RESOLUTION IN TYPE THEORY

by

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§ 1. Introduction.

In [8] J. A. Robinson introduced a complete refutation procedure called resolution for first order predicate calculus. Resolution is based on ideas in Herbrand's Theorem, and provides a very convenient framework in which to search for a proof of a wff believed to be a theorem. Moreover, it has proved possible to formulate many refinements of resolution which are still complete but are more efficient, at least in many contexts. However, when efficiency is a prime consideration, the restriction to first order logic is unfortunate, since many statements of mathematics (and other disciplines) can be expressed more simply and naturally in higher order logic than in first order logic. Also, the fact that in higher order logic (as in many-sorted first order logic) there is an explicit syntactic distinction between expressions which denote different types of intuitive objects is of great value where matching is involved, since one is automatically prevented from trying to make certain inappropriate matches. (One may contrast this with the situation in which mathematical statements are expressed in the symbolism of axiomatic set theory.)

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In this paper we shall introduce a refutation system $\mathfrak{R}$ for type theory which may be regarded as a generalization of resolution to type theory, and prove that $\mathfrak{R}$ is complete in the (weak) sense that in $\mathfrak{R}$ one can refute any sentence $\sim \underline{A}$ such that $\underline{A}$ is provable in a more conventional system $\mathfrak{T}$ of type theory. For $\mathfrak{T}$ we take the elegant and expressive formulation of type theory introduced by Church in [2], but use only Axioms 1 - 6. It should be noted that because substitution with λ-conversion is a much more complicated operation than substitution alone, the matching problem, which was completely solved for first order logic by Robinson's Unification Theorem [8], remains a major problem in the context of the system $\mathfrak{R}$. (Some appreciation of the complexity of the situation can be gained from [3]). In this sense $\mathfrak{R}$ is not as useful for refuting wffs of type theory as resolution is for refuting wffs of first order logic.

In § 2 we review certain facts about the system $\mathfrak{T}$ and λ - conversion. In § 3 we prove a theorem which is (at least in conjunction with the results of Henkin in [4]) an extension to $\mathfrak{T}$ of Smullyan's Unifying Principle in Quantification Theory ([10] and [11, Chapter VI]). Our proof relies heavily on ideas of Takahashi [12] as well as Smullyan, which is not surprising since the Unifying

Principle is closely related to cut-elimination. $\mathfrak{T}$ is a somewhat richer formulation of type theory than Schütte's formulation in [9] which Takahashi treats in [12], since in $\mathfrak{T}$ for all types α and β there is a type $(\alpha\beta)$ of functions from elements of type β to elements of type α. Therefore we verify the details of this argument rather carefully, although there is a close parallel with Takahashi's argument. We apply the theorem in § 4 to prove cut-elimination for $\mathfrak{T}$, and in § 5 to prove the completeness of $\mathfrak{R}$. (Except for the preliminary definitions, § 4 can be skipped by those interested primarily in $\mathfrak{R}$). In § 6 we present some examples of refutations in $\mathfrak{R}$.

§ 2 The System $\mathfrak{T}$.

For the convenience of the reader we here provide a condensed description of the system $\mathfrak{T}$, with a few trivial notational changes from [2]. A more complete discussion of $\mathfrak{T}$ can be found in [2] or [4]. The sustems $\mathfrak{G}$ and $\mathfrak{R}$ in § 4 and § 5 will have the same wffs as $\mathfrak{T}$.

2.1 We use α, β, γ, etc., (but not o or ι) as syntactical variables ranging over type symbols, which are defined inductively as follows:

2.1.1 o is a type symbol (denoting the type of truth values).

2.1.2 ι is a type symbol (denoting the type of individuals).

2.1.3 $(\alpha\beta)$ is a type sumbol (denoting the type of functions from elements of type β to elements of type α).

2.2 The primitive symbols of $\mathfrak{T}$ are the following:

2.2.1 Improper symbols: [ ] λ

2.2.2 For each α, a denumerable list of variables of type α:

$$f_\alpha g_\alpha h_\alpha \ldots x_\alpha y_\alpha z_\alpha f^1_\alpha g^1_\alpha \ldots z^1_\alpha f^2_\alpha \ldots$$

We shall write variable$_\alpha$ as an abbreviation for variable of type α. We shall use $\mathbf{f}_\alpha, \mathbf{g}_\alpha, \ldots, \mathbf{x}_\alpha, \mathbf{y}_\alpha, \mathbf{z}_\alpha$, etc., as syntactical variables for variables$_\alpha$.

2.2.3 Logical constants: $\sim_{(oo)}\ \vee_{((oo)o)}\ \Pi_{(o(o\alpha))}$

2.2.4. In addition there may be other constants of various types, which we call non-logical constants or parameters.

2.3 We write wff_α as an abbreviation for wff of type α, and use $\underset{\sim}{A}_\alpha, \underset{\sim}{B}_\alpha, \underset{\sim}{C}_\alpha$, etc., as syntactical variables variables ranging over wffs_α, which are defined inductively as follows:

2.3.1 A primitive variable or constant of type α is a wff_α.

2.3.2 $[\underset{\sim}{A}_{\alpha\beta}\underset{\sim}{B}_\beta]$ is a wff_α.

2.3.3 $[\lambda \underset{\sim}{x}_\beta \underset{\sim}{A}_\alpha]$ is a $\text{wff}_{(\alpha\beta)}$.

We shall assume given a fixed enumeration of the wffs of $\mathfrak{T}$. This also provides an enumeration of the variables and constants of each type.

An occurrence of $\underset{\sim}{x}_\alpha$ is bound (free) in $\underset{\sim}{B}_\beta$ iff it is (is not) in a wf part of $\underset{\sim}{B}_\beta$ of the form $[\lambda \underset{\sim}{x}_\alpha \underset{\sim}{C}_\delta]$. A wff is closed iff no variable occurs free in it. A sentence is a closed wff_o.

2.4 Definitions and abbreviations.

2.4.1 Brackets (and parentheses in type symbols) may be omitted when no ambiguity is thereby introduced. A dot stands for a left bracket whose mate is as far to the right as is consistent with the pairing of brackets already present and with the formula being well formed. Otherwise brackets and parentheses are to be restored using the convention of association to the left.

2.4.2. Type symbols may be omitted when the context indicates what they should be. The type symbol o will usually be omitted.

2.4.3. $[\mathbf{A}_o \vee \mathbf{B}_o]$ stands for $[[\vee_{((oo)o)}\mathbf{A}_o]\ \mathbf{B}_o]$.

2.4.4. $[\mathbf{A}_o \supset \mathbf{B}_o]$ stands for $[[\sim_{oo}\mathbf{A}_o] \vee \mathbf{B}_o]$.

2.4.5. $[\forall \mathbf{x}_\alpha \mathbf{A}_o]$ stands for $[\Pi_{(o(o\alpha))}[\lambda \mathbf{x}_\alpha \mathbf{A}_o]]$.

2.4.6. Other propositional connectives, and the existential quantifier, are defined in familiar ways.

2.4.7. $Q_{o\alpha\alpha}$ stands for $[\lambda x_\alpha \lambda y_\alpha \forall f_{o\alpha} \cdot f_{o\alpha} x_\alpha \supset f_{o\alpha} y_\alpha]$

2.4.8. $[\mathbf{A}_\alpha = \mathbf{B}_\alpha]$ stands for $Q_{o\alpha\alpha}\mathbf{A}_\alpha \mathbf{B}_\alpha$

2.4.9. $\exists_1 \mathbf{x}_\alpha \mathbf{A}_o$ stands for

$[\lambda p_{o\alpha} \cdot \exists y_\alpha \cdot p_{o\alpha} y_\alpha \wedge \forall z_\alpha \cdot p_{o\alpha} z_\alpha \supset z_\alpha = y_\alpha]\ [\lambda \mathbf{x}_\alpha \mathbf{A}_o]$

2.4.10. $\mathrm{S}^{\mathbf{x}_\alpha}_{\mathbf{A}_\alpha}\mathbf{B}_\beta$ $(\mathrm{S}^{\mathbf{x}_\alpha}_{\bullet\mathbf{A}_\alpha}\mathbf{B}_\beta)$ denotes the result of substituting $\mathbf{A}_\alpha$ for $\mathbf{x}_\alpha$ at all (all free) occurrences of $\mathbf{x}_\alpha$ in $\mathbf{B}_\beta$.

2.4.11. $\mathbf{A}_\alpha$ is <u>free for</u> $\mathbf{x}_\alpha$ <u>in</u> $\mathbf{B}_\beta$ iff no free occurrence of $\mathbf{x}_\alpha$ in $\mathbf{B}_\beta$ is in a wf part of $\mathbf{B}_\beta$ of the form $[\lambda \mathbf{y}_\gamma \mathbf{C}_\delta]$ such that $\mathbf{y}_\gamma$ is a free variable of $\mathbf{A}_\alpha$.

2.5 Axioms of $\mathcal{T}$.

2.5.1 $p \vee p \supset p$

2.5.2 $p \supset p \vee q$

2.5.3 $p \vee q \supset q \vee p$

2.5.4 $p \supset q \supset [r \vee p \supset r \vee q]$

2.5.5$^\alpha$ $\Pi_{o(o\alpha)} f_{o\alpha} \supset f_{o\alpha} x_\alpha$

2.5.6$^\alpha$ $\forall x_\alpha [p \vee f_{o\alpha} x_\alpha] \supset p \vee \Pi_{o(o\alpha)} f_{o\alpha}$

2.6 Rules of inference of $\mathcal{T}$.

2.6.1 Alphabetic change of bound variables.
To replace any wf part $[\lambda \mathbf{x}_\beta \mathbf{A}_\alpha]$ of a wff by $[\lambda \mathbf{y}_\beta \mathrm{S}^{\mathbf{x}_\beta}_{\mathbf{y}_\beta} \mathbf{A}_\beta]$, provided that $\mathbf{y}_\beta$ does not occur in $\mathbf{A}_\alpha$ and $\mathbf{x}_\beta$ is not bound in $\mathbf{A}_\alpha$.

2.6.2 λ-contraction. To replace any wf part $[[\lambda \mathbf{x}_\alpha \mathbf{B}_\beta]\, \mathbf{A}_\alpha]$ of a wff by $\mathrm{S}^{\mathbf{x}_\alpha}_{\mathbf{A}_\alpha} \mathbf{B}_\beta$, provided that the bound variables of $\mathbf{B}_\beta$ are distinct both from $\mathbf{x}_\alpha$ and from the free variables of $\mathbf{A}_\alpha$.

2.6.3 λ-expansion. To infer $\mathbf{C}$ from $\mathbf{D}$ if $\mathbf{D}$ can be inferred from $\mathbf{C}$ by a single application of 2.6.2.

2.6.4 Substitution. From $\mathbf{F}_{o\alpha}\mathbf{x}_\alpha$ to infer $\mathbf{F}_{o\alpha}\mathbf{A}_\alpha$, provided that $\mathbf{x}_\alpha$ is not a free variable of $\mathbf{F}_{o\alpha}$.

2.6.5 Modus Ponens. From $[\mathbf{A} \supset \mathbf{B}]$ and $\mathbf{A}$ to infer $\mathbf{B}$.

2.6.6 Generalization. From $\mathbf{F}_{o\alpha}\mathbf{x}_\alpha$ to infer $\Pi_{o(o\alpha)}\mathbf{F}_{o\alpha}$, provided that $\mathbf{x}_\alpha$ is not a free variable of $\mathbf{F}_{o\alpha}$.

Remark: It can be proved that $\Pi_{o(o\alpha)}\mathbf{F}_{o\alpha}$ is equivalent to $\forall \mathbf{x}_\alpha \mathbf{F}_{o\alpha}\mathbf{x}_\alpha$ if $\mathbf{x}_\alpha$ is not free in $\mathbf{F}_{o\alpha}$.

2.7. λ - conversion.

2.7.1. Rules 2.6.1 - 2.6.3 are <u>λ - conversion rules</u>. We write $\mathbf{A}_\alpha$ <u>conv</u> $\mathbf{B}_\alpha$ (resp. $\mathbf{A}_\alpha$ <u>conv - I - II</u> $\mathbf{B}_\alpha$) iff there is a sequence of applications of rules 2.6.1 - 2.6.3 (resp. 2.6.1 - 2.6.2) which transforms $\mathbf{A}_\alpha$ into $\mathbf{B}_\alpha$. It is well known that <u>conv</u> is an equivalence relation.

2.7.2. A <u>contractible part</u> of a wff $\mathbf{C}_\gamma$ is an occurrence of a wff of the form $[[\lambda \mathbf{x}_\alpha \mathbf{B}_\beta] \mathbf{A}_\alpha]$ in $\mathbf{C}_\gamma$. We say $\mathbf{C}_\gamma$ is in <u>λ - normal form</u> iff it has no contractible parts.

2.7.3. <u>Proposition</u>. For each wff $\mathbf{D}_\gamma$ there is a wff $\mathbf{C}_\gamma$ in λ - normal form such that $\mathbf{D}_\gamma$ conv - I - II $\mathbf{C}_\gamma$

Proof[2]:

Define $\# [[\lambda \mathbf{x}_\alpha \mathbf{B}_\beta] \mathbf{A}_\alpha]$ to be the number of occurences of (in $(\beta\alpha)$.

Let $m(\mathbf{D}_\gamma) = \max \{\# \mathbf{G}_\beta \mid \mathbf{G}_\beta$ is a contractible part of $\mathbf{D}_\gamma\}$. We say that a contractible part $\mathbf{G}_\beta$ of $\mathbf{D}_\gamma$ is <u>maximal</u> in $\mathbf{D}_\gamma$ iff $\# \mathbf{G}_\beta = m(\mathbf{D}_\gamma)$. Let $n(\mathbf{D}_\gamma)$ be the number of maximal contractible parts in $\mathbf{D}_\gamma$.

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[2] This proposition is part of the folklore of type-theoretic λ - conversion. The author first heard the idea of the proof given here from Dr. James R. Guard.

The proof is by induction on $p(\underset{\sim}{D}_\gamma) = \omega \cdot m(\underset{\sim}{D}_\gamma) + n(\underset{\sim}{D}_\gamma)$. Clearly $\underset{\sim}{D}_\gamma$ is in λ-normal form iff $p(\underset{\sim}{D}_\gamma) = 0$.

If $p(\underset{\sim}{D}_\gamma) > 0$, let $[[\lambda \underset{\sim}{x}_\alpha \underset{\sim}{B}_\beta] \underset{\sim}{A}_\alpha]$ be that maximal contractible part $\underset{\sim}{G}_\beta$ of $\underset{\sim}{D}_\gamma$ which occurs farthest to the right in $\underset{\sim}{D}_\gamma$, with the position of a contractible part being determined by the leftmost occurrence of λ in it. By applying 2.6.1 if necessary we may assume that 2.6.2 may be applied to obtain from $\underset{\sim}{D}_\gamma$ a wff $\underset{\sim}{E}_\gamma$ in which $\underset{\sim}{G}_\beta$ has been replaced by $S^{\underset{\sim}{x}_\alpha}_{\underset{\sim}{A}_\alpha} \underset{\sim}{B}_\beta$. Thus $\underset{\sim}{D}_\gamma$ conv I-II $\underset{\sim}{E}_\gamma$, and it must be shown that $p(\underset{\sim}{E}_\gamma) < p(\underset{\sim}{D}_\gamma)$.

For the sake of brevity, we shall not explicitly distinguish wffs from occurrences of wffs at certain points in the following argument.

We first prove that

(*) for each wf part $\underset{\sim}{C}_\delta$ of $\underset{\sim}{B}_\beta$, $S^{\underset{\sim}{x}_\alpha}_{\underset{\sim}{A}_\alpha} \underset{\sim}{C}_\delta$ contains no contractible part $\underset{\sim}{H}_\tau$ with $\# \underset{\sim}{H}_\tau \geq m(\underset{\sim}{D}_\gamma)$. The proof is by induction on the construction of $\underset{\sim}{C}_\delta$.

Case (a): $\mathbf{C}_\delta$ is $\mathbf{x}_\alpha$. Then $S^{\mathbf{x}_\alpha}_{\mathbf{A}_\alpha} \mathbf{C}_\delta$ is $\mathbf{A}_\alpha$, and (*) holds by virtue of the definition of $\mathbf{G}_\beta$.

Case (b): $\mathbf{C}_\delta$ is a primitive constant or variable other than $\mathbf{x}_\alpha$. $S^{\mathbf{x}_\alpha}_{\mathbf{A}_\alpha} \mathbf{C}_\delta$ is $\mathbf{C}_\delta$, so (*) holds trivially.

Case (c): $\mathbf{C}_\delta$ has the form $[\lambda \mathbf{y}_\kappa \mathbf{M}_\sigma]$. Note that $\mathbf{y}_\kappa$ cannot be $\mathbf{x}_\alpha$ by the restriction on 2.6.2. $S^{\mathbf{x}_\alpha}_{\mathbf{A}_\alpha} \mathbf{C}_\delta$ is $[\lambda \mathbf{y}_\kappa S^{\mathbf{x}_\alpha}_{\mathbf{A}_\alpha} \mathbf{M}_\sigma]$, so (*) holds by the inductive hypothesis applied to $\mathbf{M}_\sigma$.

