A SURVEY OF TOP CATEGORIES*

by

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Introduction

For a long time, general topology has not only been a study of topological spaces. Weil and Bourbaki introduced uniform spaces to study uniform continuity more than thirty years ago. The concept of a neighborhood space or closure space or "mehrstufige Topologie" is nearly fifty years old, due to Hausdorff and used largely by the Čech school. The recognition that not all structures with limits of filters are topologies led to the limit spaces of Kowalsky [21] and Fischer [9]. Feedback from other mathematical theories, and sometimes simply the urge for variations on a theme, led to yet other categories such as Hammer's extended topologies, uniform convergence spaces [5], and various quasi-uniform spaces.

Limits, continuous functions, open and closed sets, and other paraphernalia of topology were soon defined for most or all of these categories, usually by analogy to existing concepts. This search for analoga, and the companion phenomenon of carrying over proofs from one theory to another, made a general theory more and more desirable. Efforts to establish a descriptive theory of structures of topological nature were made, but they seemed overly complicated and not general enough, and thus they were largely abandoned in favor of a categorical approach.

Several categorical solutions for the problem of finding a theory of topological theories have been offered. Hušek's S-spaces [15], Katětov's M-spaces [17], Bentley's T-spaces [3], and Kennison's pullback stripping functors [18] are examples. Thus our top categories join an already large company. Not surprisingly, one finds that **any two of** these approaches are either categorically equivalent, or that one is a special case of the other. W. Shukla's thesis [29] contains a detailed study of these connections. It turns out that S-categories are equivalent, and pullback stripping functors almost equivalent, to top categories. M-spaces are less general, and T-spaces are a rather special case of top categories.

Our main reason for preferring top categories over S-spaces or other equivalent theories is the fact that top categories have a simple invariant definition which puts general topology into a larger categorical framework: a top category is "simply" a fibred category in the sense of Grothendieck [12] or Gray [11] with small complete fibres.

Top categories go back to [30]. A revised version [31], with some topics omitted and others amplified, has been submitted for publication. Meanwhile, some questions left open in [31] have been answered, new applications have been found, and experience showed that some special cases of general results are useful enough to deserve a more detailed treatment. Thus the author decided to write this report as a detailed and reasonably complete introduction to top categories, on a pre-publication level. The present report contains almost all of [31] and of [30], results from [25], [26], [8], [33], some results of Shukla's thesis, and some new results.

Section 1 discusses fibred categories in general. We give a simpler definition than that of Grothendieck and Gray. The same material occurs in [8], in somewhat more general and more condensed form, except that a converse of [8; 7.8] has been added.

Section 2 provides basic definitions and some general results.

The list of examples in section 3 could have been enlarged almost indefinitely. Shukla [29] has some very interesting examples from automata theory and information theory.

Sections 4 and 5, on the lifting of functors and of universal morphisms, are mostly from [31]. They contain results which are very useful in general topology. Section 9, also from [31], puts topological algebra into the framework of top categories. Thus new useful applications of the basic results of sections 4 and 5 can be made.

Sections 6 and 7 contain material from [25], [26], [30] in amplified and sometimes modified form. These two sections present specialized, and very useful, versions of basic results from sections [4] and [5].

The point separators of section 8 do not deal with top categories directly, but they certainly form a part of categorical topology, coming as they do from the point separation axioms T_i $(i = 0, 1, 2)$ and related topological axioms. The theory was originated in rather special form in [28], generalized to top categories over ENS in [30], and put in definitive form in this report.

Section 10 stems from an interesting theorem of Kennison [18] which was put into a general categorical framework by Herrlich [14; § 13].

Section 11, on images and relations, contains some basic definitions and results of [8], and a condensed preview of parts of [33]. No proofs have been given; this section has been inserted mainly for the convenience of the reader.

The continuous relations of section 12 are a new development of the general theory. More than any other, this section must be considered as preliminary. We do not anticipate a switch from the given definition of continuous relations to one which is not substantially equivalent, but it seems definitely indicated to find a common framework for relations and continuous relations. Other questions are posed in 12.7, and there are many more.

The study of the categories of general topology is one aspect of a new mathematical discipline for which the author has proposed the name categorical topology. Top categories seem to be a good tool for this. Categorical topology has many aspects which are not discussed at all in this report. We mention only the study of reflective, coreflective, and otherwise interesting subcategories of TOP using categorical methods (see e.g. [14]), the study of completions and compactifications (see e.g. [32]), and the study of autonomous top categories, in the sense of Linton [23]. Binz and Keller [4], and Cook and Fischer [5], have shown that limit spaces form such a category.

Due to the pre-publication level of this report, no references have been provided in the text, except occasionaly to an author by name. These elliptic references can easily be amplified from the appended bibliography. Some results and definitions appear only after they have been used, but eliminating such minor defects of the report would have delayed its appearance unduly. We hope that the present report will be useful to the reader despite these and other shortcomings.

Contents

A SURVEY OF TOP CATEGORIES

Oswald Wyler

1. FIBRATIONS. Top categories are special fibred categories in the sense of Grothendieck and Gray. We begin this survey therefore with a brief discussion of fibred categories in general.

1.1. DEFINITION. Let $P : \mathbb{A} \to \mathbb{C}$ be a functor. We call a morphism $f_1 : a \to b$ of $\mathbb{A}$ P-<u>fibred</u> if for every morphism $v : C \to P\,a$ of $\mathbb{C}$ and every morphism $u_1 : c \to b$ of $\mathbb{A}$ such that $P\,u_1 = (P\,f_1)\,v$ there is exactly one morphism $v_1 : c \to a$ of $\mathbb{A}$ such that $u_1 = f_1\,v_1$ and $P\,v_1 = v$. We call P a <u>fibration</u> if for every morphism $f : A \to B$ of $\mathbb{C}$ and every object b of $\mathbb{A}$ such that $P\,b = B$ there is a P-fibred morphism f_1 of $\mathbb{A}$ with codomain b and $P\,f_1 = f$. Dually, we say that $f_1 \in \mathbb{A}$ is P-<u>opfibred</u> if f_1 is fibred for the induced functor $P^{op} : \mathbb{A}^{op} \to \mathbb{C}^{op}$, and we say that P is an <u>opfibration</u> if P^{op} is a fibration.

1.2. EXAMPLE. The category $\mathbb{C}^2$ has morphisms of $\mathbb{C}$ as its objects, and its morphisms are quadruples $(f_o, f_1) : u \to v$ of morphisms of $\mathbb{C}$ such that $v\,f_o = u\,f_1$ in $\mathbb{C}$. The codomain functor $D_1 : \mathbb{C}^2 \to \mathbb{C}$ is defined by putting $D_1\,(f_o, f_1) = f_1$ for $(f_o, f_1) : u \to v$ in $\mathbb{C}^2$. A fibred morphism for this functor is a pullback in $\mathbb{C}$. Dually, an opfibred morphism for the domain functor $D_o : \mathbb{C}^2 \to \mathbb{C}$ is a pushout in $\mathbb{C}$. D_1 is a fibration if and only if $\mathbb{C}$

has pullbacks, and D_0 is an opfibration if and only if $\mathbb{C}$ has pushouts.

