VIBRATION OF A BEADED STRING
ANALYZED TOPOLOGICALLY

by

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Vibration of a Beaded String Analyzed Topologically*

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Abstract

Of concern are the transverse vibrations of a finite string of beads. It is shown that a periodic vibration can result when the beads are released from an inital configuration. Moreover a norm on the initial configuration can be given a prescribed value. The proof uses the Brouwer fixed point theorem.

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1. Introduction.

Consider the transverse vibration of a string of beads stretched between two fixed points. If the displacement from the rest position is infinitesimal, then the vibration is governed by linear equations. The classical analysis of these equations leads to the so called normal modes of vibration. Each normal mode is a periodic vibration.

For finite displacements the equations governing the vibration are nonlinear. In this paper Brouwer's fixed point theorem is used to show that there always exists a periodic vibration of a certain type. This type of periodic motion is characterized by adjacent beads having displacements of opposite sign. We term this an oppositional mode of vibration. For infinitesimal vibrations the oppositional mode is the mode with the highest frequency.

Attention is confined to two classes of initial conditions. In the first class of initial conditions the beads are released from displaced positions with zero velocities. This may be described as a plucked string. In the second class of initial conditions the beads start from the equilibrium position with specified velocities. This may be described as a struck string. In the oppositional mode of periodic motion adjacent beads have displacements of opposite sign. Consequently at some time all the beads will simultaneously pass through the equilibrium position. In other words the periodic motion found for the plucked string can also be regarded as a periodic motion of a struck string but starting at a later time.

In the linearized theory a periodic solution remains a periodic solution if all the displacements are increased in the same ratio. Consequently if a norm on the displacement is introduced then there is a periodic solution for any prescribed value of the norm. This latter property is also proved for the nonlinear theory. In particular the potential energy could serve as a norm. Consequently there is an oppositional mode of vibration for any given value of the energy.

As the number of beads becomes large, the behavior of a beaded string should resemble the behavior of a violin string. The infinitesimal vibrations of a violin string satisfy the wave equation, $u_{tt} = u_{xx}$. Of course the wave equation has periodic solutions. In fact all solutions satisfying the boundary conditions are periodic. The finite vibration satisfy an equation of the form $u_{tt} = f(u_x) u_{xx}$. Such equations have been studied by MacCamy and Mizel [1] and others. A surprising consequence of the MacCamy and Mizel analysis is that <u>the nonlinear string equation has no periodic solution</u>. This suggested that it would be interesting to learn more about the vibrations of the beaded string. This paper is a step in that direction.

2. Motion of a Plucked String.

Consider now a string of beads stretched between fixed points A and B as shown in Figure 1. For the sake of clarity the string is shown with three beads, however the consideration to follow readily generalize to the case of any number of beads. Let m_1, m_2, m_3 be the masses of the beads and let x, $-y$, z be the transverse displacement from the position of static equilibrium. No longitudinal motion is permitted. Then Newton's equations for the transverse accelerations are

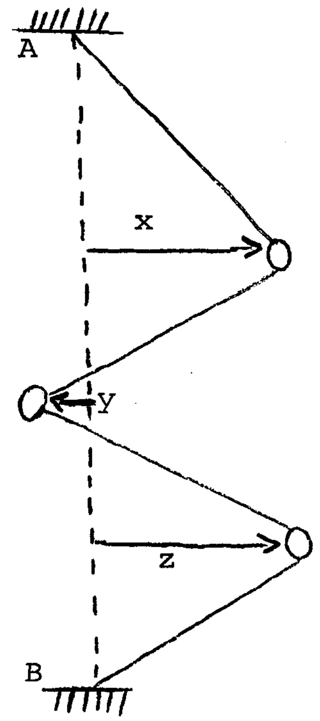

Figure 1.
A beaded string.