Case (d): $\mathbf{C}_\delta$ has the form $[\mathbf{M}_{\delta\epsilon} \mathbf{N}_\epsilon]$. Then $S^{\mathbf{x}_\alpha}_{\mathbf{A}_\alpha} \mathbf{C}_\delta$ is $[(S^{\mathbf{x}_\alpha}_{\mathbf{A}_\alpha} \mathbf{M}_{\delta\epsilon}) S^{\mathbf{x}_\alpha}_{\mathbf{A}_\alpha} \mathbf{N}_\epsilon]$, and the inductive hypothesis applies to $\mathbf{M}_{\delta\epsilon}$ and $\mathbf{N}_\epsilon$, so we need only consider the possibility that $S^{\mathbf{x}_\alpha}_{\mathbf{A}_\alpha} \mathbf{C}_\delta$ is itself a contractible part $[[\lambda \mathbf{y}_\epsilon \mathbf{P}_\delta] \; S^{\mathbf{x}_\alpha}_{\mathbf{A}_\alpha} \mathbf{N}_\epsilon]$, where $S^{\mathbf{x}_\alpha}_{\mathbf{A}_\alpha} \mathbf{M}_{\delta\epsilon} = [\lambda \mathbf{y}_\epsilon \mathbf{P}_\delta]$, with $\#\,(S^{\mathbf{x}_\alpha}_{\mathbf{A}_\alpha} \mathbf{C}_\delta) \geq m(\mathbf{D}_\gamma)$. Since $\mathbf{M}_{\delta\epsilon}$ has one of the forms 2.3.1 - 2.3.3 , $S^{\mathbf{x}_\alpha}_{\mathbf{A}_\alpha} \mathbf{M}_{\delta\epsilon}$ can have the form $[\lambda \mathbf{y}_\epsilon \mathbf{P}_\delta]$ in only two ways:

(i) $\underset{\sim}{M}_{\delta\epsilon}$ is $\underset{\sim}{x}_{\alpha}$ and $\underset{\sim}{A}_{\alpha}$ is $[\lambda \underset{\sim}{y}_{\epsilon}\ \underset{\sim}{P}_{\delta}]$. Then $\alpha = (\delta\epsilon)$ so $\#(S^{\underset{\sim}{x}_{\alpha}}_{\underset{\sim}{A}_{\alpha}}\ \underset{\sim}{C}_{\delta}) < m(\underset{\sim}{D}_{\gamma})$.

(ii) $\underset{\sim}{M}_{\delta\epsilon}$ is $[\lambda \underset{\sim}{y}_{\epsilon}\ \underset{\sim}{Q}_{\delta}]$ and $S^{\underset{\sim}{x}_{\alpha}}_{\underset{\sim}{A}_{\alpha}}\ \underset{\sim}{Q}_{\delta} = \underset{\sim}{P}_{\delta}$.

In this case $\underset{\sim}{C}_{\delta} = [[\lambda \underset{\sim}{y}_{\epsilon}\ \underset{\sim}{Q}_{\delta}]\ \underset{\sim}{N}_{\epsilon}]$ and $\#\ \underset{\sim}{C}_{\delta} \geq m(\underset{\sim}{D}_{\gamma})$. But since $\underset{\sim}{C}_{\delta}$ is a part of $\underset{\sim}{B}_{\beta}$ this contradicts the definition of $\underset{\sim}{G}_{\beta}$. Thus neither possibility can occur, and (*) holds in case (d) also.

For each wff $\underset{\sim}{C}_{\delta}$ we let $k(\underset{\sim}{C}_{\delta})$ be the number of contractible parts $\underset{\sim}{H}_{\tau}$ of $\underset{\sim}{C}_{\delta}$ with $\#\ \underset{\sim}{H}_{\tau} = m(\underset{\sim}{D}_{\gamma})$. For any wf part $\underset{\sim}{C}_{\delta}$ of $\underset{\sim}{D}_{\gamma}$ which contains $\underset{\sim}{G}_{\beta}$, we let $\underset{\sim}{C}_{\delta}'$ be the result of replacing $\underset{\sim}{G}_{\beta}$ in $\underset{\sim}{C}_{\delta}$ by $S^{\underset{\sim}{x}_{\alpha}}_{\underset{\sim}{A}_{\alpha}}\ \underset{\sim}{B}_{\beta}$, and prove

(**) $k(\underset{\sim}{C}_{\delta}') + 1 = k(\underset{\sim}{C}_{\delta})$ and $\underset{\sim}{C}_{\delta}'$ contains no contractible parts $\underset{\sim}{H}_{\tau}$ with $\#\ \underset{\sim}{H}_{\tau} > m(\underset{\sim}{D}_{\gamma})$. The proof is by induction on the construction of $\underset{\sim}{C}_{\delta}$.

<u>Case (a)</u>: $\underset{\sim}{C}_{\delta}$ is $\underset{\sim}{G}_{\beta}$, so $\underset{\sim}{C}_{\delta}'$ is $S^{\underset{\sim}{x}_{\alpha}}_{\underset{\sim}{A}_{\alpha}}\ \underset{\sim}{B}_{\beta}$. Then (**) follows directly from (*).

Case (b): $\underline{C}_\delta$ is $[\underline{G}_\beta\ \underline{N}_\epsilon]$, and β has the form $(\delta\epsilon)$. Thus $\underline{C}_\delta'$ is $[(S^{\underline{x}_\alpha}_{\underline{A}_\alpha}\ \underline{B}_\beta)\ \underline{N}_\epsilon]$. If $\underline{C}_\delta'$ is contractible, $\#\ \underline{C}_\delta'$ = the number of occurrences of (in β , which is less than $m(\underline{D}_\gamma)$, so

$$k(\underline{C}_\delta') + 1 = k(S^{\underline{x}_\alpha}_{\underline{A}_\alpha}\ \underline{B}_\beta) + k(\underline{N}_\epsilon) + 1$$
$$= \text{(by (*))}\quad 0 + k(\underline{N}_\epsilon) + 1$$
$$= k(\underline{N}_\epsilon) + k(\underline{G}_\beta) = k(\underline{C}_\delta)$$

and (**) is easily seen to be true.

The remaining cases involve trivial applications of the inductive hypothesis, and are left to the reader.

$\underline{D}_\gamma'$ is $\underline{E}_\gamma$, so by (**) $k(\underline{E}_\gamma) + 1 = k(\underline{D}_\gamma) = n(\underline{D}_\gamma)$ and $m(\underline{E}_\gamma) \leq m(\underline{D}_\gamma)$. If $m(\underline{E}_\gamma) = m(\underline{D}_\gamma)$ then $n(\underline{E}_\gamma) = k(\underline{E}_\gamma) < n(\underline{D}_\gamma)$; hence whether $m(\underline{E}_\gamma) = m(\underline{D}_\gamma)$ or $m(\underline{E}_\gamma) < m(\underline{D}_\gamma)$ we have $p(\underline{E}_\gamma) < p(\underline{D}_\gamma)$. Therefore by inductive hypothesis $\underline{E}_\gamma$ is conv - I - II to a wff in λ - normal form, so $\underline{D}_\gamma$ is also.

2.7.4 Church-Rosser Theorem: If B_γ and C_γ are in λ-normal form and B_γ conv C_γ, then B_γ can be converted into C_γ by applications of 2.6.1 alone. That is, a λ-normal form of a wff is unique up to alphabetic changes of bound variables.

This theorem was originally proved for a different system of λ-conversion without type symbols but it is known that it applies to $\mathcal{J}$ also. See [5] and the references cited therein.

2.7.5 η-wffs.

A wff A_α of $\mathcal{J}$ is an η-wff iff A_α is in λ-normal form and for each wf part $[\lambda x_\beta C_\gamma]$ of A_α, x_β is the first variable_β in alphabetic order which is distinct from the other free variables_β of C_γ. Using 2.7.3, 2.7.4, and 2.6.1 it is easy to see that for each wff A_α there is a unique η-wff B_α such that A_α conv B_α. We write $B_\alpha = \eta A_\alpha$. (To convert a wff in λ-normal form into an η-wff, proceed from left to right to decide what each bound variable should be; however some additional temporary changes of bound variables may be necessary before these changes can be made.)

η-wffs have the following pleasing properties. If A_α is an η-wff, then every wf part of A_α is an η-wff.

$\eta[A_{\alpha\beta} B_\beta] = [(\eta A_{\alpha\beta}) (\eta B_\beta)]$ if $[A_{\alpha\beta} B_\beta]$ is not contractible.

2.8 Wffs_o

2.8.1 A wff_o A_o is atomic (an atom) iff the leftmost primitive symbol of A_o which is not a bracket is a variable or parameter.

2.8.2 Every wff_o $\mathbf{D}_o$ of $\mathfrak{T}$ in λ-normal form has one of the following forms:

(a) $\mathbf{A}$, where $\mathbf{A}$ is atomic.

(b) $\sim \mathbf{B}$

(c) $\mathbf{B} \vee \mathbf{C}$

(d) $\Pi_{o(o\alpha)} \mathbf{B}_{o\alpha}$

Proof: The leftmost primitive symbol of $\mathbf{D}$ which is not a bracket cannot be λ, so it must be a variable, parameter, $\sim_{oo}$, $\vee_{ooo}$, or $\Pi_{o(o\alpha)}$.

2.9 A set $\mathcal{S}$ of wffs_o is <u>inconsistent</u> iff there is a finite subset $\{\mathbf{A}^1, \ldots, \mathbf{A}^n\}$ of $\mathcal{S}$ such that $\vdash \sim \mathbf{A}^1 \vee \ldots \vee \sim \mathbf{A}^n$; otherwise $\mathcal{S}$ is <u>consistent</u>.

§3. Abstract consistency properties, valuations, and consistency.

3.1. Definition. A property Γ of finite sets of $wffs_o$ is an abstract consistency property iff for all finite sets $\mathcal{S}$ of $wffs_o$ the following properties hold (for all wffs $\mathbf{A}, \mathbf{B}$):

3.1.1. If $\Gamma(\mathcal{S})$, then there is no atom $\mathbf{A}$ such that

$$\mathbf{A} \in \mathcal{S} \text{ and } [\sim\mathbf{A}] \in \mathcal{S}.$$

3.1.2. If $\Gamma(\mathcal{S} \cup \{\mathbf{A}\})$, then $\Gamma(\mathcal{S} \cup \{\eta\mathbf{A}\})$.

3.1.3. If $\Gamma(\mathcal{S} \cup \{\sim\sim\mathbf{A}\})$, then $\Gamma(\mathcal{S} \cup \{\mathbf{A}\})$.

3.1.4. If $\Gamma(\mathcal{S} \cup \{[\mathbf{A} \vee \mathbf{B}]\})$, then $\Gamma(\mathcal{S} \cup \{\mathbf{A}\})$ or $\Gamma(\mathcal{S} \cup \{\mathbf{B}\})$.

3.1.5. If $\Gamma(\mathcal{S} \cup \{\sim[\mathbf{A} \vee \mathbf{B}]\})$, then $\Gamma(\mathcal{S} \cup \{\sim\mathbf{A}, \sim\mathbf{B}\})$.

3.1.6. If $\Gamma(\mathcal{S} \cup \{\Pi_{o(o\alpha)}\mathbf{A}_{o\alpha}\})$, then for each wff $\mathbf{B}_\alpha$, $\Gamma(\mathcal{S} \cup \{\Pi_{o(o\alpha)}\mathbf{A}_{o\alpha}, \mathbf{A}_{o\alpha}\mathbf{B}_\alpha\})$.

3.1.7. If $\Gamma(\mathcal{S} \cup \{\sim\Pi_{o(o\alpha)}\mathbf{A}_{o\alpha}\})$, then $\Gamma(\mathcal{S} \cup \{\sim\mathbf{A}_{o\alpha}\mathbf{c}_\alpha\})$ for any variable or parameter $\mathbf{c}_\alpha$ which does not occur free in $\mathbf{A}_{o\alpha}$ or any wff in $\mathcal{S}$.

Remark: Satisfiability is an abstract consistency property.

The notion of an abstract consistency property is due to Smullyan. Our main theorem of this section will be that if Γ is an abstract consistency property and $\Gamma(\mathcal{S})$, then $\mathcal{S}$ is consistent. This is an analog for $\mathcal{T}$ of Smullyan's Unifying Principle in Quantification Theory [10].

3.2. Definition. A semi-valuation is a function V with domain some set of wffs$_o$ and range a subset of the set $\{t,f\}$ of truth values such that the following properties hold (for all wffs $\mathbf{A},\mathbf{B}$):

3.2.1. If $V\,\mathbf{A}$ is defined, then $V\,\eta\mathbf{A} = V\,\mathbf{A}$.

3.2.2. If $V[\sim\mathbf{A}] = t$, then $V\,\mathbf{A} = f$.

3.2.3. If $V[\sim\mathbf{A}] = f$, then $V\,\mathbf{A} = t$.

3.2.4. If $V[\mathbf{A} \vee \mathbf{B}] = t$, then $V\,\mathbf{A} = t$ or $V\,\mathbf{B} = t$.

3.2.5. If $V[\mathbf{A} \vee \mathbf{B}] = f$, then $V\,\mathbf{A} = f$ and $V\,\mathbf{B} = f$.

3.2.6. If $V[\Pi_{o(o\alpha)}\mathbf{A}_{o\alpha}] = t$, then for each wff $\mathbf{B}_\alpha$, $V[\mathbf{A}_{o\alpha}\mathbf{B}_\alpha] = t$.

3.2.7. If $V[\Pi_{o(o\alpha)}\mathbf{A}_{o\alpha}] = f$, then there is a wff $\mathbf{B}_\alpha$ such that $V[\mathbf{A}_{o\alpha}\mathbf{B}_\alpha] = f$.

The notion of a semi-valuation is due to Schütte [9].

3.3. Theorem. Let $\mathcal{S}$ be a finite set of wffs$_o$ and Γ be an abstract consistency property such that $\Gamma(\mathcal{S})$. There is a semi-valuation V such that $V\,\mathbf{A} = t$ for all $\mathbf{A} \in \mathcal{S}$.

Proof: (following Smullyan [10]):

We may assume $\mathcal{S}$ is non-empty, since the theorem is otherwise trivial.

3.3.1. We shall inductively define finite sequences $\mathcal{S}_1,\mathcal{S}_2,\ldots$ of wffs$_o$ such that $\mathcal{S}_i$ has at least i terms and $\mathcal{S}_i$ is an initial segment of $\mathcal{S}_{i+1}$. We let $\mathbf{E}^i$ be the i th term of $\mathcal{S}_j$. For notational convenience if $\mathcal{Y}$ is a finite sequence we let $\mathcal{Y}*\underset{\sim}{A}$ be the sequence obtained from $\mathcal{Y}$ by adding $\underset{\sim}{A}$ as an additional term; also when we use notations which suggest that $\mathcal{Y}$ is a set we refer tacitly to the set of terms of the sequence $\mathcal{Y}$. As we define $\mathcal{S}_i$ we prove $\Gamma(\mathcal{S}_i)$.

$\mathcal{S}_1$ is to be the sequence of wffs of $\mathcal{S}$ arranged in order. $\Gamma(\mathcal{S}_1)$ since $\Gamma(\mathcal{S})$.

Given $\mathcal{S}_i$ such that $\Gamma(\mathcal{S}_i)$, we define $\mathcal{S}_{i+1}$ and prove $\Gamma(\mathcal{S}_{i+1})$ in each case below:

3.3.1.1. $\underset{\sim}{E}^i$ is not an η-wff.

Let $\mathcal{S}_{i+1} = \mathcal{S}_i * \eta \underset{\sim}{E}^i$. $\underset{\sim}{E}^i \in \mathcal{S}_i$ so $\mathcal{S}_i = \mathcal{S}_i \cup \{\underset{\sim}{E}^i\}$ so $\Gamma(\mathcal{S}_{i+1})$. by 3.1.2.

In all other cases we assume $\underset{\sim}{E}^i$ is an η-wff.

3.3.1.2. $\underset{\sim}{E}^i$ is an atom or the negation of an atom.

Let $\mathcal{S}_{i+1} = \mathcal{S}_i * \underset{\sim}{E}^i$.

3.3.1.3. $\underset{\sim}{E}^i = \sim\sim \underset{\sim}{A}$.

Let $\mathcal{S}_{i+1} = \mathcal{S}_i * \underset{\sim}{A}$. $\Gamma(\mathcal{S}_{i+1})$ by 3.1.3.

3.3.1.4. $\underset{\sim}{E}^i = \underset{\sim}{A} \vee \underset{\sim}{B}$.

Let $\mathcal{S}_{i+1}$ be $\mathcal{S}_i * \underset{\sim}{A}$ if $\Gamma(\mathcal{S}_i * \underset{\sim}{A})$; otherwise let $\mathcal{S}_{i+1} = \mathcal{S}_i * \underset{\sim}{B}$. Then $\Gamma(\mathcal{S}_{i+1})$ by 3.1.4.

3.3.1.5. $\underset{\sim}{E}^i = \sim[\underset{\sim}{A} \vee \underset{\sim}{B}]$.

Let $\mathcal{S}_{i+1} = \mathcal{S}_i * \sim\underset{\sim}{A} * \sim\underset{\sim}{B}$. $\Gamma(\mathcal{S}_{i+1})$ by 3.1.5.

3.3.1.6. $\underset{\sim}{E}^i = \Pi_{o(o\alpha)}\underset{\sim}{A}_{o\alpha}$.

Let $\underset{\sim}{B}_\alpha$ be the first wff_α such that $[\underset{\sim}{A}_{o\alpha}\underset{\sim}{B}_\alpha] \notin \mathcal{S}_i$ and let $\mathcal{S}_{i+1} = \mathcal{S}_i * \underset{\sim}{A}_{o\alpha}\underset{\sim}{B}_\alpha * \underset{\sim}{E}^i$. $\Gamma(\mathcal{S}_{i+1})$ by 3.1.6.