1.3. DEFINITION. Let $P : \mathbb{A} \to \mathbb{C}$ as above. For each object A of $\mathbb{C}$, the morphisms $f_1 \in \mathbb{A}$ such that $P f_1 = \mathrm{id}\, A$ form a subcategory of $\mathbb{A}$. Let **$P^* A$** denote this subcategory and $H_A : P^* A \to \mathbb{A}$ the inclusion functor. For $f : A \to B$ in $\mathbb{C}$, we call P-<u>cleavage</u> at f a pair (f^*, φ_f) consisting of a functor $f^* : P^* B \to P^* A$ and a natural transformation $\varphi_f : H_A f^* \to H_B$ such that $\varphi_f\, b$ is P-fibred and $P(\varphi_f\, b) = f$ for every object b of $P^* B$. Dually, a P-<u>opcleavage</u> (f_*, φ_f) at f consists of a functor $f_* : P^* A \to P^* B$ and a natural transformation $\varphi_f : H_A \to H_B f^*$ such that $\varphi_f\, a$ is P-opfibred and $P(\varphi_f\, a) = f$ for every object a of $P^* A$.

1.4. PROPOSITION. $P : \mathbb{A} \to \mathbb{C}$ <u>is a fibration if and only if there is a</u> P-<u>cleavage</u> (f^*, φ_f) <u>at every morphism</u> f <u>of</u> $\mathbb{C}$.

<u>Proof</u>. The existence of a P-cleavage at every $f \in \mathbb{C}$ obviously guarantees the existence of enough P-fibred morphisms. Conversely, let P be a fibration and $f : A \to B$ in $\mathbb{C}$. Assign to every $b \in \mathrm{Ob}\, \mathbb{A}$ such that $P b = B$ a P-fibred morphism $\varphi_f\, b : f^* b \to b$ such that $P(\varphi_f\, b) = f$. If $u : b \to b'$ in $P^* B$, then $u(\varphi_f\, b) = (\varphi_f\, b')\, v$ and $P v = \mathrm{id}\, B$ for a unique morphism $v : f^* b \to f^* b'$ in $\mathbb{A}$. We put $f^* u = v$. One verifies easily, using the unicity of v, that this defines a functor $f^* : P^* B \to P^* A$, and then the morphisms $\varphi_f\, b$ define a natural transformation $\varphi_f : H_A f^* \to H_B$ ▯

1.5. PROPOSITION. <u>Let a</u> P-<u>cleavage</u> (f^*, φ_f) <u>at every</u> $f \in \mathbb{C}$ <u>be given, for</u> $P : \mathbb{A} \to \mathbb{C}$. <u>If</u> $A \in \mathrm{Ob}\, \mathbb{C}$ <u>and</u> $A \xrightarrow{f} B \xrightarrow{g} C$ <u>in</u> $\mathbb{C}$, <u>then</u>

$$\varphi_{\mathrm{id}\, A}(H_A\, c_A) = \mathrm{id}\, H_A \quad \underline{\text{and}} \quad \varphi_{gf}(H_A\, c_{f,g}) = \varphi_g(\varphi_f\, g^*)$$

<u>for unique natural equivalences</u> $c_A : \mathrm{Id}\ (P^*\ A) \longrightarrow (\mathrm{id}\ A)^*$ <u>and</u> $c_{f,g} : f^*\ g^* \longrightarrow (g\ f)^*$.

<u>Proof</u>. For $a \in \mathrm{Ob}\ P^*\ A$ there is a unique $c_A\ a$ in $P^*\ A$ such that

$$(\varphi_{\mathrm{id}\ A}\ a)(c_A\ a) = \mathrm{id}\ a \quad .$$

We have $(\varphi_{\mathrm{id}\ A}\ a)(c_A\ a)(\varphi_{\mathrm{id}\ A}\ a) = (\varphi_{\mathrm{id}\ A}\ a)(\mathrm{id}\ ((\mathrm{id}\ A)^*\ a))$ in $P^*\ A$, and thus $(c_A\ a)(\varphi_{\mathrm{id}\ A}\ a) = \mathrm{id}\ ((\mathrm{id}\ A)^*\ a)$, and $c_A\ a$ is an isomorphism of $P^*\ A$. Using the unicity of $c_A\ a$, we see that the morphisms $c_A\ a : a \longrightarrow (\mathrm{id}\ A)^*\ a$ define a natural equivalence $c_A : \mathrm{Id}\ P\ A^* \longrightarrow (\mathrm{id}\ A)^*$.

For $c \in \mathrm{Ob}\ P^*\ C$, there is a unique $c_{f,g}\ c$ in $P^*\ A$ such that

$$(\varphi_{gf}\ c)(c_{f,g}\ c) = (\varphi_g\ c)(\varphi_f\ g^*\ c)$$

in $\mathbb{A}$. Using the unicity of $c_{f,g}\ c : f^*\ g^*\ c \longrightarrow (g\ f)^*\ c$, we see that this determines a natural transformation $c_{f,g} : f^*\ g^* \longrightarrow (g\ f)^*$.

We have to show that $c_{f,g}\ c$ is an isomorphism. There is a unique morphism $v : (g\ f)^*\ c \longrightarrow g^*\ c$ in $\mathbb{A}$ with $P\ v = f$ and $(\varphi_g\ c)\ v = \varphi_{gf}\ c$, and there is a unique morphism $w : (g\ f)^*\ c \longrightarrow f^*\ g^*\ c$ such that $P\ w = \mathrm{id}\ A$ and $(\varphi_f\ g^*\ c)\ w = v$. With $c_{f,g}$ for $c_{f,g}\ c$, it follows that

$$(\varphi_{gf}\ c)\ c_{f,g}\ w = (\varphi_g\ c)(\varphi_f\ g^*\ c)\ w = (\varphi_g\ c)\ v = \varphi_{gf}\ c \quad ,$$

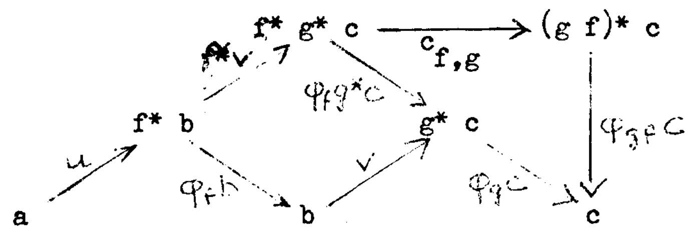
and $$(\varphi_g\ c)(\varphi_f\ g^*\ c)\ w\ c_{f,g} = (\varphi_{gf}\ c)\ c_{f,g} = (\varphi_g\ c)(\varphi_f\ g^*\ c) \quad .$$

Thus $$c_{f,g}\ w = \mathrm{id}\ (g\ f)^*\ c \quad \text{and} \quad w\ c_{f,g} = \mathrm{id}\ f^*\ g^*\ c \quad ,$$

and $c_{f,g}\ c$ is indeed an isomorphism $\square$

The natural equivalences c_A and $c_{f,g}$ satisfy coherence conditions, but we are not interested in these: we always have $(g\ f)^* = f^*\ g^*$ and $(\mathrm{id}\ A)^* = \mathrm{Id}\ (P^*\ A)$ for a top category.

1.6. A fibration $P : \mathbb{A} \to \mathbb{C}$ determines fibres $P^* A$ and embedding functors $H_A : P^* A \longrightarrow \mathbb{A}$ for objects A of $\mathbb{C}$, cleavages (f^*, φ_f) for morphisms f of $\mathbb{C}$, and coherence equivalences c_A and $c_{f,g}$. These data can be used to describe or reconstruct $\mathbb{A}$ as follows. An object A_1 of $\mathbb{A}$ is given by a pair (A,a) with $A = P\,A_1$, $a \in \mathrm{Ob}\, P^* A$, and $A_1 = H_A\, a$. A morphism $f_1 : (A,a) \longrightarrow (B,b)$ is represented by a pair (f,u), with $f = P\,f_1 : A \longrightarrow B$ in $\mathbb{C}$, $u : a \longrightarrow f^* b$ in $P^* A$, and $f_1 = (\varphi_f\, b)(H_A\, u)$ in $\mathbb{A}$. Composition of $(f,u) : (A,a) \longrightarrow (B,b)$ and $(g,v) : (B,b) \longrightarrow (C,c)$ is given as follows.