(1a) $m_1\ddot{x} = f_1(x) + f_2(x + y)$,

(1b) $m_2\ddot{y} = f_2(x + y) + f_3(y + z)$,

(1c) $m_3\ddot{z} = f_3(y + z) + f_4(z)$.

Here f_1, f_2, f_3, f_4 are force functions which arise from the tension in the four string segments. These functions are continuous but are nonlinear. The nonlinearily arises because of the geometry and also because the string may not obey Hooke's law. However the following weak restrictions are assumed:

(2) $-f_i(x) \geq hx$ for $x > 0$ and for a constant $h > 0$,

(3) $f_i(-x) = -f_i(x)$.

No further properties of these functions are needed.

It is convenient to regard (x,y,z) as coordinates of a point in configuration space. Suppose given the initial coordinates (x_o, y_o, z_o) and initial velocities $(\dot{x}_o, \dot{y}_o, \dot{z}_o)$ at time $t = t_o$. Then Newton's equations determine a trajectory in configuration space as

(4) $x = X(t)$, $y = Y(t)$, $z = Z(t)$.

From the theory of differential equations we know that the functions X, Y, and Z are jointly continuous in the seven variables $t, x_o, y_o, z_o, \dot{x}_o, \dot{y}_o, \dot{z}_o$. This property will be used in the proof of Lemma 3 to follow.

A string is said to be plucked if the beads are held in the configuration (x_o, y_o, z_o) for $t \leq t_o$ and at $t = t_o$ the beads are released with zero velocity.

Theorem 1. *A beaded string is plucked. The initial configuration point is constrained to be on a surface S of configuration space having given norm. Then there is at least one initial point on S such that the resulting trajectory passes through the origin.*

Proof. The norm is arbitrary so S may be described as an arbitrary closed surface starlike with respect to the origin.

Let D be the part of S in the first octant: $x \geq 0$, $y \geq 0$, $z \geq 0$. Since S is starlike with respect to the origin it follows that D is a topological image of a closed disk. For example if S is the surface $|x| + |y| + |z| = 1$ then D is a

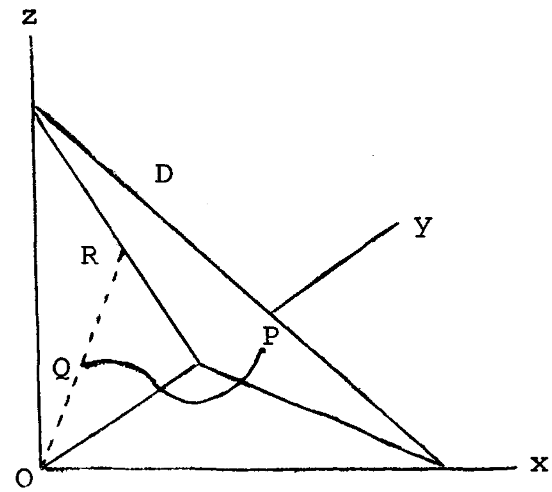

Figure 2.
A trajectory in the first octant.

triangle such as shown in Figure 2. Now we restrict consideration to trajectories starting from D. Consider a trajectory leaving a point P of D. A point Q where the trajectory first touches one of the coordinate plane is said to be a contact point. In Figure 2 the contact point Q is shown on the plane $x = 0$.

If Theorem 1 is not true it follows that the origin cannot be a contact point. This assumption will be made in the proofs to follow.

Lemma 1. A trajectory leaves the first octant at a contact point.

Proof. Since a contact point Q is not the origin one of the coordinates x_Q, y_Q, z_Q is positive and an adjacent coordinate is zero. For example, suppose $y_Q > 0$ and $x_Q = 0$. Then relations (1), and (2) give

(5) $m_1 \ddot{x}_Q = f_2(y_Q) < 0.$

If P is not a boundary point $\dot{x}_Q \leq 0$ because $x > 0$ just before contact. If P is a boundary point then $Q = P$ and so $\dot{x}_Q = 0$. Thus in any case

(6) $x_Q = 0,\ \dot{x}_Q \leq 0,\ \ddot{x}_Q < 0.$

The second derivatives are continuous so relation (6) implies that $x < 0$ just after contact. Q E D.