3.3.1.7. $\underset{\sim}{E}^i = \sim \Pi_{o(o\alpha)}\underset{\sim}{A}_{o\alpha}$.

Let $\underset{\sim}{x}_\alpha$ be the first $variable_\alpha$ which is not free in any wff of $\mathcal{S}_i$ and let $\mathcal{S}_{i+1} = \mathcal{S}_i * \sim\underset{\sim}{A}_{o\alpha}\underset{\sim}{x}_\alpha$. $\Gamma(\mathcal{S}_{i+1})$ by 3.1.7.

3.3.2. Let $\mathcal{U} = \bigcup_{i=1}^{\infty} \mathcal{S}_i$. Note that every finite subset of $\mathcal{U}$ is a subset of some set with property Γ.

3.3.3. Lemma. There is no wff $\underset{\sim}{E}$ such that $\underset{\sim}{E} \in \mathcal{U}$ and $[\sim\underset{\sim}{E}] \in \mathcal{U}$.

Proof:

Clearly by 3.3.1.1. if $\underset{\sim}{E} \in \mathcal{U}$ then $\eta\underset{\sim}{E} \in \mathcal{U}$. Also $\eta[\sim\underset{\sim}{E}] = [\sim\eta\underset{\sim}{E}]$, so it suffices to prove the lemma for η-wffs. We do this by induction on the number of occurrences of logical constants in $\underset{\sim}{E}$. In each case below we suppose $\underset{\sim}{E}$ is an η-wff and $\underset{\sim}{E} \in \mathcal{U}$ and $\sim\underset{\sim}{E} \in \mathcal{U}$.

3.3.3.1. $\underset{\sim}{E}$ is atomic.

By 3.3.2. there exists a set Z such that $\{\underset{\sim}{E}, \sim\underset{\sim}{E}\} \subseteq Z \subseteq \mathcal{U}$ and $\Gamma(Z)$. This contradicts 3.1.1.

3.3.3.2. $\underset{\sim}{E} = \sim \underset{\sim}{A}$.

Since $\sim\underset{\sim}{E} = \sim\sim\underset{\sim}{A} \in \mathcal{U}$, by 3.3.1.3. $\underset{\sim}{A} \in \mathcal{U}$, which contradicts the inductive hypothesis.

3.3.3.3. $\underline{E} = [\underline{A} \vee \underline{B}]$.

By 3.3.1.5. $\sim \underline{A} \in \mathfrak{U}$ and $\sim \underline{B} \in \mathfrak{U}$, and by 3.3.1.4. $\underline{A} \in \mathfrak{U}$ or $\underline{B} \in \mathfrak{U}$, which contradicts the inductive hypothesis.

3.3.3.4. $\underline{E} = \Pi_{o(o\alpha)}\underline{A}_{o\alpha}$.

By 3.3.1.7. there is a variable $\underline{x}_\alpha$ such that $\sim \underline{A}_{o\alpha}\underline{x}_\alpha \in \mathfrak{U}$, and from 3.3.1.6. it can be seen that $\underline{A}_{o\alpha}\underline{x}_\alpha \in \mathfrak{U}$, since there are infinitely many i such that $\underline{E}^i = \Pi_{o(o\alpha)}\underline{A}_{o\alpha}$. Hence by 3.3.1.1 $\eta[\underline{A}_{o\alpha}\underline{x}_\alpha] \in \mathfrak{U}$ and $[\sim\eta[\underline{A}_{o\alpha}\underline{x}_\alpha]] \in \mathfrak{U}$. If $\underline{A}_{o\alpha}\underline{x}_\alpha$ is not in λ-normal form, a single contradiction will make it so, and it is easy to see that $\eta[\underline{A}_{o\alpha}\underline{x}_\alpha]$ contains the same number of occurrences of logical constants as does $\underline{A}_{o\alpha}$. Thus the inductive hypothesis is contradicted.

3.3.4. We now define a function V which we shall show is a semi-valuation.

$V\,\underline{E} = t$ if $\underline{E} \in \mathfrak{U}$.

$V\,\underline{E} = f$ if $[\sim\underline{E}] \in \mathfrak{U}$.

Clearly V is well defined by 3.3.3.

3.3.5. V is a semi-valuation.

The proof is straightforward. Each clause of 3.2. is readily verified using 3.3.4. and the appropriate case of 3.3.1.

3.3.6. If $\underline{A} \in \mathfrak{S}$ then $\underline{A} \in \mathfrak{U}$ so $V\,\underline{A} = t$.

This proves 3.3.

3.4. <u>Theorem</u>. If V is any semi-valuation, then $\{\underline{A} \mid V\,\underline{A} = t\}$ is consistent.

Proof (following Takahashi [12]):

3.4.1. For each type symbol γ we define the set $\mathfrak{D}_\gamma$ of V-complexes$_\gamma$ as follows by induction on γ:

3.4.1.1. $\mathfrak{D}_o = \{\langle \underset{\sim}{A}_o, p\rangle \mid \underset{\sim}{A}_o$ is an η-wff$_o$ and p is t or f and if $V\,\underset{\sim}{A}_o$ is defined, then $p = V\,\underset{\sim}{A}_o\}$.

3.4.1.2. $\mathfrak{D}_\iota = \{\langle \underset{\sim}{A}_\iota, \iota\rangle \mid \underset{\sim}{A}_\iota$ is an η-wff$_\iota\}$.

3.4.1.3. $\mathfrak{D}_{(\alpha\beta)} = \{\langle \underset{\sim}{A}_{\alpha\beta}, p\rangle \mid \underset{\sim}{A}_{\alpha\beta}$ is an η-wff$_{(\alpha\beta)}$ and p is a function from $\mathfrak{D}_\beta$ into $\mathfrak{D}_\alpha$ such that if $\langle \underset{\sim}{B}_\beta, q\rangle$ is any member of $\mathfrak{D}_\beta$, then $p\langle \underset{\sim}{B}_\beta, q\rangle = \langle \eta[\underset{\sim}{A}_{\alpha\beta}\underset{\sim}{B}_\beta], r\rangle$ for some $r\}$.

3.4.2. Lemma. For each η-wff $\underset{\sim}{A}_\gamma$ there is an r such that $\langle \underset{\sim}{A}_\gamma, r\rangle \in \mathfrak{D}_\gamma$.

Proof:

We choose r as a function of $\underset{\sim}{A}_\gamma$ by induction on γ, and show $\langle \underset{\sim}{A}_\gamma, r(\underset{\sim}{A}_\gamma)\rangle \in \mathfrak{D}_\gamma$. This is trivial when $\gamma = \iota$ or $\gamma = o$. (If $V\,\underset{\sim}{A}_o$ is not defined, arbitrarily let $r(\underset{\sim}{A}_o) = t$.) If $\gamma = (\alpha\beta)$, let $r(\underset{\sim}{A}_{\alpha\beta})\langle \underset{\sim}{B}_\beta, q\rangle = \langle \eta[\underset{\sim}{A}_{\alpha\beta}\underset{\sim}{B}_\beta], r(\eta[\underset{\sim}{A}_{\alpha\beta}\underset{\sim}{B}_\beta])\rangle$ for each $\langle \underset{\sim}{B}_\beta, q\rangle \in \mathfrak{D}_\beta$.

3.4.3. Definitions and notations.

3.4.3.1. If $\mathfrak{C}$ is a V-complex, let $\mathfrak{C}^1$ and $\mathfrak{C}^2$ be the first and second components of $\mathfrak{C}$, so $\mathfrak{C} = \langle \mathfrak{C}^1, \mathfrak{C}^2\rangle$. If f is a function whose values are V-complexes, let f^1 and f^2 be functions with the same domain as f defined so that for any argument t, $f^i t = (ft)^i$ for $i = 1,2$. Thus $ft = \langle f^1 t, f^2 t\rangle$.

3.4.3.2. An <u>assignment</u> is a function φ defined on the variables of $\mathfrak{T}$ such that $\varphi\mathbf{x}_\alpha \in \mathfrak{D}_\alpha$ for every variable$_\alpha$ $\mathbf{x}_\alpha$.

3.4.3.3. Given an assignment φ, a variable $\mathbf{x}_\alpha$, and $\mathfrak{C} \in \mathfrak{D}_\alpha$, let $(\varphi:\mathbf{x}_\alpha/\mathfrak{C})$ be that assignment ψ such that $\psi\mathbf{y}_\beta = \varphi\mathbf{y}_\beta$ if $\mathbf{y}_\beta \neq \mathbf{x}_\alpha$ and $\psi\mathbf{x}_\alpha = \mathfrak{C}$.

3.4.3.4. If p and q are truth values, we denote by $\sim p$ and $p \vee q$ the (intuitive) negation of p and the (intuitive) disjunction of p and q, respectively. The context will show whether $\sim$ and $\vee$ are to be regarded as symbols of $\mathfrak{T}$ or of our meta-language.

3.4.4. $\mathcal{V}_\varphi$.

For each assignment φ and wff $\mathbf{C}_\gamma$ we define $\mathcal{V}_\varphi\mathbf{C}_\gamma$ and show $\mathcal{V}_\varphi\mathbf{C}_\gamma \in \mathfrak{D}_\gamma$. Thus $\mathcal{V}_\varphi\mathbf{C}_\gamma = \langle\mathcal{V}_\varphi{}^1\mathbf{C}_\gamma, \mathcal{V}_\varphi{}^2\mathbf{C}_\gamma\rangle$.

3.4.4.1. Let $\mathcal{V}_\varphi{}^1\mathbf{C}_\gamma = \eta[[\lambda\mathbf{x}^1 \ldots \lambda\mathbf{x}^n\mathbf{C}_\gamma](\varphi^1\mathbf{x}^1)\ldots(\varphi^1\mathbf{x}^n)]$, where $\mathbf{x}^1,\ldots,\mathbf{x}^n$ are the free variables of $\mathbf{C}_\gamma$.
Let $\mathcal{V}_\varphi{}^1\mathbf{C}_\gamma = \eta\mathbf{C}_\gamma$ if $\mathbf{C}_\gamma$ has no free variables.

3.4.4.2. Note that $\mathcal{V}_\varphi{}^1[\mathbf{A}_{\gamma\beta}\mathbf{B}_\beta] = \eta[(\mathcal{V}_\varphi{}^1\mathbf{A}_{\gamma\beta})\mathcal{V}_\varphi{}^1\mathbf{B}_\beta]$.
This is readily established using properties of λ-conversion.

We define $\mathcal{V}_\varphi{}^2\mathbf{C}_\gamma$, and show $\mathcal{V}_\varphi\mathbf{C}_\gamma \in \mathfrak{D}_\gamma$, simultaneously for all φ by induction on the number of occurrences of [in $\mathbf{C}_\gamma$, considering the following cases:

3.4.4.3. $\underset{\sim}{C}_\gamma$ is a parameter.

Let $\mho_\varphi{}^2\underset{\sim}{C}_\gamma = r(\underset{\sim}{C}_\gamma)$, where r is defined as in the proof of 3.4.2. Note that $\underset{\sim}{C}_\gamma$ is an η-wff, so $\mho_\varphi\underset{\sim}{C}_\gamma = \langle\underset{\sim}{C}_\gamma, r(\underset{\sim}{C}_\gamma)\rangle \in \mathfrak{D}_\gamma$ by 3.4.2.

3.4.4.4. $\underset{\sim}{C}_\gamma$ is a variable.

Let $\mho_\varphi{}^2\underset{\sim}{C}_\gamma = \varphi^2\underset{\sim}{C}_\gamma$. Thus $\mho_\varphi\underset{\sim}{C}_\gamma = \langle\varphi^1\underset{\sim}{C}_\gamma, \varphi^2\underset{\sim}{C}_\gamma\rangle = \varphi\underset{\sim}{C}_\gamma \in \mathfrak{D}_\gamma$ by 3.4.3.2., and we see that $\mho_\varphi$ extends φ.

3.4.4.5. $\underset{\sim}{C}_\gamma$ is $\sim_{oo}$.

For any $\langle\underset{\sim}{B}_o, q\rangle \in \mathfrak{D}_o$, let $(\mho_\varphi{}^2 \sim_{oo}) \langle\underset{\sim}{B}_o, q\rangle = \langle\sim_{oo}\underset{\sim}{B}_o, \sim q\rangle$. It is clear that $\sim_{oo}\underset{\sim}{B}_o$ is an η-wff since $\underset{\sim}{B}_o$ is, so to check that $\mho_\varphi \sim_{oo} \in \mathfrak{D}_{oo}$ by 3.4.1.3. we must check that $\langle\sim_{oo}\underset{\sim}{B}_o, \sim q\rangle \in \mathfrak{D}_o$. By 3.4.1.1. this is trivial if $V[\sim\underset{\sim}{B}_o]$ is not defined. If $V[\sim\underset{\sim}{B}_o] = r$ then by 3.2.2.-3.2.3. $V\underset{\sim}{B}_o = \sim r$; but $V\underset{\sim}{B}_o = q$ since $\langle\underset{\sim}{B}_o, q\rangle \in \mathfrak{D}_o$ so $\sim q = r = V[\sim\underset{\sim}{B}_o]$ and $\langle\sim\underset{\sim}{B}_o, \sim q\rangle \in \mathfrak{D}_o$.

3.4.4.6. $\underset{\sim}{C}_\gamma$ is $\vee_{(oo)o}$.

For any $\langle\underset{\sim}{B}_o, q\rangle \in \mathfrak{D}_o$, let $(\mho_\varphi{}^2 \vee_{(oo)o}) \langle\underset{\sim}{B}_o, q\rangle = \langle\vee_{(oo)o}\underset{\sim}{B}_o, h\rangle$, where h is that function from $\mathfrak{D}_o$ into $\mathfrak{D}_o$ such that for any $\langle\underset{\sim}{E}_o, r\rangle \in \mathfrak{D}_o$, $h\langle\underset{\sim}{E}_o, r\rangle = \langle[\vee_{(oo)o}\underset{\sim}{B}_o]\underset{\sim}{E}_o, q \vee r\rangle$.

Since $[[\vee_{(oo)o}\underset{\sim}{B}_o]\underset{\sim}{E}_o]$ is an η-wff whenever $\underset{\sim}{B}_o$ and $\underset{\sim}{E}_o$ are η-wffs, from 3.4.1.3. it is seen that in order to verify that $\mho_\varphi\vee_{(oo)o} \in \mathfrak{D}_{(oo)o}$ one must check that $\langle [\vee_{(oo)}\underset{\sim}{B}_o]\underset{\sim}{E}_o, q \vee r\rangle \in \mathfrak{D}_o$ whenever $\langle\underset{\sim}{B}_o, q\rangle \in \mathfrak{D}_o$ and $\langle\underset{\sim}{E}_o, r\rangle \in \mathfrak{D}_o$. If $V[\underset{\sim}{B}_o \vee \underset{\sim}{E}_o]$ is not defined this is trivial.

If $V[\underset{\sim}{B}_o \vee \underset{\sim}{E}_o]$ is defined then $V\,\underset{\sim}{B}_o = q$ and $V\,\underset{\sim}{E}_o = r$ so by 3.2.4.-3.2.5. $V[\underset{\sim}{B}_o \vee \underset{\sim}{E}_o] = q \vee r$.

3.4.4.7. $\underset{\sim}{C}_\gamma$ is $\Pi_{o(o\alpha)}$.

For any $\langle \underset{\sim}{A}_{o\alpha}, p\rangle \in \mathfrak{D}_{o\alpha}$, let $(\mathfrak{V}_\varphi{}^2 \Pi_{o(o\alpha)})\ \langle \underset{\sim}{A}_{o\alpha}, p\rangle = \langle \Pi_{o(o\alpha)}\underset{\sim}{A}_{o\alpha}, r\rangle$, where r is t if $p^2\mathfrak{C} = t$ for every $\mathfrak{C} \in \mathfrak{D}_\alpha$, and r is f otherwise.

It must be shown that $V[\Pi_{o(o\alpha)}\underset{\sim}{A}_{o\alpha}] = r$ if $V[\Pi_{o(o\alpha)}\underset{\sim}{A}_{o\alpha}]$ is defined, so suppose it is defined.

Suppose $V[\Pi_{o(o\alpha)}\underset{\sim}{A}_{o\alpha}] = t$, and let $\langle \underset{\sim}{B}_\alpha, q\rangle \in \mathfrak{D}_\alpha$. By 3.2.6. and 3.2.1. $t = V[\underset{\sim}{A}_{o\alpha}\underset{\sim}{B}_\alpha] = V\eta[\underset{\sim}{A}_{o\alpha}\underset{\sim}{B}_\alpha]$, so $p\ \langle \underset{\sim}{B}_\alpha, q\rangle = \langle \eta[\underset{\sim}{A}_{o\alpha}\underset{\sim}{B}_\alpha], V\eta[\underset{\sim}{A}_{o\alpha}\underset{\sim}{B}_\alpha]\rangle = \langle \eta[\underset{\sim}{A}_{o\alpha}\underset{\sim}{B}_\alpha], t\rangle$ by 3.4.1.3. and 3.4.1.1. Thus $r = t = V[\Pi_{o(o\alpha)}\underset{\sim}{A}_{o\alpha}]$ in this case.