Thus $(g,v)\,(f,u) = (g\,f,\, w)$ with $w = c_{f,g}\,(f^*\, v)\, u$ in $P^* A$.

We note that (f,u) does not describe f_1 quite **fully: b is not given.**

1.7. THEOREM. <u>If</u> $P : \mathbb{A} \to \mathbb{C}$ <u>is a fibration with cleavages</u> (f^*, φ_f), <u>then the following are equivalent for</u> $f : A \longrightarrow B$ in $\mathbb{C}$.

1.7.1. (f_*, ψ_f) <u>is an opcleavage for</u> P <u>at</u> f.

1.7.2. $f_* : P^* A \longrightarrow P^* B$ <u>is left adjoint to</u> f^*, <u>and</u> $\psi_f = (\varphi_f\, f_*)(H_A\, \eta)$ <u>for a front adjunction</u> $\eta : \mathrm{Id}\, P^* A \longrightarrow f^* f_*$ <u>for</u> f^*.

<u>Proof</u>. If (f_*, ψ_f) is an opcleavage at f, consider the equation

$$(\varphi_f\, b)\, u = v\, (\psi_f\, a)$$

for objects a of $P^* A$ and b of $P^* b$. If $u : a \longrightarrow f^* b$ in $P^* A$ is given, this determines $v : f_*\, a \longrightarrow b$ in $P^* B$ uniquely since (f_*, ψ_f) is an opcleavage. Dually, v determines u uniquely. Thus we have bijections

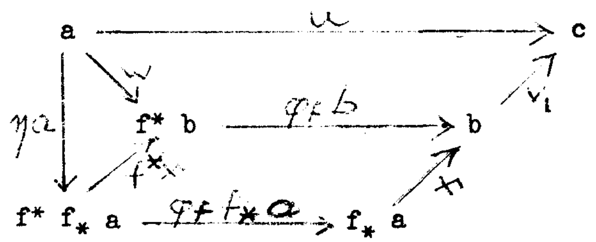
$$\vartheta_{a,b} \; : \; (P^* B)(f_* a, b) \longrightarrow (P^* A)(a, f^* b) \quad ,$$

one for each pair (a,b) of objects. One verifies easily that $\vartheta_{a,b}$ is natural in a and in b. We obtain a front adjunction η for this adjoint situation by putting $\eta\, a = u$ for $b = f_* a$ and $v = \mathrm{id}\, b$. For this front adjunction,

$$\psi_f\, a \;=\; (\varphi_f\, f_*\, a)(\eta\, a) \quad ,$$

for $a \in \mathrm{Ob}\, P^* A$, and thus $\psi_f = (\varphi_f\, f_*)(H_A\, \eta)$.

Conversely, we have to **show** that $\psi_f\, a$ above is P-opfibred if the adjoint situation is given. Thus let $u : a \longrightarrow c$ with $P\, u = v\, f$. If $v_1 : b \longrightarrow c$ is P-fibred and $P\, v_1 = v$, then $u = v_1\, f_1$ for a unique $f_1 : a \longrightarrow b$ such that $P\, f_1 = f$, and $f_1 = (\varphi_f\, b)\, w$ for a unique $w : a \longrightarrow f^* b$ in $P^* A$. Now $w = (f^* x)(\eta\, a)$ for a unique $x : f_* a \longrightarrow b$ in $P^* B$. Since φ_f is natural, the remaining square in the diagram below commutes.



Thus $u = v_1'\, (\psi_f\, a)$ for $v_1' = v_1\, x$ with $P\, v_1' = v$. If also $u = v_1''\, (\psi_f\, a)$ with $P\, v_1'' = v$, then $v_1'' = v_1\, x'$ with $x' : f_* a \longrightarrow b$ in $P^* B$. This sets up the diagram above with $w = (f^* x')(\eta\, a)$ and $f_1 = (\varphi_f\, b)\, w$. Thus $x' = x$, and $v_1'' = v_1$ follows $\square$

2. TOP CATEGORIES. Much of what follows remains valid for general fibrations, but we specialize to a simpler and important case.

2.1. DEFINITION. A <u>top category</u> over a category $\mathbb{C}$ is a pair $(\mathbb{A}, P)$ consisting of a category $\mathbb{A}$ and a functor $P : \mathbb{A} \to \mathbb{C}$, with the following two properties.

2.1.1. P is both a fibration and an opfibration.

2.1.2. Every fibre $P^* A$, for $A \in \mathrm{Ob}\,\mathbb{A}$, is a complete ordered set.

We call P the <u>projection functor</u> of the top category $(\mathbb{A}, P)$, and by abuse of language, we often identify the top category with $\mathbb{A}$.

2.2. **PROPOSITION.** <u>If</u> $(\mathbb{A}, P)$ <u>is a top category over</u> $\mathbb{C}$, <u>then the dual pair</u> $(\mathbb{A}^{op}, P^{op})$ <u>is a top category over</u> $\mathbb{C}^{op}$.

<u>Proof</u>. Definition 2.1 is self-dual $\square$

We shall use this self-duality of our theory freely in what follows.

2.3. DISCUSSION. We have noted in 1.6 that a fibration $P : \mathbb{A} \to \mathbb{C}$ determines, and is in turn determined by, the fibres $P^* A$ for $A \in \mathrm{Ob}\,\mathbb{C}$, the cleavages (f^*, φ_f) for $f \in \mathbb{C}$, and the coherence maps c_A and $c_{f,g}$. For ordered sets as fibres, all equivalences are identities. Thus $(\mathrm{id}\, A)^* = \mathrm{id}\, P^* A$ and $(g\, f)^* = f^*\, g^*$, if $g\, f$ is defined. In other words, the fibres and cleavages determine a contravariant functor $P^* : \mathbb{C}^{op} \to \mathrm{ORD}$, from $\mathbb{C}$ to the category ORD of ordered sets. In this situation, $u : a \to f^*\, b$ in $P^* A$ means that $a \leq f^*\, b$. Thus a morphism $(f, u) : (A, a) \to (B, b)$ of $\mathbb{A}$ is a morphism $f : A \to B$ of $\mathbb{C}$ with $a \leq f^*\, b$, and we can suppress u from the notation. With this convention, composition in $\mathbb{A}$ becomes simply composition of "underlying" morphisms in $\mathbb{C}$. The cleavage $\varphi_f\, b$ becomes a morphism $f : (A, f^*\, b) \to (B, b)$ which we call <u>coarse</u>. Clearly every contravariant functor $P^* : \mathbb{C}^{op} \to \mathrm{ORD}$ sets up a fibration $P : \mathbb{A} \to \mathbb{C}$ in this way.

A complete ordered set is one which is complete as a category, or in other words a complete lattice. Categorical limits are infima, and categorical colimits are suprema. By 1.7, a fibration P is an opfibration if and only if all functors f^* have left adjoints. Thus we are led to the following theorem.

2.3. **THEOREM.** A top category $(\mathbb{A}, P)$ over a category $\mathbb{C}$ is determined by a contravariant functor $P^* : \mathbb{C}^{op} \longrightarrow \mathrm{ORD}$, from $\mathbb{C}$ to the category of ordered sets, with the following properties.