Lemma 2. *A trajectory starting from an arbitrary point* P *makes contact in a time* $t_Q \leq K$ *where* K *is a positive constant independent of* P.

Proof. Assume the trajectory starts at time $t_o = 0$. Then relations (1) and (2) give the inequality

(7) $m_1\ddot{x} + hx \leq 0$ for $0 \leq t \leq t_Q$.

Let $m_1\ddot{x} + hx = \varphi(t)$ and integrate this differential equation to obtain

(8) $x = x_o \cos(t\,\Pi/2K) + \int_o^t \sin[(t - s)\,\Pi/2K]\,\varphi(s)\,ds/\sqrt{m_1 h}$

where $K = (\Pi/2)\,(m_1/h)^{1/2}$. So for $t = K$

(9) $x = \int_o^K \cos(s\,\Pi/2K)\,\varphi(s)\,ds/\sqrt{m_1 h}$.

If $t_Q > K$ then $\varphi(s) \leq 0$ in (9). Thus $x \leq 0$ for $t = K$. This contradiction proves the lemma.

Lemma 3. *The position of the contact point* Q *is a continuous function of the position of the initial point* P.

Proof. Let J be a trajectory from P to Q. According to Lemma 1 the trajectory J goes through Q and extends to some point M exterior to the octant at time t_M. Let P' be another point of D and let J' be the associated trajectory. Then J' can be made uniformly close to J for time t in the range $0 \leq t \leq t_M$ if P' is taken sufficiently close to P.

First suppose that P is not on the boundary of D so consequently $t_Q > 0$. Thus given an ϵ in the range $0 < \epsilon < t_Q$ there exists a δ such that if $|P' - P| \leq \delta$ then J' is in the interior of the octant for time t in the range $0 \leq t \leq t_Q - \epsilon$ and J' is in the exterior of the octant for time t in the range $t_Q + \epsilon \leq t \leq t_M$. Hence at some time $t_{Q'}$ in the interval $(t_Q - \epsilon, t_Q + \epsilon)$ the trajectory J' has a contact point Q'.

Next let $\epsilon \to 0$ then $P' \to P$ and $t_{Q'} \to t_Q$. This implies that $Q' \to Q$. The same relation follows by a similar argument if P is on the boundary of D, so the proof of the lemma is complete.

Let Q be a contact point then the <u>shadow point</u> R is the projection of Q onto the boundary of the disk D by a ray from the origin through Q. Such a point is shown in Figure 2.

<u>Lemma 4</u>. <u>The postion of the shadow point</u> R <u>is a continuous function of the position of the initial point</u> P.

<u>Proof</u>. The image of the closed set D under the continuous transformation $P \to Q$ is a closed set D_Q. The origin is not in D_Q and since D_Q is closed it follows that the origin is at a finite distance from D_Q.

It is then seen by simple geometry that the transformation $Q \to R$ is continuous. Then the transformation $P \to R$ is continuous because it is the composition of the continuous transformations $P \to Q$ and $Q \to R$. Q E D.

Lemma 1 and Lemma 4 together state that the transformation $P \to R$ is a continuous transformation of the disk D into its boundary with the boundary fixed. This statement contradicts the Brouwer fixed point theorem. The contradiction shows that there is at least one trajectory through the origin. This completes the proof of Theorem 1.

Theorem 2. A beaded string can be plucked so that the resulting vibration is periodic. Moreover the norm of the initial configuration can be prescribed arbitrarily.

Proof. It is now to be shown that the trajectory of Theorem 1 finally returns to the initial point P. The complete trajectory is of the type shown in Figure 3. The configuration point moves back and forth on the arc $P\ O\ P'$.