Suppose $V[\Pi_{o(o\alpha)}\underset{\sim}{A}_{o\alpha}] = f$. By 3.2.7. and 3.2.1. there is a wff $\underset{\sim}{B}_\alpha$ such that $V\eta[\underset{\sim}{A}_{o\alpha}\underset{\sim}{B}_\alpha] = f$. By 3.4.2. there is a q such that $\langle \eta\underset{\sim}{B}_\alpha, q\rangle \in \mathfrak{D}_\alpha$. Thus $p\ \langle \eta\underset{\sim}{B}_\alpha, q\rangle = \langle \eta[\underset{\sim}{A}_{o\alpha}\underset{\sim}{B}_\alpha], f\rangle$, so $r = f = V[\Pi_{o(o\alpha)}\underset{\sim}{A}_{o\alpha}]$ in this case.

3.4.4.8. $\underset{\sim}{C}_\gamma$ has the form $[\underset{\sim}{A}_{\gamma\beta}\underset{\sim}{B}_\beta]$.

Let $\mathfrak{V}_\varphi{}^2[\underset{\sim}{A}_{\gamma\beta}\underset{\sim}{B}_\beta] = ((\mathfrak{V}_\varphi{}^2\underset{\sim}{A}_{\gamma\beta})(\mathfrak{V}_\varphi\underset{\sim}{B}_\beta))^2$. Note that $\mathfrak{V}_\varphi[\underset{\sim}{A}_{\gamma\beta}\underset{\sim}{B}_\beta] = (\mathfrak{V}_\varphi{}^2\underset{\sim}{A}_{\gamma\beta})(\mathfrak{V}_\varphi\underset{\sim}{B}_\beta) \in \mathfrak{D}_\gamma$, since by inductive hypothesis $\mathfrak{V}_\varphi\underset{\sim}{A}_{\gamma\beta} \in \mathfrak{D}_{\gamma\beta}$ and $\mathfrak{V}_\varphi\underset{\sim}{B}_\beta \in \mathfrak{D}_\beta$ so

$((\mathfrak{V}_\varphi{}^2\underset{\sim}{A}_{\gamma\beta})(\mathfrak{V}_\varphi\underset{\sim}{B}_\beta))^1 = \eta[(\mathfrak{V}_\varphi{}^1\underset{\sim}{A}_{\gamma\beta})\mathfrak{V}_\varphi{}^1\underset{\sim}{B}_\beta]$ (by 3.4.1.3.)

$= \mathfrak{V}_\varphi{}^1[\underset{\sim}{A}_{\gamma\beta}\underset{\sim}{B}_\beta]$ (by 3.4.4.2.).

3.4.4.9. $\underset{\sim}{C}_\gamma$ has the form $[\lambda \underset{\sim}{x}_\beta \underset{\sim}{A}_\alpha]$.

Let $\mathcal{V}_\varphi{}^2[\lambda \underset{\sim}{x}_\beta \underset{\sim}{A}_\alpha]$ be that function from $\mathcal{D}_\beta$ into $\mathcal{D}_\alpha$ whose value on each $\mathfrak{C} \in \mathcal{D}_\beta$ is $\mathcal{V}_{(\varphi:\underset{\sim}{x}_\beta/\mathfrak{C})}\underset{\sim}{A}_\alpha$.

To satisfy 3.4.1.3. we must show that if $\mathfrak{C} = \langle \underset{\sim}{B}_\beta, q\rangle \in \mathcal{D}_\beta$, then $(\mathcal{V}_{(\varphi:\underset{\sim}{x}_\beta/\mathfrak{C})}\underset{\sim}{A}_\alpha)^1 = \eta[(\mathcal{V}_\varphi{}^1[\lambda \underset{\sim}{x}_\beta \underset{\sim}{A}_\alpha])\underset{\sim}{B}_\beta]$. Let $\underset{\sim}{y}^1,\ldots,\underset{\sim}{y}^n$ be the free variables of $[\lambda \underset{\sim}{x}_\beta \underset{\sim}{A}_\alpha]$. Then $\mathcal{V}_\varphi{}^1[\lambda \underset{\sim}{x}_\beta \underset{\sim}{A}_\alpha] = \eta[[\lambda \underset{\sim}{y}^1 \ldots \lambda \underset{\sim}{y}^n \lambda \underset{\sim}{x}_\beta \underset{\sim}{A}_\alpha](\varphi^1 \underset{\sim}{y}^1)\ldots(\varphi^1 \underset{\sim}{y}^n)]$. Also whether or not $\underset{\sim}{x}_\beta$ is free in $\underset{\sim}{A}_\alpha$, $\mathcal{V}^1{}_{(\varphi:\underset{\sim}{x}_\beta/\mathfrak{C})}\underset{\sim}{A}_\alpha$ conv $[[\lambda \underset{\sim}{y}^1 \ldots \lambda \underset{\sim}{y}^n \lambda \underset{\sim}{x}_\beta \underset{\sim}{A}_\alpha](\varphi^1 \underset{\sim}{y}^1)\ldots(\varphi^1 \underset{\sim}{y}^n)\underset{\sim}{B}_\beta]$ by 3.4.4.1. and 3.4.3.3. The desired result follows by λ-conversion.

3.4.5. Remark. In the terminology of [4] we have now essentially shown that the set of V-complexes constitute a general model for $\mathcal{T}$ in which the axioms of extensionality (Axioms 10 of [4]) do not necessarily hold. Of course in order to permit the axioms of extensionality to fail we have avoided making $\mathcal{D}_o = \{t,f\}$, and we have avoided making $\mathcal{D}_{\alpha\beta}$ contain genuine functions from $\mathcal{D}_\beta$ into $\mathcal{D}_\alpha$. Instead we have in essence indexed these truth values and functions p by wffs $\underset{\sim}{A}$ and called the indexed entity $\langle \underset{\sim}{A}, p\rangle$ a V-complex.

Since the theorems of $\mathcal{T}$ are known to be valid in all general models, the unsceptical reader will readily believe Lemma 3.4.9. below, and may proceed directly to 3.4.10. after noting 3.4.8.

3.4.6. Lemma. If φ and ψ are assignments which agree on the free variables of A_α, then $\mathcal{V}_\varphi A_\alpha = \mathcal{V}_\psi A_\alpha$.

This follows in a straightforward way from 3.4.4.

3.4.7. Lemma. If $\mathbf{D}_\gamma$ conv $\mathbf{E}_\gamma$ and φ is any assignment, then $\mathcal{V}_\varphi \mathbf{D}_\gamma = \mathcal{V}_\varphi \mathbf{E}_\gamma$.

Proof: We first establish several subsidiary lemmas.

3.4.7.1. Lemma. If $\mathbf{D}_\gamma$ conv $\mathbf{E}_\gamma$ and φ is any assignment, then $\mathcal{V}_\varphi^1 \mathbf{D}_\gamma = \mathcal{V}_\varphi^1 \mathbf{E}_\gamma$.

This follows easily from 3.4.4.1. and properties of λ-conversion, using the fact that if $\mathbf{y}^1,\ldots,\mathbf{y}^m$ are the variables which occur free in $\mathbf{D}_\gamma$ or $\mathbf{E}_\gamma$, then

$\mathcal{V}_\varphi^1 \mathbf{D}_\gamma$ conv $[[\lambda \mathbf{y}^1 \ldots \lambda \mathbf{y}^m \mathbf{D}_\gamma](\varphi^1 \mathbf{y}^1)\ldots(\varphi^1 \mathbf{y}^m)]$.

3.4.7.2. Lemma. If the bound variables of $\mathbf{B}_\beta$ are distinct from $\mathbf{x}_\alpha$ and from the free variables of $\mathbf{A}_\alpha$, φ is an assignment, and $\psi = (\varphi : \mathbf{x}_\alpha / \mathcal{V}_\varphi \mathbf{A}_\alpha)$, then $\mathcal{V}_\varphi \mathrm{S}^{\mathbf{x}_\alpha}_{\mathbf{A}_\alpha} \mathbf{B}_\beta = \mathcal{V}_\psi \mathbf{B}_\beta$.

Proof:

3.4.7.2.1. First treating $\mathcal{V}_\varphi^1$ we have

$$\begin{aligned}
\mathcal{V}_\varphi^1 \mathrm{S}^{\mathbf{x}_\alpha}_{\mathbf{A}_\alpha} \mathbf{B}_\beta &= \mathcal{V}_\varphi^1 [[\lambda \mathbf{x}_\alpha \mathbf{B}_\beta] \mathbf{A}_\alpha] && \text{(by 3.4.7.1.)} \\
&= \eta[(\mathcal{V}_\varphi^1 [\lambda \mathbf{x}_\alpha \mathbf{B}_\beta]) \mathcal{V}_\varphi^1 \mathbf{A}_\alpha] && \text{(by 3.4.4.2.)} \\
&= \eta[(\mathcal{V}_\psi^1 [\lambda \mathbf{x}_\alpha \mathbf{B}_\beta]) \mathcal{V}_\psi^1 \mathbf{x}_\alpha] && \text{(by 3.4.6. and 3.4.4.4.)} \\
&= \mathcal{V}_\psi^1 \mathbf{B}_\beta && \text{(by 3.4.4.2. and 3.4.7.1.).}
\end{aligned}$$

Next we prove the lemma by induction on the number of occurrences of [in $\mathbf{B}_\beta$, and consider the following cases:

3.4.7.2.2. $\mathbf{B}_\beta$ is $\mathbf{x}_\alpha$.

Then $\mathcal{V}_\varphi \mathrm{S}^{\mathbf{x}_\alpha}_{\mathbf{A}_\alpha}\mathbf{B}_\beta = \mathcal{V}_\varphi \mathbf{A}_\alpha = \psi\mathbf{x}_\alpha = \mathcal{V}_\psi \mathbf{B}_\beta$.

3.4.7.2.3. $\mathbf{x}_\alpha$ does not occur in $\mathbf{B}_\beta$.

Then $\mathcal{V}_\varphi \mathrm{S}^{\mathbf{x}_\alpha}_{\mathbf{A}_\alpha}\mathbf{B}_\beta = \mathcal{V}_\varphi \mathbf{B}_\beta = \mathcal{V}_\psi \mathbf{B}_\beta$ by 3.4.6.

3.4.7.2.4. $\mathbf{B}_\beta$ has the form $[\mathbf{G}_{\beta\delta}\mathbf{H}_\delta]$.

This is straightforward using 3.4.4.8. and the inductive hypothesis.

3.4.7.2.5. $\mathbf{B}_\beta$ has the form $[\lambda\mathbf{y}_\delta\mathbf{E}_\epsilon]$.

Note that $\mathbf{y}_\delta$ must be distinct from $\mathbf{x}_\alpha$ and from the free variables of $\mathbf{A}_\alpha$. Let $\mathfrak{C} \in \mathfrak{D}_\delta$. Let $\varphi' = (\varphi:\mathbf{y}_\delta/\mathfrak{C})$ and $\psi' = (\psi:\mathbf{y}_\delta/\mathfrak{C})$. Then $\mathcal{V}_\varphi \mathbf{A}_\alpha = \mathcal{V}_{\varphi'}\mathbf{A}_\alpha$ by 3.4.6., so $\psi' = (\varphi':\mathbf{x}_\alpha/\mathcal{V}_{\varphi'}\mathbf{A}_\alpha)$. Thus

$$
\begin{aligned}
(\mathcal{V}_\varphi{}^2 \mathrm{S}^{\mathbf{x}_\alpha}_{\mathbf{A}_\alpha}\mathbf{B}_\beta)\mathfrak{C} &= (\mathcal{V}_\varphi{}^2 [\lambda\mathbf{y}_\delta \mathrm{S}^{\mathbf{x}_\alpha}_{\mathbf{A}_\alpha}\mathbf{E}_\epsilon])\mathfrak{C} \\
&= \mathcal{V}_{\varphi'} \mathrm{S}^{\mathbf{x}_\alpha}_{\mathbf{A}_\alpha}\mathbf{E}_\epsilon && \text{(by 3.4.4.9.)} \\
&= \mathcal{V}_{\psi'}\mathbf{E}_\epsilon && \text{(by inductive hypothesis)} \\
&= (\mathcal{V}_\psi{}^2 \mathbf{B}_\beta)\mathfrak{C} && \text{(by 3.4.4.9.)}.
\end{aligned}
$$

Thus $\mathcal{V}_\varphi{}^2 \mathrm{S}^{\mathbf{x}_\alpha}_{\mathbf{A}_\alpha}\mathbf{B}_\beta$ is the same function as $\mathcal{V}_\psi{}^2 \mathbf{B}_\beta$.

3.4.7.3. <u>Lemma</u>. If the bound variables of $\mathbf{B}_\beta$ are distinct from $\mathbf{x}_\alpha$ and the free variables of $\mathbf{A}_\alpha$, and φ is an assignment, then $\mathcal{V}_\varphi[[\lambda\mathbf{x}_\alpha\mathbf{B}_\beta]\mathbf{A}_\alpha] = \mathcal{V}_\varphi \mathrm{S}^{\mathbf{x}_\alpha}_{\mathbf{A}_\alpha}\mathbf{B}_\beta$.

Proof:

Let ψ be as in 3.4.7.2. Then

$$\mathcal{V}_{\varphi}[[\lambda \underset{\sim}{x}_{\alpha}\underset{\sim}{B}_{\beta}]\underset{\sim}{A}_{\alpha}] = (\mathcal{V}_{\varphi}^{2}[\lambda \underset{\sim}{x}_{\alpha}\underset{\sim}{B}_{\beta}])\mathcal{V}_{\varphi}\underset{\sim}{A}_{\alpha} \qquad \text{(by 3.4.4.8.)}$$

$$= \mathcal{V}_{\psi}\underset{\sim}{B}_{\beta} \qquad \text{(by 3.4.4.9.)}$$

$$= \mathcal{V}_{\varphi}S^{\underset{\sim}{x}_{\alpha}}_{\underset{\sim}{A}_{\alpha}}\underset{\sim}{B}_{\beta} \qquad \text{(by 3.4.7.2.).}$$

3.4.7.4. Lemma. If $\underset{\sim}{y}_{\beta}$ does not occur in $\underset{\sim}{A}_{\alpha}$ and $\underset{\sim}{x}_{\beta}$ is not bound in $\underset{\sim}{A}_{\alpha}$ and φ is an assignment, then

$$\mathcal{V}_{\varphi}[\lambda \underset{\sim}{x}_{\beta}\underset{\sim}{A}_{\alpha}] = \mathcal{V}_{\varphi}[\lambda \underset{\sim}{y}_{\beta}S^{\underset{\sim}{x}_{\beta}}_{\underset{\sim}{y}_{\beta}}\underset{\sim}{A}_{\alpha}].$$

Proof:

We assume $\underset{\sim}{x}_{\beta} \neq \underset{\sim}{y}_{\beta}$, since otherwise the result is trivial.

$\mathcal{V}_{\varphi}^{1}[\lambda \underset{\sim}{x}_{\beta}\underset{\sim}{A}_{\alpha}] = \mathcal{V}_{\varphi}^{1}[\lambda \underset{\sim}{y}_{\beta}S^{\underset{\sim}{x}_{\beta}}_{\underset{\sim}{y}_{\beta}}\underset{\sim}{A}_{\alpha}]$ by 3.4.7.1.

Considering $\mathcal{V}_{\varphi}^{2}$, for all $\mathfrak{C} \in \mathcal{D}_{\beta}$ we have

$$(\mathcal{V}_{\varphi}^{2}[\lambda \underset{\sim}{x}_{\beta}\underset{\sim}{A}_{\alpha}])\mathfrak{C} = \mathcal{V}_{(\varphi:\underset{\sim}{x}_{\beta}/\mathfrak{C})}^{2}\underset{\sim}{A}_{\alpha} \qquad \text{(by 3.4.4.9.)}$$

$$= \mathcal{V}_{((\varphi:\underset{\sim}{y}_{\beta}/\mathfrak{C}):\underset{\sim}{x}_{\beta}/\mathfrak{C})}^{2}\underset{\sim}{A}_{\alpha} \qquad \text{(by 3.4.6.)}$$

$$= \mathcal{V}_{(\varphi:\underset{\sim}{y}_{\beta}/\mathfrak{C})}^{2}S^{\underset{\sim}{x}_{\beta}}_{\underset{\sim}{y}_{\beta}}\underset{\sim}{A}_{\alpha} \qquad \text{(by 3.4.7.2.)}$$

$$= (\mathcal{V}_{\varphi}^{2}[\lambda \underset{\sim}{y}_{\beta}S^{\underset{\sim}{x}_{\beta}}_{\underset{\sim}{y}_{\beta}}\underset{\sim}{A}_{\alpha}])\mathfrak{C} \qquad \text{(by 3.4.4.9.)}$$

so the indicated functions are the same.

3.4.7.5. The proof of 3.4.7. now follows easily from 3.4.7.3. and 3.4.7.4. One may assume that $\underset{\sim}{E}_{\gamma}$ is obtained from $\underset{\sim}{D}_{\gamma}$ by a single application of a rule of λ-conversion, and proceed by induction on the number of occurrences of [in $\underset{\sim}{D}_{\gamma}$.

3.4.8. Lemma. Let φ be any assignment.

3.4.8.1. $\mathcal{V}_{\varphi}^{2}[\sim \underset{\sim}{A}_{o}] = \sim \mathcal{V}_{\varphi}^{2}\underset{\sim}{A}_{o}.$

3.4.8.2. $\mathcal{V}_\varphi^2[\mathbf{A}_o \vee \mathbf{B}_o] = (\mathcal{V}_\varphi^2\mathbf{A}_o) \vee (\mathcal{V}_\varphi^2\mathbf{B}_o)$.