2.3.1. Every fibre $P^* A$, $A \in \mathrm{Ob}\,\mathbb{C}$, is a complete lattice.

2.3.2. Every map $P^* f = f^*$, for $f \in \mathbb{C}$, preserves infima $\square$

2.4. If $f : A \longrightarrow B$ and $g : B \longrightarrow A$ are functors of ordered sets, i.e. order preserving maps, then g is left adjoint to f if and only if

$$g\,b \leq a \iff b \leq f\,a \quad ,$$

i.e. if and only if f and g form a (covariant) Galois correspondence. This is of course well known. If A is complete and $f : A \longrightarrow B$ is given, then g with this property exists if and only if f preserves infima. This is equally well known, and at the basis of 2.3.2.

2.5. NOTATIONS. Let $P : \mathbb{A} \longrightarrow \mathbb{C}$ define a top category $(\mathbb{A}, P)$ over $\mathbb{C}$. We put $p\,A$ for $P^* A$ and f^p for f^* if $A \in \mathrm{Ob}\,\mathbb{C}$ and $f \in \mathbb{C}$. Thus $f^p : p\,B \longrightarrow p\,A$ for $f : A \longrightarrow B$ in $\mathbb{C}$. We identify objects of $\mathbb{A}$ with pairs (A,a) with $A \in \mathrm{Ob}\,\mathbb{C}$ and $a \in p\,A$, and morphisms $f : (A,a) \longrightarrow (B,b)$ of $\mathbb{A}$ with morphisms $f : A \longrightarrow B$ of $\mathbb{C}$ such that $a \leq f^p\,b$. By 2.3.2 and 2.4,

$$f_p\,a \leq b \iff a \leq f^p\,b \quad ,$$

for all $a \in p\,A$ and $b \in p\,B$, defines $f_p : p\,A \longrightarrow p\,B$ if $f : A \longrightarrow B$.

We have noted that $f : (A, f^p b) \longrightarrow (B, b)$ is P-fibred (or <u>coarse</u>); by 1.7 the morphism $f : (A, a) \longrightarrow (B, f_p a)$ is P-opfibred (or <u>fine</u>). **The following** result expresses this situation.

2.6. PROPOSITION. *Let $(\mathbb{A}, P)$ be a top category over $\mathbb{C}$. If $f : A \longrightarrow C$ and $g : C \longrightarrow B$ in $\mathbb{C}$, and if $a \in p\,A$ and $b \in p\,B$, then the following three statements are logically equivalent.*

2.6.1. $g\,f : (A,a) \longrightarrow (B,b)$ *in* $\mathbb{A}$.

2.6.2. $f : (A,a) \longrightarrow (C, g^p b)$ *in* $\mathbb{A}$.

2.6.3. $g : (C, f_p a) \longrightarrow (B,b)$ *in* $\mathbb{A}$.

Proof. $a \leq (g\,f)^p b \iff a \leq f^p g^p b \iff f_p a \leq g^p b$ ▯

We note that $P_* A = p\,A$ and $P_* f = f_p$ defines a covariant functor $P_* : \mathbb{C} \longrightarrow \mathrm{ORD}$. This follows easily from the dual of 1.5 and the fact, noted in 2.3, that coherence maps are identity maps if fibres are ordered sets.

2.7. DEFINITION. Let $(\mathbb{A}, P)$ be a top category over $\mathbb{C}$. We denote by α_A and ω_A, or by α_A^p and ω_A^p if the situation requires it, the least and the greatest element of $p\,A$, for $A \in \mathrm{Ob}\,\mathbb{C}$. Elements of $p\,A$ are usually known as <u>structures</u> of A, and α_A and ω_A then are called the <u>discrete</u> and the <u>indiscrete</u> structure on A. We say that a structure a is <u>finer</u> than a structure a' if $a \leq a'$, or equivalently if $\mathrm{id}\,A : (A,a) \longrightarrow (A,a')$ in $\mathbb{A}$.

2.8. THEOREM. *If $(\mathbb{A}, P)$ is a top category over $\mathbb{C}$, then the functor P is faithful and has a left adjoint right inverse α_p, obtained by putting $\alpha_p A = (A, \alpha_A)$ for $A \in \mathrm{Ob}\,\mathbb{C}$, and $\alpha_p f = f : \alpha_p A \longrightarrow \alpha_p B$ for $f : A \longrightarrow B$ in $\mathbb{C}$. Dually, P has a right adjoint right inverse ω_p, obtained similarly.*

Proof. Since in particular $\alpha_A \leq f^p \alpha_B$ for $f : A \longrightarrow B$ in $\mathcal{C}$, α_p is a functor. We have $f : \alpha_p A \longrightarrow (B,b)$ in $\mathcal{A}$ if and only if $f : A \longrightarrow B$ in $\mathcal{C}$ since $\alpha_A \leq f^p\, b$ in any case. The bijection

$$(\alpha_p A, (B,b)) \xrightarrow{\approx} (A, P(B,b))$$

thus obtained clearly is natural in A and in (B,b). This proves the **first** part, and the second part **is strictly dual** to this $\square$

3. EXAMPLES OF TOP CATEGORIES. We give examples from general topology, from topological algebra, and purely set-theoretic examples, and one theorem.

3.1. TOPOLOGICAL SPACES. Let $t\,E$ be the complete lattice of all topologies on a set E, with $\tau \leq \tau'$ if τ is finer, i.e. has more open sets. For $f : E \longrightarrow F$ and a topology σ of F, let $f^t(\sigma)$ be the topology of E with all sets $f^{-1}(V)$, V open for σ, as open sets. This satisfies 2.3.2, and the resulting top category over ENS clearly is the category TOP of topological spaces.

3.2. A <u>convergence structure</u> on a set E is a relation q from proper filters on E to E which satisfies the two Fréchet axioms in filter form.

L 1. $\dot{x}\, q\, x$ for $x \in E$ and the filter $\dot{x}$ on E with basis $\{\{x\}\}$.

L 2. If $\varphi\, q\, x$ and ψ is finer than φ, then $\psi\, q\, x$.

We put $q \leq q'$, for convergence structures q and q' on E, if always $\varphi\, q\, x \Longrightarrow \varphi\, q'\, x$. With this order relation, convergence structures on E form a complete lattice $Q\,E$. For a mapping $f : E \longrightarrow F$ and a filter φ on E, define a filter $f_*(\varphi)$ on F by $Y \in f_*(\varphi) \Longleftrightarrow f^{-1}(Y) \in \varphi$, for $Y \subset F$. Then $\varphi\, q^*\, x \Longleftrightarrow f_*(\varphi)\, q\, f(x)$ defines a convergence structure

$q^* = f^Q(q)$ on E for a convergence structure q on F . The maps f^Q preserve infima, and a top category CONV over ENS , the <u>category of convergence spaces</u>, results.

There are many similar examples, such as uniform spaces, limit spaces, closure spaces, proximity spaces, uniform convergence spaces, and others.

3.3. TOPOLOGICAL GROUPS. Let $\mathcal{C}$ = GRP , the category of groups. For a group G , let tg G be the set of all topologies of G which are compatible with the group structure of G . Define $\leq$ and f^{tg} as $\leq$ and f^t in 3.1. The resulting top category is the category of topological groups and continuous group transformations.

This is a theme with many variations.

3.4. For any category $\mathcal{C}$, the identity functor Id $\mathcal{C}$ is both a fibration and an opfibration. The fibres are singletons and thus trivially complete lattices. Thus every category $\mathcal{C}$ is a top category over itself.

