, and compact spaces or ultracompact[3] Hausdorff spaces for absolutely closed.)

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3 Countable sets have compact closure [B].

REFERENCES

[B]   A. R. Bernstein, A new kind of compactness for topological spaces, to appear.

[F]   S. P. Franklin, <u>Categorical</u> <u>Topology</u>, Lecture notes, Carnegie-Mellon University, 1970, p.38.

[Ha] W. S. Hatcher, Complete subcategories of the category of compact Hausdorff spaces, to appear.

[He] H. Herrlich, Algebraic categories: an axiomatic approach, to appear.

[H-S] H. Herrlich and G. Strecker, Algebra 0 Topology = Compactness, to appear.

[W]   E. Wattel, <u>The compactness operator in set theory and topology</u>, Mathematical Center Tracts No.21, Amsterdam, 1968, p.51.

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