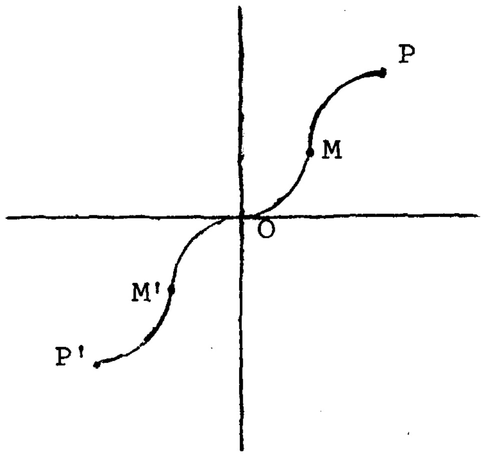

Figure 3.

Trajectory of a periodic motion.

The arc $O\ P'$ is constructed to be the reflection of $O\ P$ in the origin. Let the configuration point traverse the arc $O\ P'$ so that the velocities at point M and at the reflected point M' are equal. Clearly this will result in the acceleration at point M' being the negative of that at M. By hypothesis (3) the force functions are antisymmetric so the force at M' is the negative of the force at M. Hence Newton's equations (1) are satisfied at M'. Thus arc $O\ P'$ is a trajectory. Moreover $P\ O\ P'$ is a trajectory because the velocity is continuous on this arc.

By the same continuation method the trajectory P O P' leads to the new trajectory P' O P. Thus P O P' O P is a trajectory because the velocity is zero at P and P' and hence continuous over the whole trajectory. Q E D.

3. Refinements and Applications.

The following extension of the theory are worth noting:

A. The assumption of Theorem 1 that the surface S is starlike can be weakened. The proof requires only the property that D is equivalent to a closed disk.

B. The proof of Theorem 1 made use of property (2) but not property (3).

C. The property (2) can be replaced by the following weaker restriction

(2a) $-f_i(x) > 0$ for $x > 0$.

This merely makes the proof of Lemma 2 somewhat longer.

D. A norm on the initial velocities of the struck string can be imposed. Presumably analogous theorems hold.

E. The beaded string was assumed to be free of damping forces. Of course a damped system cannot have a periodic motion, however Theorem 1 might remain valid. Inspection of the proof shows that Lemma 2 holds if the damping constants are not too large. With this further assumption Theorem 1 is still valid.

The following applications are of some interest:

F. It is well known that a beaded string has an electrical analogy in a ladder filter of inductors and capacitors. Such a filter is shown in Figure 4. Then if the capacitors exhibit nonlinear behavior the network equations for charge on the capacitors can be put in the form (1) and properties (2) and (3) hold. Thus a periodic state of current flow can exist in such a network.

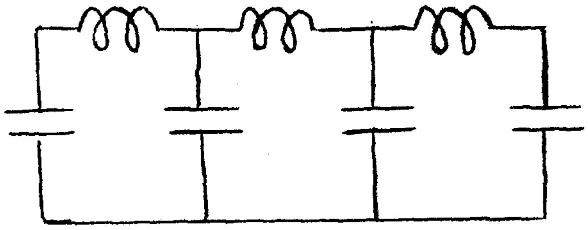

Figure 4.
A ladder filter.

G. A multiple pendulum is shown in Figure 5. This is somewhat analogous to a string of beads. Presumably a similar analysis can be applied to show that such a pendulum has a periodic swing.

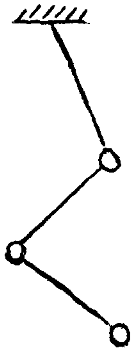

Figure 5.
A multiple pendulum.

Reference

1. R. C. MacCamy and V. J. Mizel, 'Existence and nonexistence in the large of solutions of quasilinear wave equations', Arch. for Rat. Mech. 25(1967), pp. 299-320.