These follow directly from 3.4.4.8., 3.4.4.5., and 3.4.4.6.

3.4.9. Lemma. If φ is any assignment and $\vdash_{\mathfrak{T}} \mathbf{A}_o$, then $\mathcal{V}_\varphi^2\mathbf{A}_o = t$.

Proof: We show that $\mathfrak{W} = \{\mathbf{A}_o \mid \mathcal{V}_\varphi^2\mathbf{A}_o = t$ for all assignments $\varphi\}$ contains the axioms of $\mathfrak{T}$ and is closed under the rules of inference. This follows immediately from 3.4.8. for Axioms 2.5.1.-2.5.4. and Modus ponens, and from 3.4.7. for the rules of λ-conversion. We leave to the reader the routine calculations for Axioms 2.5.5. and 2.5.6., using 3.4.4. and 3.4.8.

For 2.6.4. (Substitution) and 2.6.6. (Generalization) we suppose $\mathbf{x}_\alpha$ is not free in $\mathbf{F}_{o\alpha}$ and that $[\mathbf{F}_{o\alpha}\mathbf{x}_\alpha] \in \mathfrak{W}$; we must show $\Pi_{o(o\alpha)}\mathbf{F}_{o\alpha} \in \mathfrak{W}$ and $[\mathbf{F}_{o\alpha}\mathbf{A}_\alpha] \in \mathfrak{W}$. Given φ, we let $\mathfrak{C} \in \mathcal{D}_\alpha$ and $\psi = (\varphi : \mathbf{x}_\alpha/\mathfrak{C})$.

Then $t = \mathcal{V}_\psi^2[\mathbf{F}_{o\alpha}\mathbf{x}_\alpha]$

$= ((\mathcal{V}_\psi^2\mathbf{F}_{o\alpha})\mathcal{V}_\psi\mathbf{x}_\alpha)^2$ (by 3.4.4.8.)

$= ((\mathcal{V}_\psi^2\mathbf{F}_{o\alpha})\mathfrak{C})^2$

$= ((\mathcal{V}_\varphi^2\mathbf{F}_{o\alpha})\mathfrak{C})^2$ (by 3.4.6.)

for all $\mathfrak{C} \in \mathcal{D}_\alpha$ so

$t = ((\mathcal{V}_\varphi^2\Pi_{o(o\alpha)})\mathcal{V}_\varphi\mathbf{F}_{o\alpha})^2$ (by 3.4.4.7.)

$= \mathcal{V}_\varphi^2[\Pi_{o(o\alpha)}\mathbf{F}_{o\alpha}]$ (by 3.4.4.8.)

so $[\Pi_{o(o\alpha)}\mathbf{F}_{o\alpha}] \in \mathfrak{W}$. Also, if we let $\mathfrak{C} = \mathcal{V}_\varphi\mathbf{A}_\alpha$ then

$t = ((\mathcal{V}_\varphi^2\mathbf{F}_{o\alpha})\mathcal{V}_\varphi\mathbf{A}_\alpha)^2 = \mathcal{V}_\varphi^2[\mathbf{F}_{o\alpha}\mathbf{A}_\alpha]$ (by 3.4.4.8)

so $[\mathbf{F}_{o\alpha}\mathbf{A}_\alpha] \in \mathfrak{W}$.

3.4.10. We now complete the proof of 3.4. If 3.4. is not true, there are η-wffs $\mathbf{A}^1, \ldots, \mathbf{A}^n$ such that $V\,\mathbf{A}^i = t$ for $1 \leq i \leq n$ but $\vdash_{\mathfrak{T}} \sim \mathbf{A}^1 \vee \ldots \vee \sim\mathbf{A}^n$. By 3.4.2. we can define an assignment φ so that $\varphi^1 \mathbf{x}_\alpha = \mathbf{x}_\alpha$ for all variables $\mathbf{x}_\alpha$. Then $\mathcal{V}_\varphi{}^1 \mathbf{A} = \mathbf{A}$ for all η-wffs $\mathbf{A}$ by 3.4.4.1. so by 3.4.4. and 3.4.1.1. for $1 \leq i \leq n$ $\mathcal{V}_\varphi \mathbf{A}^i = \langle \mathbf{A}^i, V\,\mathbf{A}^i \rangle = \langle \mathbf{A}^i, t \rangle$ and $\mathcal{V}_\varphi{}^2 \mathbf{A}^i = t$. Hence by 3.4.8. $\mathcal{V}_\varphi{}^2 [\sim\mathbf{A}^1 \vee \ldots \vee \sim\mathbf{A}^n] = f$, contradicting 3.4.9.

3.5. <u>Theorem</u>. If Γ is an abstract consistency property and $\mathcal{S}$ is a finite set of wffs$_o$ such that $\Gamma(\mathcal{S})$, then $\mathcal{S}$ is consistent.

Proof: by 3.3. and 3.4.

<u>Remark</u>. Our analogy with [10] suggests that the conclusion of 3.5. should be that $\mathcal{S}$ has a denumerable general model. By the remark 3.4.5. we have actually shown that $\mathcal{S}$ has a general model (although we have not actually defined what is meant by a general model when axioms of extensionality are not assumed). Of course we have not dealt with the question of denumerability.

§ 4. Cut-elimination.

4.1. Preliminary definitions.

4.1.1. The disjunctive components of a wff_o are defined inductively as follows:

4.1.2.1. $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are disjunctive components of $[\underset{\sim}{A} \vee \underset{\sim}{B}]$.

4.1.2.2. $\underset{\sim}{A}$ is a disjunctive component of $\underset{\sim}{A}$.

4.1.2.3. If $\underset{\sim}{A}$ is a disjunctive component of $\underset{\sim}{B}$, and $\underset{\sim}{B}$ is a disjunctive component of $\underset{\sim}{C}$, then $\underset{\sim}{A}$ is a disjunctive component of $\underset{\sim}{C}$.

We regard disjunctive components as occurrences of $wffs_o$.

4.1.2. We now find it convenient to modify our conventions concerning syntactical variables so that $\underset{\sim}{A} \vee \underset{\sim}{B}$ and $\underset{\sim}{B} \vee \underset{\sim}{A}$ may simply stand for $\underset{\sim}{A}$ in appropriate contexts. To this end we introduce a "pseudo-wff" , the constant $\Box$, which may be interpreted as the empty disjunction, and therefore denotes falsehood. We henceforth let $\underset{\sim}{A}_o$, $\underset{\sim}{B}_o$, $\underset{\sim}{C}_o$, (etc.) take $\Box$ as value when these syntactic variables occur as disjunctive components of an expression which stands for a wff . Then we regard $\underset{\sim}{A} \vee \Box$ and $\Box \vee \underset{\sim}{A}$ as abbreviations for $\underset{\sim}{A}$. $\Box$ standing alone may be regarded as an abbreviation for $\forall p_o p_o$.

4.2. The System $\mathcal{G}$

4.2.1. Axioms: $\sim A \vee A$, where A is atomic.

4.2.2. Rules of Inference

4.2.2.1. Conversion - I-III. Apply 2.6.1. or 2.6.3.

4.2.2.2. Disjunction Rules. To replace a disjunctive component D of a wff by E, where

D is $[[A \vee B] \vee C]$ and E is $[A \vee [B \vee C]]$, or

D is $[A \vee [B \vee C]]$ and E is $[[A \vee B] \vee C]$, or

D is $[A \vee B]$ and E is $[B \vee A]$.

4.2.2.3. Weakening. From M to infer $M \vee A$ (where M is not $\square$).

4.2.2.4. Negation Introduction. From $M \vee A$ to infer $M \vee \sim\sim A$.

4.2.2.5. Conjunction Introduction. From $M \vee \sim A$ and $M \vee \sim B$ to infer $M \vee \sim [A \vee B]$.

4.2.2.6. Existential Generalization. From $M \vee \sim \Pi_{o(o\alpha)} A_{o\alpha} \vee \sim A_{o\alpha} B_{\alpha}$ to infer $M \vee \sim \Pi_{o(o\alpha)} A_{o\alpha}$.

4.2.2.7. Universal Generalization. From $M \vee A_{o\alpha} x_{\alpha}$ to infer $M \vee \Pi_{o(o\alpha)} A_{o\alpha}$, provided x_{α} is not free in M or $A_{o\alpha}$.

4.3. Proposition. If $\vdash_{\mathcal{G}} A$, then $\vdash_{\mathcal{T}} A$. This is readily established by showing that the rules of inference of $\mathcal{G}$ are derived rules of inference of $\mathcal{T}$.

We next establish some subsidiary lemmas. We shall discuss their proofs together since they all have the same form.

4.4. Lemma. If $\underset{\sim}{P}$ conv $\underset{\sim}{Q}$ then $\vdash_{\mathcal{G}} \underset{\sim}{P}$ iff $\vdash_{\mathcal{G}} \underset{\sim}{Q}$.

4.5. Lemma. If $\vdash_{\mathcal{G}} \underset{\sim}{P}$, and $\underset{\sim}{P}$ has a disjunctive component of the form $\sim\sim \underset{\sim}{D}$, and $\underset{\sim}{Q}$ is the result of replacing this component of $\underset{\sim}{P}$ by $\underset{\sim}{D}$, then $\vdash_{\mathcal{G}} \underset{\sim}{Q}$.

4.6. Lemma. If $\vdash_{\mathcal{G}} \underset{\sim}{P}$, and $\underset{\sim}{P}$ has a disjunctive component of the form $\sim[\underset{\sim}{D} \vee \underset{\sim}{E}]$, and $\underset{\sim}{Q}$ is the result of replacing this component of $\underset{\sim}{P}$ by $\sim \underset{\sim}{D}$ or by $\sim \underset{\sim}{E}$, then $\vdash_{\mathcal{G}} \underset{\sim}{Q}$.

4.7. Lemma. If $\vdash_{\mathcal{G}} \underset{\sim}{P}$, and $\underset{\sim}{y}^1, \ldots, \underset{\sim}{y}^n$ are distinct variables and $\underset{\sim}{z}^j$ is a variable of the same type as $\underset{\sim}{y}^j$ for $1 \leq j \leq n$, then there is a wff $\underset{\sim}{Q}$ such that $\underset{\sim}{P}$ conv-I $\underset{\sim}{Q}$ and $\underset{\sim}{z}^j$ is free for $\underset{\sim}{y}^j$ in $\underset{\sim}{Q}$ for $1 \leq j \leq n$ and $\vdash_{\mathcal{G}} \mathrm{S}^{\underset{\sim}{y}^1 \ldots \underset{\sim}{y}^n}_{\underset{\sim}{z}^1 \ldots \underset{\sim}{z}^n} \underset{\sim}{Q}$.

4.8. Lemma. If $\vdash_{\mathcal{G}} \underset{\sim}{P}$, and $\underset{\sim}{P}$ has a disjunctive component of the form $\Pi_{o(o\beta)} \underset{\sim}{B}_{o\beta}$, and $\underset{\sim}{Q}$ is the result of replacing this component of $\underset{\sim}{P}$ by $\underset{\sim}{B}_{o\beta} \underset{\sim}{z}_{\beta}$, then $\vdash_{\mathcal{G}} \underset{\sim}{Q}$.

Proofs of 4.4-4.8:

Note that to prove 4.4 it suffices to prove 4.4': if $\vdash_{\mathcal{G}} \underset{\sim}{P}$ then $\vdash_{\mathcal{G}} \eta \underset{\sim}{P}$. For when this is established one knows that if $\vdash \underset{\sim}{P}$, then $\vdash \eta \underset{\sim}{P}$, so $\vdash \eta \underset{\sim}{Q}$, so $\vdash \underset{\sim}{Q}$ by 4.2.2.1.

To prove 4.4' and 4.5-4.8 let $P^1,\ldots,P^m$ be a proof in $\mathcal{G}$. We prove by induction on i that the lemmas hold for P^i for $1 \leq i \leq m$. Each lemma is trivial when P^i is an axiom. If P^i is not an axiom one considers how P^i was inferred and applies the inductive hypothesis (if necessary) to the wff(s) from which it was inferred. The proofs of lemmas 4.4', 4.5, and 4.6 are routine in all cases.

The proof of 4.7 is trivial except when P^i is inferred by 4.2.2.1 or 4.2.2.7. Suppose P^i is inferred from P^k by 4.2.2.1. One easily defines a wff Q^i such that P^i conv—I Q^i and z^j is free for y^j for $1 \leq j \leq n$. Let Q^k be a wff whose existence is assured by the inductive hypothesis. Then Q^k conv P^k conv P^i conv Q^i so

$$S^{y^1\ldots y^n}_{z^1\ldots z^n} Q^k \text{ conv } [[\lambda y^1\ldots\lambda y^n Q^k]\ z^1\ldots z^n]$$

$$\text{conv } [[\lambda y^1\ldots\lambda y^n Q^i]\ z^1\ldots z^n] \text{ conv } S^{y^1\ldots y^n}_{z^1\ldots z^n} Q^i,$$

so the latter wff is a theorem of $\mathcal{G}$ by 4.4 and the inductive hypothesis.

Suppose P^i is $M \vee \Pi_{o(o\alpha)} A_{o\alpha}$ and is inferred by 4.2.2.7 from $M \vee A_{o\alpha} x_\alpha$. Since x_α is not free in P^i we may assume x_α is distinct from $y^1,\ldots,y^n$, but we must allow for the possibility that some z^j is x_α. Let g_α be distinct from $z^1,\ldots,z^n$ and all variables free in P^i. By the inductive hypothesis 4.7 there is a wff $[M' \vee A_{o\alpha}' x_\alpha]$ conv-I $[M \vee A_{o\alpha} x_\alpha]$ such that z^j

is free for $\underset{\sim}{y}^j$ in $\underset{\sim}{M}' \vee \underset{\sim}{A}_{o\alpha}{}'\underset{\sim}{x}_\alpha$ for $1 \leq j \leq n$ and

$\vdash_{\mathcal{G}} S^{\underset{\sim}{y}^1 \ldots \underset{\sim}{y}^n \underset{\sim}{x}_\alpha}_{\underset{\sim}{z}^1 \ldots \underset{\sim}{z}^n \underset{\sim}{g}_\alpha} [\underset{\sim}{M}' \vee \underset{\sim}{A}_{o\alpha}{}'\underset{\sim}{x}_\alpha]$. It is readily seen that one may apply 4.2.2.7 to obtain $\vdash_{\mathcal{G}} S^{\underset{\sim}{y}^1 \ldots \underset{\sim}{y}^n}_{\underset{\sim}{z}^1 \ldots \underset{\sim}{z}^n} [\underset{\sim}{M}' \vee \Pi_{o(o\alpha)}\underset{\sim}{A}_{o\alpha}{}']$, which completes the proof of 4.7.

The proof of 4.8 is trivial except when $\underset{\sim}{P}^i$ is inferred by 4.2.2.7, so suppose $\underset{\sim}{P}^i$ has the form $\underset{\sim}{M} \vee \Pi_{o(o\alpha)}\underset{\sim}{A}_{o\alpha}$ and is inferred from $\underset{\sim}{M} \vee \underset{\sim}{A}_{o\alpha}\underset{\sim}{x}_\alpha$. If the component $\Pi_{o(o\beta)}\underset{\sim}{B}_{o\beta}$ referred to in 4.8 is the component $\Pi_{o(o\alpha)}\underset{\sim}{A}_{o\alpha}$ introduced by this application of 4.2.2.7, one obtains $\underset{\sim}{M} \vee \underset{\sim}{B}_{o\beta}\underset{\sim}{z}_\beta$ from $\underset{\sim}{M} \vee \underset{\sim}{A}_{o\alpha}\underset{\sim}{x}_\alpha$ by 4.7 and 4.2.2.1. Otherwise one may assume without real loss of generality that $\underset{\sim}{P}^i$ has the form $\underset{\sim}{N} \vee \Pi_{o(o\beta)}\underset{\sim}{B}_{o\beta} \vee \Pi_{o(o\alpha)}\underset{\sim}{A}_{o\alpha}$ and is inferred from $\underset{\sim}{N} \vee \Pi_{o(o\beta)}\underset{\sim}{B}_{o\beta} \vee \underset{\sim}{A}_{o\alpha}\underset{\sim}{x}_\alpha$. Let $\underset{\sim}{y}_\beta$ be distinct from all variables in the latter wff.

$\vdash_{\mathcal{G}} \underset{\sim}{N} \vee \underset{\sim}{B}_{o\beta}\underset{\sim}{y}_\beta \vee \underset{\sim}{A}_{o\alpha}\underset{\sim}{x}_\alpha$ by inductive hypothesis

$\vdash_{\mathcal{G}} \underset{\sim}{N} \vee \underset{\sim}{B}_{o\beta}\underset{\sim}{y}_\beta \vee \Pi_{o(o\alpha)}\underset{\sim}{A}_{o\alpha}$ by 4.2.2.7.

$\vdash_{\mathcal{G}} \underset{\sim}{N} \vee \underset{\sim}{B}_{o\beta}\underset{\sim}{z}_\beta \vee \Pi_{o(o\alpha)}\underset{\sim}{A}_{o\alpha}$ by 4.7 and 4.2.2.1.

This completes the proof of 4.8.

4.9. Lemma. If $\vdash_{\mathcal{G}} \underset{\sim}{M} \vee \underset{\sim}{D} \vee \underset{\sim}{D}$ then $\vdash_{\mathcal{G}} \underset{\sim}{M} \vee \underset{\sim}{D}$.