3.5. If we include the null filter or improper filter with basis $\{\emptyset\}$, then filters on a set E form a complete lattice. The proper order relation for this lattice turns out to be the one opposite to set inclusion. For a mapping $f : E \to F$, we have defined a mapping f_* from filters on E to filters on F in 3.2. We obtain f^* in the opposite direction by letting $f^*(\psi)$ be the filter on E generated by all sets $f^{-1}(Y)$ with $Y \in \psi$. If f is not surjective, $f^*(\psi)$ may well be the null filter on E for a proper filter ψ . One sees easily that $f_*(\varphi) \leq \psi \iff \varphi \leq f^*(\psi)$, where $\leq$ means finer; see above. Thus f^* preserves infima, and a top category over ENS results, with pairs (E,φ) , where φ is a filter on E , as objects.

3.6. The graphs of equivalence relations on a set E, ordered by set inclusion, form a complete lattice $e\,E$, with set intersections as infima. If $f : E \longrightarrow F$ and $v \subset F \times F$ is the graph of an equivalence relation, then $f^e(v) = (f \times f)^{-1}(v)$ defines the graph of an equivalence relation on E. This preserves infima, and a top category on ENS, the <u>category of equivalence relations</u>, is obtained.

A <u>category of reflective relations</u> is defined in the same way.

3.7. Subsets of a set E form a complete lattice, with set inclusion as order relation. If $f : E \longrightarrow F$, then f^{-1} maps subsets of F into subsets of E, preserving intersections. Thus a top category on ENS is obtained. Objects are pairs (A,X) of sets with $X \subset A$, and maps $f : (A,X) \longrightarrow (B,Y)$ are mappings $f : A \longrightarrow B$ with $X \subset f^{-1}(Y)$, i.e. with $f(X) \subset Y$. Thus this category is the category of pairs of sets.

The same construction works if we replace sets and subsets by groups and subgroups, or by topological spaces and subspaces. Thus pairs of groups and pairs of topological spaces form top categories over GRP and TOP respectively.

3.8. THEOREM. <u>If</u> $(\mathbb{A}, P)$ <u>is a top category over</u> $\mathbb{C}$ <u>and</u> K <u>a small category, then</u> $(\mathbb{A}^K, P^K)$ <u>is a top category over the functor category</u> $\mathbb{C}^K$.

<u>Proof</u>. Let $I = \mathrm{Ob}\,K$. If $F : K \longrightarrow \mathbb{C}$ is a functor, then we define an F-family as a family $(a_i)_{i \in I}$ with $a_i \in p\,(F\,i)$ for each $i \in I$, and with $a_i \leq (F\varphi)^p(a_j)$ for every $\varphi : i \longrightarrow j$ in K. F-families form a set $p\,F$. If $(a_i)_{i \in I}$ is an F-family, then $\Phi i = (F\,i, a_i)$ for $i \in I$, and $\Phi\varphi = F\varphi : \Phi i \longrightarrow \Phi j$ for $\varphi : i \longrightarrow j$ in K, clearly defines a functor $\Phi :$ $K \longrightarrow \mathbb{A}$ such that $P^K\Phi = P\Phi = F$. Every functor $\Phi : K \longrightarrow \mathbb{A}$ with $P\Phi = F$

is obtained in this way.

We order F-families $a = (a_i)_{i\in I}$ by putting $a \leq a'$ if $a_i \leq a'_i$ for all $i \in I$. For F-families $a_\mu = (a_{\mu i})_{i\in I}$, we claim that $a_i = \bigcap_\mu a_{\mu i}$ for $i \in I$ defines an F-family $a = \bigcap a_\mu$. If $\varphi : i \longrightarrow j$ in K, then indeed

$$\bigcap a_{\mu i} \leq \bigcap (F\varphi)^p(a_{\mu j}) = (F\varphi)^p(\bigcap a_{\mu j}) ,$$

as required. Thus F-families form a complete lattice p F.

For $\lambda : F \longrightarrow G$ in $\mathcal{C}^K$ and a G-family $b = (b_i)_{i\in I}$, we claim that $a_i = (\lambda i)^p(b_i)$ for $i \in I$ defines an F-family $a = \lambda^p(b)$. Indeed,

$$(\lambda i)^p(b_i) \leq (\lambda i)^p(G\varphi)^p(b_j) = (F\varphi)^p(\lambda j)^p(b_j)$$

for $\varphi : i \longrightarrow j$ in K. Thus $\lambda^p(b)$ is an F-family. Now one sees easily that $\lambda^p : p\,G \longrightarrow p\,F$ preserves infima.

Let Φ and Ψ be functors from K to $\mathbb{A}$, with $\Phi i = (F\,i, a_i)$ and $\Psi i = (G\,i, b_i)$ for $i \in I$. If $\Lambda : \Phi \to \Psi$ in $\mathbb{A}^K$ and $P^K\Lambda = P\Lambda = \lambda :$ $F \longrightarrow G$ in $\mathcal{C}^K$, then $\Lambda i = \lambda i : (F\,i, a_i) \longrightarrow (G\,i, b_i)$ in $\mathbb{A}$ for $i \in I$. This requires $a_i \leq (\lambda i)^p(b_i)$ for all $i \in I$, or $a \leq \lambda^p(b)$. Conversely, if $a \leq \lambda^p(b)$, then the morphisms given above exist, and one sees easily that they define $\Lambda : \Phi \to \Psi$ in $\mathbb{A}^K$ with $P^K\Lambda = \lambda$ ▯

4. LIFTING FUNCTORS AND NATURAL TRANSFORMATIONS. We consider in this section top categories $(\mathbb{A}^p, P)$ over $\mathbb{A}$ and $(\mathbb{B}^q, Q)$ over $\mathbb{B}$, and we use the notations of 2.5 in the obvious way for both categories.

4.1. DEFINITION. We say that a functor $\Phi : \mathbb{A}^p \to \mathbb{B}^q$ <u>lifts</u> a functor $F : \mathbb{A} \to \mathbb{B}$ if $Q\Phi = F\,P$. If Φ and Ψ lift F and G, then we say that a natural transformation $\Lambda : \Phi \to \Psi$ <u>lifts</u> a natural transformation λ :

$F \longrightarrow G$ if $Q\Lambda = \lambda P$.

If Φ lifts F, then $F = Q\,\Phi\alpha_p = Q\,\Phi\omega_p$, and thus Φ determines F. The case $F = \mathrm{Id}\,\mathbb{A}$ is important in general topology (where $\mathbb{A} = \mathrm{ENS}$) and elsewhere; we shall consider this case separately.

4.2. PROPOSITION. Let $F : \mathbb{A} \to \mathbb{B}$ be a functor. If maps $\varphi_A : p\,A \longrightarrow q\,F\,A$ in ORD are given, one for every object A of $\mathbb{A}$, and if

$$(4.2.1) \qquad \varphi_A\, f^p\, y \leqslant (F\,f)^q \varphi_B\, y$$

in $q\,F\,A$ whenever $f : A \longrightarrow B$ in $\mathbb{A}$ and $y \in p\,B$, then

$$(4.2.2) \qquad \Phi(A,x) = (F\,A, \varphi_A\, x)\ , \quad \Phi f = F\,f : \Phi(A,x) \longrightarrow \Phi(B,y)$$

for objects (A,x) and morphisms $f : (A,x) \longrightarrow (B,y)$ of $\mathbb{A}^p$, defines a functor $\Phi : \mathbb{A}^p \longrightarrow \mathbb{B}^q$ which lifts F. Every functor $\Phi : \mathbb{A}^p \longrightarrow \mathbb{B}^q$ which lifts F is obtained in this way.