Proof:

The proof is by induction on the number of occurrences of logical constants in $\eta\underset{\sim}{D}$. We consider the following cases, assuming that $\underset{\sim}{D}$ is an η-wff in cases 4.9.1-4.9.4:

4.9.1. $\underset{\sim}{D}$ has the form $[\underset{\sim}{B} \vee \underset{\sim}{C}]$.

$\vdash \underset{\sim}{M} \vee \underset{\sim}{B} \vee \underset{\sim}{C} \vee \underset{\sim}{B} \vee \underset{\sim}{C}$ given.

$\vdash \underset{\sim}{M} \vee \underset{\sim}{B} \vee \underset{\sim}{B} \vee \underset{\sim}{C}$ by 4.2.2.2 and inductive hypothesis.

$\vdash \underset{\sim}{M} \vee \underset{\sim}{D}$ by 4.2.2.2 and inductive hypothesis.

4.9.2. $\underset{\sim}{D}$ has the form $\sim\sim \underset{\sim}{E}$.

$\vdash \underset{\sim}{M} \vee \sim\sim \underset{\sim}{E} \vee \sim\sim \underset{\sim}{E}$ given.

$\vdash \underset{\sim}{M} \vee \underset{\sim}{E} \vee \underset{\sim}{E}$ by 4.5 (twice).

$\vdash \underset{\sim}{M} \vee \underset{\sim}{E}$ by inductive hypothesis.

$\vdash \underset{\sim}{M} \vee \underset{\sim}{D}$ by 4.2.2.4.

4.9.3. D has the form $\sim[\underset{\sim}{B} \vee \underset{\sim}{C}]$

$\vdash \underset{\sim}{M} \vee \sim [\underset{\sim}{B} \vee \underset{\sim}{C}] \vee \sim[\underset{\sim}{B} \vee \underset{\sim}{C}]$ given.

$\vdash \underset{\sim}{M} \vee \sim \underset{\sim}{B} \vee \sim \underset{\sim}{B}$ by 4.6 (twice).

$\vdash \underset{\sim}{M} \vee \sim \underset{\sim}{B}$ by inductive hypothesis.

$\vdash \underset{\sim}{M} \vee \sim \underset{\sim}{C}$ similarly.

$\vdash \underset{\sim}{M} \vee \underset{\sim}{D}$ by 4.2.2.5.

4.9.4. $\underset{\sim}{D}$ has the form $\Pi_{o(o\alpha)}\underset{\sim}{A}_{o\alpha}$.

Let $\underset{\sim}{x}_\alpha$ be a variable which does not occur in $\underset{\sim}{D}$.

$\vdash \underset{\sim}{M} \vee \Pi_{o(o\alpha)}\underset{\sim}{A}_{o\alpha} \vee \Pi_{o(o\alpha)}\underset{\sim}{A}_{o\alpha}$ given.

$\vdash \underset{\sim}{M} \vee \underset{\sim}{A}_{o\alpha}\underset{\sim}{x}_\alpha \vee \underset{\sim}{A}_{o\alpha}\underset{\sim}{x}_\alpha$ by 4.8 (twice).

$\vdash \underset{\sim}{M} \vee \eta[\underset{\sim}{A}_{o\alpha}\underset{\sim}{x}_\alpha] \vee \eta[\underset{\sim}{A}_{o\alpha}\underset{\sim}{x}_\alpha]$ by 4.4.

Since $\underset{\sim}{A}_{o\alpha}$ is an η-wff, it is easy to see as in 3.3.3.4 that $\eta[\underset{\sim}{A}_{o\alpha}\underset{\sim}{x}_\alpha]$ contains the same number of occurrences of logical constants as does $\underset{\sim}{A}_{o\alpha}$, so

$\vdash \underset{\sim}{M} \vee \eta[\underset{\sim}{A}_{o\alpha}\underset{\sim}{x}_\alpha]$ by inductive hypothesis.

$\vdash \underset{\sim}{M} \vee \underset{\sim}{D}$ by 4.2.2.1 and 4.2.2.7.

4.9.5. $\eta\underset{\sim}{D}$ is an atom, the negation of an atom, or is of the form $\sim \Pi_{o(o\alpha)}\underset{\sim}{A}_{o\alpha}$.

We prove that if $\underset{\sim}{P}^1,\ldots,\underset{\sim}{P}^m$ is any proof in $\mathcal{G}$, and $\underset{\sim}{P}^i$ has disjunctive components $\underset{\sim}{H}$ and $\underset{\sim}{K}$ such that $\eta\underset{\sim}{H} = \eta\underset{\sim}{K}$ and $\eta\underset{\sim}{H}$ has one of these three forms, then the result of dropping $\underset{\sim}{K}$ from $\underset{\sim}{P}^i$ (i.e., replacing a component of $\underset{\sim}{P}^i$ of the form $[\underset{\sim}{C} \vee \underset{\sim}{K}]$ or $[\underset{\sim}{K} \vee \underset{\sim}{C}]$ by $\underset{\sim}{C}$) is a theorem of $\mathcal{G}$. The proof is straightforward by induction on i.

4.10. <u>Theorem</u>. If $\vdash_{\mathcal{J}}\underset{\sim}{A}$ then $\vdash_{\mathcal{G}}\underset{\sim}{A}$.

Proof:

4.10.1. Let $\Gamma\{\underset{\sim}{C}_o^{\,1},\ldots,\underset{\sim}{C}_o^{\,n}\}$ mean not $\vdash_{\mathcal{G}} \sim\underset{\sim}{C}_o^{\,1} \vee\ldots\vee \sim\underset{\sim}{C}_o^{\,n}$. Note that by 4.2.2.2 this definition is independent of the order in which the wffs $\underset{\sim}{C}^i$ are listed. Also by 4.2.2.3 and 4.9 , $\Gamma(\{\underset{\sim}{C}^1,\ldots,\underset{\sim}{C}^n\} \cup \{\underset{\sim}{D}^1,\ldots,\underset{\sim}{D}^m\})$ is equivalent to not $\vdash_{\mathcal{G}} \sim\underset{\sim}{C}^1 \vee\ldots\vee \sim\underset{\sim}{C}^n \vee \sim\underset{\sim}{D}^1 \vee\ldots\vee \sim\underset{\sim}{D}^m$ whether or not some $\underset{\sim}{C}^i$ is the same as some $\underset{\sim}{D}^j$.

4.10.2. We verify that Γ is an abstract consistency property by checking the contrapositive of 3.1.k in step 4.10.2.k below:

4.10.2.1. If $\underset{\sim}{A}$ is an atom, $\vdash \underset{\sim}{M} \vee \sim\sim\underset{\sim}{A} \vee \sim\underset{\sim}{A}$ by 4.2.1 , 4.2.2.4 , and 4.2.2.3.

4.10.2.2. If $\vdash \underset{\sim}{M} \vee \sim \eta\underset{\sim}{A}$ then $\vdash \underset{\sim}{M} \vee \sim\underset{\sim}{A}$ by 4.2.2.1.

4.10.2.3. If $\vdash \underset{\sim}{M} \vee \sim\underset{\sim}{A}$ then $\vdash \underset{\sim}{M} \vee \sim\sim\sim\underset{\sim}{A}$ by 4.2.2.4.

4.10.2.4. If $\vdash \underset{\sim}{M} \vee \sim \underset{\sim}{A}$ and $\vdash \underset{\sim}{M} \vee \sim \underset{\sim}{B}$ then $\vdash \underset{\sim}{M} \vee \sim [\underset{\sim}{A} \vee \underset{\sim}{B}]$ by 4.2.2.5.

4.10.2.5. If $\vdash \underset{\sim}{M} \vee \sim\sim \underset{\sim}{A} \vee \sim\sim \underset{\sim}{B}$ then $\vdash M \vee \underset{\sim}{A} \vee \underset{\sim}{B}$ by 4.5, so $\vdash \underset{\sim}{M} \vee \sim\sim [\underset{\sim}{A} \vee \underset{\sim}{B}]$ by 4.2.2.4.

4.10.2.6. If $\vdash \underset{\sim}{M} \vee \sim \Pi_{o(o\alpha)} \underset{\sim}{A}_{o\alpha} \vee \sim \underset{\sim}{A}_{o\alpha} \underset{\sim}{B}_{\alpha}$ then $\vdash \underset{\sim}{M} \vee \sim \Pi_{o(o\alpha)} \underset{\sim}{A}_{o\alpha}$ by 4.2.2.6.

4.10.2.7. Suppose there is a variable or parameter $\underset{\sim}{c}_{\alpha}$ which does not occur free in $\underset{\sim}{M}$ or in $\underset{\sim}{A}_{o\alpha}$ such that $\vdash \underset{\sim}{M} \vee \sim\sim \underset{\sim}{A}_{o\alpha} \underset{\sim}{c}_{\alpha}$. By choosing an appropriate variable $\underset{\sim}{x}_{\alpha}$ and substituting it for $\underset{\sim}{c}_{\alpha}$ throughout the proof we obtain $\vdash \underset{\sim}{M} \vee \sim\sim \underset{\sim}{A}_{o\alpha} \underset{\sim}{x}_{\alpha}$, where $\underset{\sim}{x}_{\alpha}$ is a variable not free in $\underset{\sim}{M}$ or $\underset{\sim}{A}_{o\alpha}$. Hence $\vdash \underset{\sim}{M} \vee \underset{\sim}{A}_{o\alpha} \underset{\sim}{x}_{\alpha}$ by 4.5, so $\vdash \underset{\sim}{M} \vee \Pi_{o(o\alpha)} \underset{\sim}{A}_{o\alpha}$ by 4.2.2.7, so $\vdash \underset{\sim}{M} \vee \sim\sim \Pi_{o(o\alpha)} \underset{\sim}{A}_{o\alpha}$ by 4.2.2.4.

4.10.3. Suppose $\vdash_{\mathfrak{J}} \underset{\sim}{A}$. Then $\{\sim \underset{\sim}{A}\}$ is inconsistent (in $\mathfrak{J}$) so by 3.5 , not $\Gamma\{\sim \underset{\sim}{A}\}$, i.e., $\vdash_{\mathcal{G}} \sim\sim \underset{\sim}{A}$, so $\vdash_{\mathcal{G}} \underset{\sim}{A}$ by 4.5.

4.11. <u>Corollary</u>. If $\vdash_{\mathcal{G}} \underset{\sim}{M} \vee \underset{\sim}{A}$ and $\vdash_{\mathcal{G}} \sim \underset{\sim}{A} \vee \underset{\sim}{N}$ then $\vdash_{\mathcal{G}} \underset{\sim}{M} \vee \underset{\sim}{N}$.

Proof: by 4.3 and 4.10 , since this result is easy to establish for $\mathfrak{J}$.

§5. The Resolution System $\mathfrak{R}$.

5.1. Definition. Let $\mathfrak{S}$ be a finite set of sentences. For each type symbol γ choose a parameter $\underset{\sim}{c}_{\gamma(o\gamma)}$ (henceforth called an existential parameter) which does not occur in $\mathfrak{S}$. For this choice of existential parameters, a derivation in $\mathfrak{R}$ of $\underset{\sim}{E}$ from $\mathfrak{S}$ is a finite sequence $\underset{\sim}{D}^1,\ldots,\underset{\sim}{D}^n$ such that $\underset{\sim}{D}^n$ is $\underset{\sim}{E}$ and each $\underset{\sim}{D}^i$ is a member of $\mathfrak{S}$ or is obtained from preceding members of the sequence by one of the following rules of inference:

5.1.1. Conversion - I-II. Apply 2.6.1. or 2.6.2.

5.1.2. Disjunction Rules. (4.2.2.2.)

5.1.3. Simplification. From $\underset{\sim}{M} \vee \underset{\sim}{A} \vee \underset{\sim}{A}$ to infer $\underset{\sim}{M} \vee \underset{\sim}{A}$.

5.1.4. Negation Elimination. From $\underset{\sim}{M} \vee \sim\sim \underset{\sim}{A}$ to infer $\underset{\sim}{M} \vee \underset{\sim}{A}$.

5.1.5. Conjunction Elimination. From $\underset{\sim}{M} \vee \sim [\underset{\sim}{A} \vee \underset{\sim}{B}]$ to infer $\underset{\sim}{M} \vee \sim \underset{\sim}{A}$ and $\underset{\sim}{M} \vee \sim \underset{\sim}{B}$.

5.1.6. Existential Instantiation. From $\underset{\sim}{M} \vee \sim \Pi_{o(o\alpha)}\underset{\sim}{A}_{o\alpha}$ to infer $\underset{\sim}{M} \vee \sim \underset{\sim}{A}_{o\alpha}[\underset{\sim}{c}_{\alpha(o\alpha)}\underset{\sim}{A}_{o\alpha}]$.

5.1.7. Universal Instantiation. From $\underset{\sim}{M} \vee \Pi_{o(o\alpha)}\underset{\sim}{A}_{o\alpha}$ to infer $\underset{\sim}{M} \vee \underset{\sim}{A}_{o\alpha}\underset{\sim}{x}_\alpha$.

5.1.8. Substitution. From $\underset{\sim}{A}$ to infer $[\lambda \underset{\sim}{x}_\alpha \underset{\sim}{A}]\underset{\sim}{B}_\alpha$.

5.1.9. Cut. From $\underset{\sim}{M} \vee \underset{\sim}{A}$ and $\underset{\sim}{N} \vee \sim \underset{\sim}{A}$ to infer $\underset{\sim}{M} \vee \underset{\sim}{N}$.

A derivation of $\square$ from $\mathcal{S}$ is a __refutation of__ $\mathcal{S}$. In $\mathfrak{R}$ one proves a sentence $\underset{\sim}{A}$ by refuting $\sim\underset{\sim}{A}$ (i.e., $\{\sim\underset{\sim}{A}\}$). More generally, one shows that $\underset{\sim}{A}$ follows from a set $\mathcal{H}$ of sentences by refuting $\mathcal{H} \cup \{\sim\underset{\sim}{A}\}$.

5.2. __Remarks__.

For convenience, $\mathfrak{R}$ has been formulated so that only sets of __sentences__ may be refuted in $\mathfrak{R}$, but clearly this involves no real loss of generality.

We write $\mathcal{S} \vdash_{\mathfrak{R}} \underset{\sim}{E}$ (resp. $\mathcal{S} \vdash_{\mathfrak{J}} \underset{\sim}{E}$) iff there is a derivation of $\underset{\sim}{E}$ from $\mathcal{S}$ in $\mathfrak{R}$ (resp. in $\mathfrak{J}$). For $\mathfrak{J}$ this notion is defined, and the deduction theorem is proved in [2, section 5]. In a proof in $\mathfrak{J}$ from assumptions $\mathcal{S}$ one may not generalize upon or substitute for a variable which is free in a wff of $\mathcal{S}$.

The reader may be bothered by the presence of the cut rule 5.1.9. among the rules of inference for $\mathfrak{R}$, since we showed in 4.9. that this need not be taken as a primitive rule of inference in $\mathfrak{G}$. However, since one proves wffs in $\mathfrak{G}$, but refutes them in $\mathfrak{R}$, the role of the cut rule is quite different in the two systems. One is tempted to establish the completeness of $\mathfrak{G}$ and $\mathfrak{R}$ directly with a proof by induction on i that if $\underset{\sim}{D}^1,\ldots,\underset{\sim}{D}^n$ is a proof in $\mathfrak{J}$, then $\vdash_{\mathfrak{G}} \underset{\sim}{D}^i$ and $\sim\overline{\underset{\sim}{D}}^i \vdash_{\mathfrak{R}} \square$, where $\overline{\underset{\sim}{D}}^i$ is obtained from $\underset{\sim}{D}^i$

upon replacing free variables by new parameters in one-one fashion. In each case the crucial difficulty arises when $\underset{\sim}{D}^i$ is inferred by modus ponens. In $\mathcal{G}$ one can overcome this difficulty by proving that the cut rule is a derived rule of inference. However in $\mathcal{R}$ the analogous meta-theorem is that if $\mathcal{S} \cup \{\underset{\sim}{A}\} \vdash \square$ and $\mathcal{S} \cup \{\sim\underset{\sim}{A}\} \vdash \square$ then $\mathcal{S} \vdash \square$.

The wffs $\underset{\sim}{c}_{\alpha(o\alpha)}\underset{\sim}{A}_{o\alpha}$ introduced by 5.1.6. are essentially Herbrand-Skolem functors whose arguments are the free variables of $\underset{\sim}{A}_{o\alpha}$. Suppose one is given $\underset{\sim}{M} \vee \exists\underset{\sim}{x}_\alpha\underset{\sim}{B}_{o\alpha}\underset{\sim}{x}_\alpha$, where the free variables of $\underset{\sim}{B}_{o\alpha}$ are $\underset{\sim}{y}_{\beta_1}^1,\ldots,\underset{\sim}{y}_{\beta_n}^n$ and do not include $\underset{\sim}{x}_\alpha$. (Matters may be so arranged that one may assume $\underset{\sim}{y}_{\beta_1}^1,\ldots,\underset{\sim}{y}_{\beta_n}^n$ were previously introduced by 5.1.7.). The given wff is $\underset{\sim}{M} \vee \sim \Pi_{o(o\alpha)}[\lambda\underset{\sim}{x}_\alpha \sim \underset{\sim}{B}_{o\alpha}\underset{\sim}{x}_\alpha]$, so by 5.1.6., 5.1.1., and 5.1.4. one obtains $\underset{\sim}{M} \vee \underset{\sim}{B}_{o\alpha}[\underset{\sim}{c}_{\alpha(o\alpha)}\cdot\lambda\underset{\sim}{x}_\alpha\cdot\sim\underset{\sim}{B}_{o\alpha}\underset{\sim}{x}_\alpha]$. One may write $[\underset{\sim}{c}_{\alpha(o\alpha)}\cdot\lambda\underset{\sim}{x}_\alpha\cdot\sim\underset{\sim}{B}_{o\alpha}\underset{\sim}{x}_\alpha]$ as $\underset{\sim}{f}_{\alpha\beta_n\ldots\beta_1}\underset{\sim}{y}_{\beta_1}^1\ldots\underset{\sim}{y}_{\beta_n}^n$, where $\underset{\sim}{f}_{\alpha\beta_n\ldots\beta_1}$ is a new function symbol. Thus one replaces $\underset{\sim}{M} \vee \exists\underset{\sim}{x}_\alpha\underset{\sim}{B}_{o\alpha}\underset{\sim}{x}_\alpha$ by $\underset{\sim}{M} \vee \underset{\sim}{B}_{o\alpha}[\underset{\sim}{f}_{\alpha\beta_n\ldots\beta_1}\underset{\sim}{y}_{\beta_1}^1\ldots\underset{\sim}{y}_{\beta_n}^n]$.

When one sets out to refute a set of sentences by resolution [8] in first order logic, one eliminates all propositional connectives except negation, conjunction, and disjunction, and pushes negations in so that they have the smallest possible scope, with double negations being dropped. Then one eliminates existential quantifiers by the method of Herbrand-Skolem functors, and drops universal quantifiers. The resulting quantifier-free wffs are put into conjunctive normal form, whose conjuncts are called clauses. One then derives $\Box$ from this set of clauses by an operation called resolution, which is an elegant combination of substitution and cut (with 5.1.2 and 5.1.3. used implicitly). (An important open problem concerning resolution in type theory is to find an equally elegant way of combining 5.1.8., 5.1.1., and 5.1.9.). However, in type theory one may introduce new occurrences of logical constants by the substitution rule, so one must continually have available the rules 5.1.4.-5.1.7. which correspond to the preliminary processing in first order logic. However, 5.1.8. and 5.1.9. (in conjunction with the subsidiary rules 5.1.1-5.1.3.) remain the crucial rules of inference.

When applying Rule 5.1.7., one might as well choose $\underset{\sim}{x}_\alpha$ to be distinct from the free variables of $\underset{\sim}{M} \vee \Pi_{o(o\alpha)}\underset{\sim}{A}_{o\alpha}$, since one can identify $\underset{\sim}{x}_\alpha$ with another variable later by a substitution, if desired. If $\underset{\sim}{x}_\alpha$ is so chosen, one might as well apply 5.1.3.-5.1.7. immediately whenever these rules are applicable, and then discard the wffs to which these rules are applied, since they need not be used again.

5.3 Theorem. Let $\mathcal{S}$ be a finite set of sentences. If $\mathcal{S} \vdash_{\mathfrak{T}} \Box$ then $\mathcal{S} \vdash_{\mathfrak{R}} \Box$.

Proof:

5.3.1. For any finite set $\mathcal{S}$ of wffs_o, let $\Gamma(\mathcal{S})$ mean not $\mathcal{S}' \vdash_{\mathfrak{R}} \Box$, where $\mathcal{S}'$ is obtained from $\mathcal{S}$ by replacing the free variables in wffs of $\mathcal{S}$ by new parameters in a one-one fashion. We shall show that Γ is an abstract consistency property, so if $\mathcal{S}$ is a set of sentences such that $\mathcal{S} \vdash_{\mathfrak{T}} \Box$, then $\mathcal{S}$ is inconsistent in $\mathfrak{T}$, so by 3.5 not $\Gamma(\mathcal{S})$, i.e. $\mathcal{S} \vdash_{\mathfrak{R}} \Box$.

5.3.2. We verify that Γ is an abstract consistency property by checking the contrapositive of 3.1.k in step 5.3.2k below. For the sake of brevity we shall be rather informal about the distinction between $\mathcal{S}$ and $\mathcal{S}'$, simply assuming that wffs are closed when appropriate.

5.3.2.1. If there is an atom $\mathbf{A}$ such that $\mathbf{A} \in \mathcal{S}$ and $\sim \mathbf{A} \in \mathcal{S}$ then $\mathcal{S} \vdash_{\mathfrak{R}} \Box$ by 5.1.9.

5.3.2.2. If $\mathcal{S} \cup \{\eta\mathbf{A}\} \vdash_{\mathfrak{R}} \Box$ then $\mathcal{S} \cup \{\mathbf{A}\} \vdash_{\mathfrak{R}} \Box$ by 5.1.1.

5.3.2.3. If $\mathcal{S} \cup \{\mathbf{A}\} \vdash_{\mathfrak{R}} \Box$ then $\mathcal{S} \cup \{\sim\sim \mathbf{A}\} \vdash_{\mathfrak{R}} \Box$ by 5.1.4.

5.3.2.4. Suppose $\mathcal{S} \cup \{\mathbf{A}\} \vdash_{\mathfrak{R}} \Box$ and $\mathcal{S} \cup \{\mathbf{B}\} \vdash_{\mathfrak{R}} \Box$. We may assume given refutations $\mathbf{C}^1, \ldots, \mathbf{C}^n$ of $\mathcal{S} \cup \{\mathbf{B}\}$ and $\mathbf{E}^1, \ldots, \mathbf{E}^m$ of $\mathcal{S} \cup \{\mathbf{A}\}$ using the same existential parameters. We define which of the wffs $\mathbf{C}^i$ are derived from $\mathbf{B}$ in the given refutation in the obvious inductive fashion: if $\mathbf{C}^i$ is in $\mathcal{S} \cup \{\mathbf{B}\}$, then $\mathbf{C}^i$ is derived from $\mathbf{B}$ iff $\mathbf{C}^i$ is $\mathbf{B}$; if $\mathbf{C}^i$ is inferred from $\mathbf{C}^j$ (and $\mathbf{C}^k$), then $\mathbf{C}^i$ is derived from $\mathbf{B}$ iff $\mathbf{C}^j$ (or $\mathbf{C}^k$) is derived from $\mathbf{B}$. We define $\underset{\sim}{D}^i$ (for $1 \leq i \leq n$) to be $\underset{\sim}{A} \vee \underset{\sim}{C}^i$ if $\underset{\sim}{C}^i$ is derived from $\underset{\sim}{B}$; otherwise $\underset{\sim}{D}^i$ is $\underset{\sim}{C}^i$. By examining the rules of inference of $\mathfrak{R}$ it is easy to see that $\mathcal{S} \cup \{\underset{\sim}{A} \vee \underset{\sim}{B}\} \vdash \eta\underset{\sim}{D}^i$ for $1 \leq i \leq n$ by induction on i. If $\underset{\sim}{D}^n$ is $\Box$ we are done. Otherwise $\underset{\sim}{D}^n$ is $\underset{\sim}{A}$ so $\mathcal{S} \cup \{\underset{\sim}{A} \vee \underset{\sim}{B}\} \vdash \eta\underset{\sim}{A}$. Now we readily establish $\mathcal{S} \cup \{\underset{\sim}{A} \vee \underset{\sim}{B}\} \vdash \eta\underset{\sim}{E}^i$ for $1 \leq i \leq m$ by induction on i, so $\mathcal{S} \cup \{\underset{\sim}{A} \vee \underset{\sim}{B}\} \vdash \Box$.

5.3.2.5. If $\mathcal{S} \cup \{\sim \underset{\sim}{A}, \sim \underset{\sim}{B}\} \vdash_{\mathfrak{R}} \Box$ then $\mathcal{S} \cup \{\sim [\underset{\sim}{A} \vee \underset{\sim}{B}]\} \vdash_{\mathfrak{R}} \Box$ by 5.1.5.

5.3.2.6. If there exists a wff $\underset{\sim}{B}_\alpha$ such that $\mathcal{S} \cup \{\Pi_{o(o\alpha)}\underset{\sim}{A}_{o\alpha}, \underset{\sim}{A}_{o\alpha}\underset{\sim}{B}_\alpha\} \vdash_{\mathfrak{R}} \Box$ then $\mathcal{S} \cup \{\Pi_{o(o\alpha)}\underset{\sim}{A}_{o\alpha}\} \vdash_{\mathfrak{R}} \Box$ by 5.1.7, 5.1.8, and 5.1.1.

5.3.2.7. Suppose there is a parameter $\underset{\sim}{d}_\alpha$ which does not occur in $\underset{\sim}{A}_{o\alpha}$ or any wff of $\mathcal{S}$ such that $\mathcal{S} \cup \{\sim \underset{\sim}{A}_{o\alpha}d_\alpha\} \vdash_{\mathfrak{R}} \Box$. Let a refutation of $\mathcal{S} \cup \sim A_{o\alpha}\underset{\sim}{d}_\alpha$ be given with existential parameter $\underset{\sim}{c}_{\alpha(o\alpha)}$. Since $[\underset{\sim}{c}_{\alpha(o\alpha)}\underset{\sim}{A}_{o\alpha}]$ is a closed wff it is easy to see that one can replace $\underset{\sim}{d}_\alpha$ by $[\underset{\sim}{c}_{\alpha(o\alpha)}\underset{\sim}{A}_{o\alpha}]$ everywhere in the given refutation to obtain a refutation of $\mathcal{S} \cup \{\sim \Pi_{o(o\alpha)}\underset{\sim}{A}_{o\alpha}\}$, using 5.1.6 to infer $\sim \underset{\sim}{A}_{o\alpha}[\underset{\sim}{c}_{\alpha(o\alpha)}\underset{\sim}{A}_{o\alpha}]$.

5.4 <u>Theorem</u>. Let $\mathcal{S}$ be a finite set of sentences. If $\mathcal{S} \vdash_{\mathfrak{R}} \Box$ then $\mathcal{S} \vdash_{\mathfrak{T}} \Box$.

Proof:

5.4.1. <u>Definition</u>. A derivation in $\mathfrak{R}$ is <u>standard</u> iff the premiss of each application of 5.1.6 in this derivation is a sentence and an η-wff.

5.4.2. <u>Lemma</u>. If $\mathcal{S} \vdash_{\mathfrak{R}} \Box$, then there is a standard refutation of $\mathcal{S}$ in $\mathfrak{R}$.

Proof:

Let $\underset{\sim}{D}^1,\ldots,\underset{\sim}{D}^n$ be a refutation of $\mathcal{S}$. We prove by induction on i that (*) if all the free variables of $\underset{\sim}{D}^i$ are among $\underset{\sim}{y}^1_{\gamma_1},\ldots,\underset{\sim}{y}^k_{\gamma_k}$, and $\underset{\sim}{E}^1_{\gamma_1},\ldots,\underset{\sim}{E}^k_{\gamma_k}$ are closed wffs, then there is a standard derivation in $\mathcal{R}$ from $\mathcal{S}$ of $\eta(S^{\underset{\sim}{y}^1\ldots\underset{\sim}{y}^k}_{\underset{\sim}{E}^1\ldots\underset{\sim}{E}^k}\,\underset{\sim}{D}^i)$.

For the sake of brevity we let θ be the substitution $S^{\underset{\sim}{y}^1\ldots\underset{\sim}{y}^k}_{\underset{\sim}{E}^1\ldots\underset{\sim}{E}^k}$.

(*) is immediate using 5.1.1 if $\underset{\sim}{D}^i \in \mathcal{S}$, since $\underset{\sim}{D}^i$ is then a sentence. If $\underset{\sim}{D}^i$ is obtained from $\underset{\sim}{D}^j$ (where $j < i$) by any of 5.1.2-5.1.6 or 5.1.9, (*) follows easily from the inductive hypothesis; as an example we treat the case where $\underset{\sim}{D}^i$ is $\underset{\sim}{M} \vee \sim \underset{\sim}{A}_{o\alpha}[\underset{\sim}{c}_{\alpha(o\alpha)}\underset{\sim}{A}_{o\alpha}]$ and is obtained by 5.1.6 from $\underset{\sim}{M} \vee \sim \Pi_{o(o\alpha)}\underset{\sim}{A}_{o\alpha}$. By inductive hypothesis there is a standard derivation of $(\eta\ \theta\ \underset{\sim}{M}) \vee \sim \Pi_{o(o\alpha)}(\eta\ \theta\ \underset{\sim}{A}_{o\alpha})$, from which by 5.1.6 one may infer $\eta\ \theta\ \underset{\sim}{M} \vee \sim (\eta\ \theta\ \underset{\sim}{A}_{o\alpha})\ [\underset{\sim}{c}_{\alpha(o\alpha)}\,\eta\ \theta\ \underset{\sim}{A}_{o\alpha}]$, from which one may infer $\eta\ \theta\ \underset{\sim}{D}^i$ by 5.1.1. It is easy to see that one thus obtains a standard derivation of $\eta\ \theta\ \underset{\sim}{D}^i$.

If $\underset{\sim}{D}^i$ is inferred from $\underset{\sim}{D}^j$ by 5.1.1 we have $\theta\ \underset{\sim}{D}^j$ conv $[[\lambda\underset{\sim}{y}^1\ldots\lambda\underset{\sim}{y}^k\underset{\sim}{D}^j]\ \underset{\sim}{E}^1\ldots\underset{\sim}{E}^k]$ conv $[[\lambda\underset{\sim}{y}^1\ldots\lambda\underset{\sim}{y}^k\underset{\sim}{D}^i]\ \underset{\sim}{E}^1\ldots\underset{\sim}{E}^k$ conv $\theta\ \underset{\sim}{D}^i$ so $\eta\ \theta\ \underset{\sim}{D}^j = \eta\ \theta\ \underset{\sim}{D}^i$. Thus the inductive hypothesis suffices.

Suppose $\underset{\sim}{D}^i$ is $\underset{\sim}{M} \vee \underset{\sim}{A}_{o\alpha}\underset{\sim}{x}_\alpha$ and is inferred from $\underset{\sim}{M} \vee \Pi_{o(o\alpha)}\underset{\sim}{A}_{o\alpha}$ by 5.1.7. Let $\underset{\sim}{x}_\alpha$ be $\underset{\sim}{y}^r$. By inductive hypothesis there is a standard derivation of $\eta\ \theta\ \underset{\sim}{M} \vee \Pi_{o(o\alpha)}\,\eta\ \theta\ \underset{\sim}{A}_{o\alpha}$, which is a sentence. From this one can derive $\eta\ \theta\ \underline{M} \vee (\eta\ \theta\ \underline{A}_{o\alpha})\underline{x}_\alpha$ by 5.1.7, then $[\lambda \underline{x}_\alpha \cdot \eta\ \theta\ \underline{M} \vee (\eta\ \theta\ \underline{A}_{o\alpha})\underline{x}_\alpha]\underline{E}^r$ by 5.1.8, then $\eta\ \theta\ \underline{D}^i$ by 5.1.1. One thus obtains a standard derivation of $\eta\ \theta\ \underline{D}^i$.

Suppose D^i is $[\lambda \underline{x}_\alpha \underline{D}^j]\underline{B}_\alpha$ and is inferred from $\underline{D}^j$ by 5.1.8. Let σ be that substitution which simultaneously replaces free occurrences of $\underline{x}_\alpha$ by occurrences of $\theta\ \underline{B}_\alpha$, and free occurrences of $\underline{y}^r$ (for $1 \leq r \leq k$) by occurrences of $\underline{E}^r$ if $\underline{y}^r \neq \underline{x}_\alpha$. Since $\theta\ B_\alpha$ is closed, by inductive hypothesis there is a standard derivation of $\eta\ \sigma\ \underline{D}^j$. However it can be seen that

$$\theta\ \underline{D}^i = (\theta\ [\lambda \underline{x}_\alpha \underline{D}^j])\ \theta\ \underline{B}_\alpha = (\sigma\ [\lambda \underline{x}_\sigma \underline{D}^j])\ \sigma\ \underline{x}_\alpha$$

$$= \sigma[[\lambda \underline{x}_\alpha \underline{D}^j]\ \underline{x}_\alpha] \text{ conv } \sigma\ \underline{D}^j, \text{so } \eta\ \theta\ \underline{D}^i = \eta\ \sigma\ \underline{D}^j.$$

This completes the proof of (*), and hence of 5.4.2.

5.4.3. We now prove 5.4. We may assume given a standard refutation $\underline{D}^1,\ldots,\underline{D}^n$ of $\mathcal{S}$ in $\mathfrak{R}$. Let $[\underline{c}_{\alpha_1(o\alpha_1)}\underline{A}^1_{o\alpha_1}],\ldots,$

$[\underline{c}_{\alpha(o\alpha_k)}\underline{A}^k_{o\alpha_k}]$ be the wffs introduced by 5.1.6 in this refutation. (Note that any of these wffs may be introduced by several different applications of 5.1.6.) We henceforth write

$\underline{c}_{\alpha_j(o\alpha_j)}$ as $\underline{c}^j$ and $\underline{A}^j_{o\alpha_j}$ as $\underline{A}^j$. We may assume that the wffs $[\underline{c}^j\underline{A}^j]$ are ordered in such a way that if $[\underline{c}^j\underline{A}^j]$ occurs in $\underline{A}^i$, then $j < i$. Let $\underline{E}^j$ be the wff $[\underline{A}^j[\underline{c}^j\underline{A}^j] \supset \Pi_{o(o\alpha_j)}\underline{A}^j]$

for $1 \leq j \leq k$. Note that since each rule of inference of $\mathfrak{R}$ except 5.1.6 is a derived rule of inference of $\mathfrak{J}$, the sequence $\underline{D}^1,\ldots,\underline{D}^n$, and hence the sequence $\eta\underline{D}^1,\ldots,\eta\underline{D}^n$, can be regarded as a proof in $\mathcal{T}$ from the assumptions $\mathcal{S} \cup \{\underset{\sim}{E}^1,\ldots,\underset{\sim}{E}^k\}$ using derived rules of inference.