We call the maps φ_A the structure maps of the functor Φ.

Proof. If the maps φ_A in ORD are given and satisfy (4.2.1), then also

$$(4.2.3) \qquad x \leqslant f^p\, y \implies \varphi_A\, x \leqslant (F\,f)^q \varphi_B\, y\ ,$$

for $f : A \longrightarrow B$ in $\mathbb{A}$, $x \in p\,A$, $y \in p\,B$, and thus (4.2.2) defines a functor $\Phi : \mathbb{A}^p \longrightarrow \mathbb{B}^q$ which lifts F. Conversely, a functor Φ which lifts F obviously is of the form (4.2.2), with $\varphi_A\, x \in q\,F\,A$ for $x \in p\,A$, and with (4.2.3) satisfied. For $f = \mathrm{id}\,A$, (4.2.3) shows that φ_A preserves order, so that $\varphi_A : p\,A \longrightarrow q\,F\,A$ in ORD, and (4.2.1) is (4.2.3) for $x = f^p\, y$ ▯

4.3. PROPOSITION. $\mathrm{Id}\,\mathbb{A}^p$ lifts $\mathrm{Id}\,\mathbb{A}$, with structure morphisms $\mathrm{id}\,p\,A$. If $\Phi : \mathbb{A}^p \longrightarrow \mathbb{B}^q$ and $\Psi : \mathbb{B}^q \longrightarrow \mathbb{C}^r$ lift $F : \mathbb{A} \to \mathbb{B}$ and $G : \mathbb{B} \to \mathbb{C}$,

<u>with structure morphisms</u> φ_A <u>and</u> ψ_B <u>respectively, then</u> $\Psi\Phi$ <u>lifts</u> $G\,F$, <u>with structure morphisms</u> $\psi_{FA}\varphi_A$ ▯

4.4. PROPOSITION. <u>If functors</u> Φ <u>and</u> Ψ <u>from</u> $\mathbb{A}^p$ <u>to</u> $\mathbb{B}^q$ <u>lift functors</u> F <u>and</u> G <u>from</u> $\mathbb{A}$ <u>to</u> $\mathbb{B}$, <u>with structure maps</u> φ_A <u>and</u> ψ_A, <u>then every natural transformation</u> $\Lambda : \Phi \to \Psi$ <u>lifts a natural transformation</u> $\lambda : F \to G$, <u>and in this situation</u> Λ <u>and</u> λ <u>determine each other</u>.

<u>Proof</u>. If Λ lifts λ, then $\lambda = Q\Lambda\omega_p$, and Λ determines λ. Conversely, let $\Lambda(A,x) = f : (F\,A, \varphi_A\, x) \to (G\,A, \psi_A\, x)$ for an object (A,x) of $\mathbb{A}^p$ and let $\Lambda(\omega_p\, A) = \lambda_A : \Phi\omega\, A \to \Psi\omega A$. Since $\mathrm{id}\, A : (A,x) \to \omega\, A$ in $\mathbb{A}^p$, we have $\lambda_A\,(\mathrm{id}\; F\, A) = (\mathrm{id}\; G\, A)\; f : \Phi(A,x) \to \Psi\omega A$ in $\mathbb{B}^q$, and thus $f = \lambda_A$. Since $\lambda_A = Q\Lambda(\omega_p\, A)$, the morphisms λ_A define a natural transformation $\lambda = Q\Lambda\omega_p : F \to G$. λ clearly determines Λ, and $Q\Lambda = \lambda P$ ▯

4.5. COROLLARY. Λ <u>in</u> 4.4 <u>is a natural equivalence if and only if</u> λ <u>is a natural equivalence and</u> $\varphi_A = (\lambda A)^q \psi_A$ <u>for every</u> $A \in \mathrm{Ob}\,\mathbb{A}$.

<u>Proof</u>. This follows immediately from 4.4 and from 7.2 below▯

4.6. EXAMPLE. One does not expect every functor $\Phi : \mathbb{A}^p \to \mathbb{B}^q$ to be lifted from a functor $F : \mathbb{A} \to \mathbb{B}$. The following simple example sustains this expectation. Let $\mathbb{A}^p$ be the category of pairs of sets (3.7) and $\mathbb{B}^q = \mathrm{ENS}$, both considered as top categories over ENS. Put $T\,(A,X) = X$ for a pair of sets, and let $T\,f : X \to Y$ be the restriction of f for $f : (A,X) \to (B,Y)$. Since $P\,(A,X) = A$ and $Q\,X = X$ in this situation, there can be no functor F : $\mathrm{ENS} \to \mathrm{ENS}$ such that $Q\,T = F\,P$.

The following theorem of Shukla generalizes our Theorem 3.8.

4.7. THEOREM. Let $\mathbb{A}^p$ and $\mathbb{B}^q$ be top categories. If $\mathbb{A}$ is small, then the functors $\Phi : \mathbb{A}^p \to \mathbb{B}^q$ which lift functors $F : \mathbb{A} \to \mathbb{B}$, and the natural transformations of these functors, form a top category over $\mathbb{B}^{\mathbb{A}}$.

Proof. If Φ lifts F, then we write $\Phi = (F, \varphi)$, where φ is the family of all structure maps $\varphi_A : p\,A \to q\,F\,A$ of Φ. If $\mathbb{A}$ is small, then these families form a set $t\,A$. We order $t\,A$ by putting $\varphi \le \varphi'$ if $\varphi_A\,x \le \varphi'_A\,x$ in $q\,F\,A$ for every object (A,x) of $\mathbb{A}^p$. For a family of families φ^μ in $t\,F$, we put $(\bigcap p^\mu)_A\,x = \bigcap (p^\mu_A\,x)$ for every object (A,x) of $\mathbb{A}$. This defines $\bigcap \varphi^\mu$ in $t\,F$; see the proof of 3.8 for this and other details. If $\lambda : F \to G$ in $\mathbb{B}^{\mathbb{A}}$ and $\psi \in t\,G$, put $(\lambda^t(\psi))_A\,x = (\lambda A)^p(\psi_A\,x)$ for every object (A,x) of $\mathbb{A}^p$. This defines $\lambda^t : t\,G \to t\,F$. One sees easily that λ^t preserves infima, and that a natural transformation $\lambda : F \to G$ can be lifted to a natural transformation $\Lambda : (F,\varphi) \to (G,\psi)$ (which is unique by 4.4) if and only if $\varphi \le \lambda^t(\psi)$ []

5. LIFTING UNIVERSAL MORPHISMS. We consider in this section the lifting of categorical limits and colimits, and of adjoint functor situations, from categories $\mathbb{A}$ and $\mathbb{B}$ to top categories $\mathbb{A}^p$ and $\mathbb{B}^q$.

5.1. DEFINITION. We say that $\Phi : \mathbb{A}^p \to \mathbb{B}^q$ is taut over $F : \mathbb{A} \to \mathbb{B}$, or that Φ lifts F tautly, if Φ lifts F, all structure maps $\varphi_A : p\,A \to q\,F\,A$ of Φ preserve infima, and $\varphi_A\,f^p = (F\,f)^q\varphi_B$ for every morphism $f : A \to B$ of $\mathbb{A}$. We say dually that Φ is cotaut over F if Φ lifts F, all structure maps φ_A preserve suprema, and always $\varphi_B\,f_p = (F\,f)_q\,\varphi_A$.

Everything in this section follows from our next result.

5.2. THEOREM. Let a functor $\Phi : \mathbb{A}^p \to \mathbb{B}^q$ lift a functor $F : \mathbb{A} \to \mathbb{B}$, with structure maps φ_A. If Φ is taut over F, then $h : (B,y) \to \Phi(C,z)$ is a universal morphism for Φ if and only if $h : B \to F\,C$ is a universal morphism for F and $z = \bigcap \{x \in p\,C : y \leq h^q \varphi_C\, x\}$.

Proof. If $h : (B,y) \to \Phi(C,z)$ is universal for Φ, let $g : B \to F\,A$ in $\mathbb{B}$. If Φ is taut, then $\Phi(A,\omega_A) = (F\,A, \omega_{FA})$, and thus $g : (B,y) \to \Phi(A,\omega_A)$ in $\mathbb{B}^q$, and $g = (F\,f)\,h$ for a unique $f : (C,z) \to (A,\omega_A)$ in $\mathbb{A}^p$. As $f : (C,z) \to (A,\omega_A)$ in $\mathbb{A}^p$ if and only if $f : C \to A$ in $\mathbb{A}$, it follows that $h : B \to F\,C$ is universal for F.

Conversely, let $h : B \to F\,C$ be universal for F. By the definitions, $h : (B,y) \to \Phi(C,z)$ is universal for Φ if and only if $y \leq g^q \varphi_A\, x \iff z \leq f^p\, x$ for $f : C \to A$ in $\mathbb{A}$, $x \in p\,A$, and $g = (F\,f)\,h$ in $\mathbb{B}$. For $f = \mathrm{id}\,C$, it follows that we must have $z = \bigcap \{x \in p\,C : y \leq h^q \varphi_C\, x\}$.

For this z, we have $h^q \varphi_C\, z = \bigcap \{h^q \varphi_C\, x : y \leq h^q \varphi_C\, x\}$ since h^q and (by request) φ_C preserve infima. It follows that $y \leq h^q \varphi_C\, x \iff z \leq x$ for $x \in p\,C$. If $f : C \to A$ in $\mathbb{A}$ and $g = (F\,f)\,h$, then

$$g^q \varphi_A = h^q (F\,f)^q \varphi_A = h^q \varphi_C\, f^p$$

for φ taut over F, and then

$$y \leq g^q \varphi_A\, x \iff y \leq h^q \varphi_C\, f^p\, x \iff z \leq f^p\, x ,$$

for $x \in p\,A$, as required $\square$

5.3. THEOREM. A diagram $D : K \to \mathbb{A}^p$ has a limit in $\mathbb{A}^p$ if and only if the diagram $P\,D : K \to \mathbb{A}$ has a limit in $\mathbb{A}$. If $\lambda : A^K \to P\,D$ is a limit of $P\,D$, then $\lambda = P^K \Lambda = P \Lambda$ for a unique limit $\Lambda : (A,x)^K \to D$ of D,

<u>and all limits of</u> D <u>are obtained in this way</u>.

Dually, all colimits in $\mathbb{A}^p$ are obtained by lifting colimits in $\mathbb{A}$.

<u>Proof</u>. We recall that $A^K : K \longrightarrow \mathbb{A}$ is obtained by putting $A^K\varphi = \text{id } A$ for every $\varphi \in K$, and that $f^K i = f$, for all $i \in \text{Ob } K$, defines a natural transformation $f^K : A^K \longrightarrow B^K$ for $f : A \longrightarrow B$ in $\mathbb{A}$. Putting $\Gamma f = f^K$ for $f \in \mathbb{A}$ defines a constant diagram functor $\Gamma : \mathbb{A} \longrightarrow \mathbb{A}^K$, and a limit of $P\,D$ is a couniversal morphism $\lambda : \Gamma A \longrightarrow P\,D$ for this functor. One sees easily that the constant diagram functor $\Gamma : \mathbb{A}^p \longrightarrow (\mathbb{A}^p)^K$, with $(\mathbb{A}^p)^K$ regarded as top category over $\mathbb{A}^K$ by 3.8, lifts the functor $\Gamma : \mathbb{A} \longrightarrow \mathbb{A}^K$, with structure maps γ_A given by $\gamma_A\, x = (x_i)$, the constant family with $x_i = x$ for every $i \in \text{Ob } i$, for $x \in p\,A$. This is obviously taut and cotaut, and thus we obtain 5.3 by applying the dual of 5.2 to this situation.

5.4. THEOREM. <u>If</u> $\Phi : \mathbb{A}^p \longrightarrow \mathbb{B}^q$ <u>is a functor of top categories which lifts a functor</u> $F : \mathbb{A} \longrightarrow \mathbb{B}$, <u>with structure maps</u> $\varphi_A : p\,A \longrightarrow q\,F\,A$, <u>then the following three statements are logically equivalent</u>.

5.4.1. F <u>has a left adjoint functor and</u> Φ <u>is taut over</u> F.

5.4.2. Φ <u>has a left adjoint, and</u> $\Phi\omega_p = \omega_q F$.

5.4.3. Φ <u>has a left adjoint which lifts a functor</u> $G : \mathbb{B} \longrightarrow \mathbb{A}$.

<u>Proof</u>. We prove 5.4.3 $\Longrightarrow$ 5.4.2 $\Longrightarrow$ 5.4.1 $\Longrightarrow$ 5.4.3.

If $\Psi \dashv \Phi$ and $P\Psi' = G\,Q$, then $G = P\,\Psi\alpha_q \dashv Q\,\Phi\omega_p = F$. Thus $P\,\Psi \dashv \Phi\,\omega_p$, and $G\,Q \dashv \omega_q\,F$, so that $\Phi\,\omega_p$ and $\omega_q\,F$ are naturally equivalent. If $u_A : (F\,A, \varphi_A\,\omega_A) \longrightarrow (F\,A, \omega_{FA})$ is an equivalence, then u_A is isomorphic in $\mathbb{B}$, and $\varphi_A\,\omega_A = u_A{}^q\,\omega_{FA} = \omega_{FA}$, with 7.2 below. But then $\Phi\,\omega A = (F\,A, \varphi_A\omega_A) = (F\,A, \omega_{FA}) = \omega F\,A$, and 5.4.3 $\Longrightarrow$ 5.4.2.

If $\Psi \dashv \Phi$, then $P\Psi\alpha_q \dashv F$, as above, and Φ preserves limits. For an object A of $\mathbb{A}$ and a family $(x_i)_{i\in I}$ of elements of pA, the object $(A, \bigcap x_i)$ of $\mathbb{A}^p$ is a limit of a diagram with one vertex (A, ω_A), and with arrows $\mathrm{id}\,A : (A, x_i) \longrightarrow (A, \omega_A)$. If Φ satisfies 5.4.2, then Φ preserves this situation, and thus $\varphi_A(\bigcap x_i) = \bigcap(\varphi_A\, x_i)$. Similarly, a diagram

$$\begin{array}{ccc} (A,x) & \xrightarrow{\ \mathrm{id}\,A\ } & (A,\omega_A) \\ \downarrow f & & \downarrow f \\ (B,y) & \xrightarrow{\ \mathrm{id}\,B\ } & (B,\omega_B) \end{array}$$

in $\mathbb{A}^p$, with $f : A \longrightarrow B$ in $\mathbb{A}$, is a pullback if and only if $x = f^p\, y$. This is easily verified. If Φ satisfies 5.4.2, then Φ preserves this pullback situation, and $\varphi_A\, f^p\, y = (F\, f)^q\, \varphi_B\, y$ follows. Thus 5.4.2 $\Longrightarrow$ 5.4.1.