Let $\underset{\sim}{y}^1_{\alpha_1},\ldots,\underset{\sim}{y}^k_{\alpha_k}$ be distinct variables which do not occur in any wff of $\mathcal{S} \cup \{\underset{\sim}{E}^1,\ldots,\underset{\sim}{E}^k\} \cup \{\eta\underset{\sim}{D}^1,\ldots,\eta\underset{\sim}{D}^n\}$. For any wff $\underset{\sim}{B}$, let $\rho\underset{\sim}{B}$ be the result of replacing all occurrences of $[\underset{\sim}{c}^j\underset{\sim}{A}^j]$ in $\underset{\sim}{B}$ by occurrences of $\underset{\sim}{y}^j$ for $j = k,\ldots,1$, starting with $j = k$ and working downward. It can be seen that $\mathcal{S} \cup \{\rho\underset{\sim}{E}^1,\ldots\rho\underset{\sim}{E}^k\} \vdash_{\mathcal{T}} \rho\eta\underset{\sim}{D}^i$ for $1 \leq i \leq n$ by induction on i. We leave the details to the reader; in each case consider how $\underset{\sim}{D}^i$ was inferred in the original refutation of $\mathcal{S}$. (In the case where $\underset{\sim}{D}^i$ was inferred by 5.1.8 one uses the fact that the $\underset{\sim}{A}^j$ are closed wffs.) When $i = n$ we have $\mathcal{S} \cup \{\rho\underset{\sim}{E}^1,\ldots,\rho\underset{\sim}{E}^k\} \vdash_{\mathcal{T}} \Box$.

Let $\mathcal{E}^0 = \emptyset$ and $\mathcal{E}^i = \{\rho\underset{\sim}{E}^1,\ldots,\rho\underset{\sim}{E}^i\}$ for $1 \leq i \leq k$. We prove $\mathcal{S} \cup \mathcal{E}^{k-j} \vdash_{\mathcal{T}} \Box$ for $0 \leq j \leq k$ by induction on j. This is clear for $j = 0$. For the induction step we prove $\mathcal{S} \cup \mathcal{E}^{i-1} \vdash_{\mathcal{T}} \Box$ from

(a) $\mathcal{S} \cup \mathcal{E}^{i-1} \cup \{\rho\underset{\sim}{E}^i\} \vdash_{\mathcal{T}} \Box$ (the inductive hypothesis).

$\rho\underset{\sim}{E}^i$ has the form $\underset{\sim}{B}_{o\alpha}\underset{\sim}{y}_\alpha \supset \Pi_{o(o\alpha)}\underset{\sim}{B}_{o\alpha}$, where $\underset{\sim}{y}_\alpha$ is $\underset{\sim}{y}^i$ and $\underset{\sim}{B}_{o\alpha}$ is $\rho\underset{\sim}{A}^i$. Note that $\underset{\sim}{y}_\alpha$ does not occur in $\underset{\sim}{B}_{o\alpha}$ or in any wff of $\mathcal{S} \cup \mathcal{E}^{i-1}$. Hence by the deduction theorem and propositional calculus we obtain

(b) $\mathcal{S} \cup \mathcal{E}^{i-1} \vdash_{\mathcal{T}} \sim \Pi_{o(o\alpha)}\underset{\sim}{B}_{o\alpha}$ and

(c) $\mathcal{S} \cup \mathcal{E}^{i-1} \vdash_{\mathcal{T}} \underset{\sim}{B}_{o\alpha}\underset{\sim}{y}_\alpha$, from which we obtain

(d) $\mathcal{S} \cup \mathcal{E}^{i-1} \vdash_{\mathcal{J}} \Pi_{o(o\alpha)} \underline{B}_{o\alpha}$ by 2.6.6, so

(e) $\mathcal{S} \cup \mathcal{E}^{i-1} \vdash_{\mathcal{J}} \Box$ from (b) and (d).

Thus $\mathcal{S} \vdash_{\mathcal{J}} \Box$ and 5.4 is proved.

§ 6 Remarks and Examples for $\mathfrak{R}$

6.1. When one sets out to prove in $\mathfrak{R}$ a theorem of some branch of mathematics, one of course assumes as hypotheses the postulates of that branch of mathematics. In addition certain assumptions which are used in all branches of mathematics, and which in other contexts would be regarded as axioms of the underlying logic, should be taken as hypotheses. Among these we mention the axioms of extensionality:

6.1.1^{o} $\forall p_o \forall q_o . [p_o \equiv q_o] \supset . p_o = q_o$

6.1.1$^{(\alpha\beta)}$ $\forall f_{\alpha\beta} \forall g_{\alpha\beta} . \forall x_\beta [f_{\alpha\beta} x_\beta = g_{\alpha\beta} x_\beta] \supset . f_{\alpha\beta} = g_{\alpha\beta}$

and the axiom of descriptions:

6.1.2 $\exists i_{\iota(o\iota)} \forall f_{o\iota} . \exists_1 x_\iota f_{o\iota} x_\iota \supset f_{o\iota}[i_{\iota(o\iota)} f_{o\iota}]$. In addition one may wish to assume some formulation of the axiom of choice (in which case 6.1.2 is dispensable) and an axiom of infinity.

Of course there are infinitely many axioms of extensionality, and it may not be obvious which of these may be needed to prove a particular theorem. However, when implementing the system it should be possible to treat the α and β of 6.1.1$^{(\alpha\beta)}$ as special variables (type variables, in the terminology of [1]) for which one can substitute particular type symbols as necessary.

6.2. In the examples below we shall use letters with bars over them for parameters. Thus $\bar{o}_\iota$ and $\bar{s}_{\iota\iota}$ in 6.3 are parameters. For the sake of brevity we shall introduce Herbrand-Skolem functors as abbreviations in the manner discussed in 5.2. We shall call such a functor with its arguments an existential term. Since applications of 5.1.1-5.1.7 are routine we shall usually leave it to the reader to determine which of these rules are being used. However we shall indicate (at the right-hand margin) from which line(s) a given line is inferred if it is not inferred from the line immediately preceding it. The reader will quickly discover the advantage of formulating derived rules of inference to speed up these manipulations. We here discuss only two such rules, which we shall need in 6.4.

6.2.1 If $\mathcal{S} \vdash_{\mathcal{R}} \sim [A_\alpha = A_\alpha] \vee B$ then $\mathcal{S} \vdash_{\mathcal{R}} B$.

Proof: From the given wff by 2.4.8 and 5.1.1 we obtain

.1 $\sim \forall f_{o\alpha}[\sim f_{o\alpha}A_\alpha \vee f_{o\alpha}A_\alpha] \vee B$ where $f_{o\alpha}$ is not free in A_α.

.2 $\sim[\sim F_{o\alpha}A_\alpha \vee F_{o\alpha}A_\alpha] \vee B$ where $F_{o\alpha}$ is an existential term.

.3 $F_{o\alpha}A_\alpha \vee B$ .2.

.4 $\sim F_{o\alpha}A_\alpha \vee B$ .2.

.5 B cut: .3,.4.

6.2.2 If A_α and B_α are free for x_α in C, and $\mathcal{S} \vdash_{\mathcal{R}} N \vee S^{x_\alpha}_{A_\alpha}C$, and $\mathcal{S} \vdash_{\mathcal{R}} M \vee [A_\alpha = B_\alpha]$ or $\mathcal{S} \vdash_{\mathcal{R}} M \vee [B_\alpha = A_\alpha]$, then $\mathcal{S} \vdash_{\mathcal{R}} M \vee N \vee S^{x_\alpha}_{B_\alpha}C$.

Proof for the case $[A_\alpha = B_\alpha]$:

Let $f_{o\alpha}$ be a variable not free in A_α, B_α, or M.

.1 $\underset{\sim}{M} \vee \forall \underset{\sim}{f}_{o\alpha} \centerdot \sim \underset{\sim}{f}_{o\alpha}\underset{\sim}{A}_{\alpha} \vee \underset{\sim}{f}_{o\alpha}\underset{\sim}{B}_{\alpha}$ given

.2 $\underset{\sim}{M} \vee \sim \underset{\sim}{f}_{o\alpha}\underset{\sim}{A}_{\alpha} \vee \underset{\sim}{f}_{o\alpha}\underset{\sim}{B}_{\alpha}$

.3 $\underset{\sim}{M} \vee \sim[\lambda \underset{\sim}{x}_{\alpha}\underset{\sim}{C}]\ \underset{\sim}{A}_{\alpha} \vee [\lambda \underset{\sim}{x}_{\alpha}\underset{\sim}{C}]\ \underset{\sim}{B}_{\alpha}$ Sub

.4 $\underset{\sim}{M} \vee \sim(\overset{\bullet}{S}{}^{\underset{\sim}{x}_{\alpha}}_{\underset{\sim}{A}_{\alpha}}\underset{\sim}{C}) \vee \overset{\bullet}{S}{}^{\underset{\sim}{x}_{\alpha}}_{\underset{\sim}{B}_{\alpha}}\underset{\sim}{C}$

.5 $\underset{\sim}{N} \vee \overset{\bullet}{S}{}^{\underset{\sim}{x}_{\alpha}}_{\underset{\sim}{A}_{\alpha}}\underset{\sim}{C}$ given

.6 $\underset{\sim}{M} \vee \underset{\sim}{N} \vee \overset{\bullet}{S}{}^{\underset{\sim}{x}_{\alpha}}_{\underset{\sim}{B}_{\alpha}}\underset{\sim}{C}$ cut: .4,.5

In the case $[\underset{\sim}{B}_{\alpha} = \underset{\sim}{A}_{\alpha}]$ substitute $[\lambda \underset{\sim}{x}_{\alpha} \sim \underset{\sim}{C}]$ for $\underset{\sim}{f}_{o\alpha}$ in the line corresponding to .2.

6.3 <u>Example</u>

Let $\mathbb{N}_{o\iota}$ stand for

$[\lambda n_{\iota} \centerdot \forall p_{o\iota} \centerdot [p\bar{o}_{\iota} \wedge \forall x_{\iota} \centerdot px \supset p \centerdot \bar{s}_{\iota\iota}x] \supset pn]$.

$\mathbb{N}_{o\iota}$ denotes the set of natural numbers when $\bar{o}_{\iota}$ denotes zero and $\bar{s}_{\iota\iota}$ denotes the successor function. We prove $\forall y_{\iota}[\mathbb{N}y \supset \mathbb{N}.\bar{s}y]$ by refuting its negation in $\mathfrak{R}$.

.1 $\sim \forall y_{\iota} \centerdot \mathbb{N}y \supset \mathbb{N} \centerdot \bar{s}y$ given

.2 $\mathbb{N}\bar{y}_{\iota}$ .1

.3 $\sim \mathbb{N} \centerdot \bar{s}\bar{y}_{\iota}$ .1

.4 $\sim \centerdot [\bar{p}_{o\iota}\bar{o} \wedge \forall x_{\iota} \centerdot \bar{p}x \supset \bar{p}\bar{s}x] \supset \bar{p}\bar{s}\bar{y}$

*.5 $\bar{p}_{o\iota}\bar{o}$ .4

*.6 $\sim \bar{p}_{o\iota}x_{\iota} \vee \bar{p}\bar{s}x$ .4

*.7 $\sim \bar{p}_{o\iota}\bar{s}\bar{y}_{\iota}$ .4

.8 $\sim p_{o\iota}\bar{o} \vee \sim \forall x_{\iota}[\sim px \vee p.\bar{s}x] \vee p\bar{y}_{\iota}$ .2

*.9 $\sim p_{o\iota}\bar{o} \vee \sim[\sim p[\bar{x}_{\iota(o\iota)}p] \vee p.\bar{s}.\bar{x}p] \vee p\bar{y}_{\iota}$

Lines .5-.7 and .9 were obtained routinely from .1, and □ must be derived from these. We could apply 5.1.5 to .9, but it is convenient to postpone this.

.10 $\sim \bar{p}_{o\iota}\bar{y}_{\iota}$ Sub: .6;cut: .7

.11 $\sim.\sim \bar{p}_{o\iota}[\bar{x}_{\iota(o\iota)}\bar{p}] \vee \bar{p}.\bar{s}.\bar{x}\bar{p}$ Sub: .9;cut: .5,.10

.12 $\bar{p}_{o\iota}[\bar{x}_{\iota(o\iota)}\bar{p}]$ .11

.13 $\sim \bar{p}_{o\iota}.\bar{s}.\bar{x}_{\iota(o\iota)}\bar{p}$ .11

.14 □ Sub: .6;cut: .12,.13

6.4. <u>Example</u>

For a somewhat less trivial example, we prove that if some iterate of a function f has a unique fixed point, then f has a fixed point. (This example is suggested by [6].)

Let $J_{o(\iota\iota)(\iota\iota)}$ stand for

$[\lambda f_{\iota\iota}\lambda g_{\iota\iota}\forall p_{o(\iota\iota)}.[pf \wedge \forall h_{\iota\iota}.ph \supset p.\lambda t_{\iota}.f.ht] \supset pg]$.

Then $Jf_{\iota\iota}g_{\iota\iota}$ means g is an iterate of f, i.e., g is in the intersection of all sets p which contain f such that p contains f∘h whenever p contains h.

We wish to prove

(*) $\forall f_{\iota\iota}.\exists g_{\iota\iota}[Jfg \wedge \exists_1 x_{\iota}.gx = x] \supset \exists y_{\iota}.fy = y$

.1 $\sim(*)$ given

.2 $J\bar{f}_{\iota\iota}\bar{g}_{\iota\iota}$ .1

.3 $\exists_1 x_{\iota}.\bar{g}_{\iota\iota}x = x$ .1

*.4 $\sim \bar{f}_{\iota\iota} y_{\iota} = y$ .1

*.5 $\bar{g}_{\iota\iota}\bar{x}_{\iota} = \bar{x}_{\iota}$ .3

*.6 $\sim \bar{g}_{\iota\iota} z_{\iota} = z_{\iota} \vee z_{\iota} = \bar{x}_{\iota}$ .3

*.7 $\sim p_{o(\iota\iota)}\bar{f}_{\iota\iota} \vee \sim[\sim p[\bar{h}_{\iota\iota(o(\iota\iota))}p] \vee p . \lambda t_{\iota} . \bar{f} . [\bar{h}p]\, t] \vee p\bar{g}_{\iota\iota}$ .2

We must derive □ from .4,.5,.6, and .7. We could break down .4,.5, and .6 further using the definition of equality, but we prefer to rely on 6.2.1 and 6.2.2.

Next we substitute $[\lambda k_{\iota\iota} . k[\bar{f}_{\iota\iota}\bar{x}_{\iota}] = \bar{f} . k\bar{x}]$ for $p_{o(\iota\iota)}$ in .7, and write the existential term corresponding to $\bar{h}_{\iota\iota(o(\iota\iota))}p_{o(\iota\iota)}$ simply as $\bar{h}_{\iota\iota}$ to obtain

.8 $\sim \bar{f}_{\iota\iota}\bar{f}\bar{x}_{\iota} = \bar{f}\bar{f}\bar{x} \vee \sim[\sim \bar{h}_{\iota\iota}\bar{f}\bar{x} = \bar{f}\bar{h}\bar{x} \vee \bar{f}\bar{h}\bar{f}\bar{x} = \bar{f}\bar{f}\bar{h}\bar{x}]$
$\vee\ \bar{g}_{\iota\iota}\bar{f}\bar{x} = \bar{f}\bar{g}\bar{x}$ .7

Applying 6.2.1 to .8 we obtain .9 and .10 below:

.9 $\bar{h}_{\iota\iota}\bar{f}_{\iota\iota}\bar{x}_{\iota} = \bar{f}\bar{h}\bar{x} \vee \bar{g}_{\iota\iota}\bar{f}\bar{x} = \bar{f}\bar{g}\bar{x}$ .8

.10 $\sim \bar{f}_{\iota\iota}\bar{h}_{\iota\iota}\bar{f}_{\iota\iota}\bar{x}_{\iota} = \bar{f}\bar{f}\bar{h}\bar{x} \vee \bar{g}_{\iota\iota}\bar{f}\bar{x} = \bar{f}\bar{g}\bar{x}$ .8

.11 $\sim \bar{f}_{\iota\iota}\bar{f}\bar{h}_{\iota\iota}\bar{x}_{\iota} = \bar{f}\bar{f}\bar{h}\bar{x} \vee \bar{g}_{\iota\iota}\bar{f}\bar{x} = \bar{f}\bar{g}\bar{x}$ 6.2.2: .9,.10

.12 $\bar{g}_{\iota\iota}\bar{f}_{\iota\iota}\bar{x}_{\iota} = \bar{f}\bar{g}\bar{x}$ 6.2.1: .11

.13 $\bar{g}_{\iota\iota}\bar{f}_{\iota\iota}\bar{x}_{\iota} = \bar{f}\bar{x}$ 6.2.2: .5,.12

.14 $\bar{f}_{\iota\iota}\bar{x}_{\iota} = \bar{x}_{\iota}$ Sub: .6;cut: .13

.15 □ Sub: .4;cut: .14

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