Finally, 5.4.1 $\Longrightarrow$ 5.4.3 follows immediately from the following result $\square$

5.5. PROPOSITION. If $G : \mathbb{B} \to \mathbb{A}$ is left adjoint to $F : \mathbb{A} \to \mathbb{B}$, with front adjunction $\eta : \mathrm{Id}\,\mathbb{B} \to F\,G$, and if $\Phi : \mathbb{A}^p \longrightarrow \mathbb{B}^q$ lifts F tautly, then there is a unique left adjoint functor $\Psi : \mathbb{B}^q \longrightarrow \mathbb{A}^p$ of Φ which lifts G and for which η can be lifted to a front adjunction $H : \mathrm{Id}\,\mathbb{B}^q \longrightarrow \Phi\Psi$.

Proof. Let (B,y) be an object of $\mathbb{B}^q$. By 5.2, there is a unique universal morphism $\eta_A : (B,y) \longrightarrow \Phi(G\,B,\ z)$ for Φ which lifts the universal morphism $\eta_A : B \longrightarrow F\,G\,B$ for F. Taken together, the universal morphisms η_A for Φ determine a left adjoint functor Ψ of Φ and a front adjunction H : $\mathrm{Id}\,\mathbb{B}^q \longrightarrow \Phi\Psi$ in the usual way. One verifies easily that Ψ lifts G and that H lifts η $\square$

5.6. REMARKS. If Φ in 5.4 and 5.5 has a left adjoint functor Ψ which lifts a functor $G : \mathcal{B} \to \mathcal{A}$, then G is left adjoint to F by the proof of 5.4. In this situation, a front adjunction $H : \mathrm{Id}\,\mathcal{B}^q \to \Phi \Psi$ lifts a natural transformation $\eta : \mathrm{Id}\ B \to F\ G$. One verifies easily that η is a front adjunction for F . Thus 5.5 has a converse which we have not stated.

5.4, and in particular 5.4.2, raises the following question: is it possible that Φ lifts F and has a left adjoint Ψ, but not a left adjoint which lifts a functor $G : \mathcal{B} \to \mathcal{A}$? Shukla has shown that this is indeed possible; his example follows.

5.7. EXAMPLE. If Q is the forgetful functor from TOP to ENS , then (TOP, Q) is a top category over ENS . Let TOP^p be the category of pairs of topological spaces and define $P : \mathrm{TOP}^p \to \mathrm{TOP}$ by $P\,(A,X) = A$ and $P\,f = f$. Then (TOP^p , P) is a top category over TOP ; see 3.7. If $\Phi = P$ and $F = Q$ in this situation, then Φ obviously lifts F , and Φ has a left adjoint functor $\alpha_p : \mathrm{TOP} \to \mathrm{TOP}^p$. The following diagram illustrates the situation.

$$\begin{array}{ccc} \mathrm{TOP}^p & \xrightarrow{\ \Phi = P\ } & \mathrm{TOP} \\ \downarrow{\scriptstyle P} & & \downarrow{\scriptstyle Q} \\ \mathrm{TOP} & \xrightarrow{\ F = Q\ } & \mathrm{ENS} \end{array}$$

If (E,τ) is a topological space, then $\omega_q\ F\,(E,\tau) = (E,\omega_E)$, the set E with the indiscrete topology, and $\Phi\omega_p\ (E,\tau) = (E,\tau)$. Thus 5.4.2 is false, and Φ cannot have a left adjoint functor which lifts the left adjoint functor $\alpha_q : \mathrm{ENS} \to \mathrm{TOP}$ of F (or any other functor $G : \mathrm{ENS} \to \mathrm{TOP}$).

6. TOP SUBCATEGORIES AND FUNCTORS. We apply our results to a special situation which occurs often enough to merit consideration.

6.1. DEFINITION. Let $\mathbb{A}^p$ and $\mathbb{A}^q$ be top categories over a category $\mathbb{A}$. A functor $T : \mathbb{A}^p \to \mathbb{A}^q$ is called a <u>top functor</u> over $\mathbb{A}$ if T is taut over $\mathrm{Id}\,\mathbb{A}$. We call $\mathbb{A}^p$ a <u>top subcategory</u> of $\mathbb{A}^q$ if $\mathbb{A}^p$ is a subcategory of $\mathbb{A}^q$, and the embedding functor is a top functor over $\mathbb{A}$. Dually, a functor $S : \mathbb{A}^q \to \mathbb{A}^p$ is called <u>cotop</u> over $\mathbb{A}$ if S lifts $\mathrm{Id}\,\mathbb{A}$ cotautly, and a <u>cotop subcategory</u> is one with a cotop embedding functor.

6.2. PROPOSITION. <u>A top functor</u> $T : \mathbb{A}^p \to \mathbb{A}^q$ <u>has a unique cotop left adjoint functor</u> $S : \mathbb{A}^q \to \mathbb{A}^p$, <u>and then</u> $S\,T\,S = S$ <u>and</u> $T\,S\,T = T$.

<u>Proof</u>. By 5.5, T has a left adjoint S which lifts $\mathrm{Id}\,\mathbb{A}$, and S is cotop by the dual of 5.4. If $T\,(A,x) = (A, \tau_A\, x)$ and $S\,(A,y) = (A, \sigma_A\, y)$, then

$$\mathrm{id}\,A : (A,y) \to (A, \tau_A\, x) \iff \mathrm{id}\,A : (A, \sigma_A\, y) \to (A,x)$$

since $S \dashv T$. In other words, $y \leq \tau_A\, x \iff \sigma_A\, y \leq x$. Thus T determines the structure maps σ_A, and hence S, uniquely. Moreover, $\sigma_A \tau_A \sigma_A = \sigma_A$ and $\tau_A \sigma_A \tau_A = \tau_A$, as for every Galois correspondence, and $S\,T\,S = S$ and $T\,S\,T = T$ follow $\square$

6.3. THEOREM. <u>If</u> $\mathbb{B}$ <u>is a subcategory of a top category</u> $\mathbb{A}^p$, <u>then the following statements are logically equivalent</u>.

6.3.1. $\mathbb{B}$ <u>is a full reflective subcategory of</u> $\mathbb{A}^p$, <u>and every object</u> (A,x) <u>of</u> $\mathbb{A}^p$ <u>has a reflection</u> $\mathrm{id}\,A : (A,x) \to (A,\hat{x})$ <u>for</u> $\mathbb{B}$.

6.3.2. $\mathbb{B}$ <u>is a full subcategory of</u> $\mathbb{A}^p$ <u>and satisfies the following two conditions</u>. (i) <u>If</u> $A \in \mathrm{Ob}\,\mathbb{A}$ <u>and</u> $(x_i)_{i \in I}$ <u>is a family of elements of</u> $p\,A$ <u>such that</u> $(A,x_i) \in \mathrm{Ob}\,\mathbb{B}$ <u>for all</u> $i \in I$, <u>then</u> $(A, \bigcap x_i)$ <u>is in</u> $\mathrm{Ob}\,\mathbb{B}$. (ii) <u>If</u> $f : A \to B$ <u>in</u> $\mathbb{A}$ <u>and</u> $(B,y) \in \mathrm{Ob}\,\mathbb{B}$, <u>then</u> $(A, f^p\, y) \in \mathrm{Ob}\,\mathbb{B}$.

6.3.3. $\mathbb{B}$ <u>is a top subcategory of</u> $\mathbb{A}^p$.

6.3.4. $\mathbb{B}$ <u>is a full, reflective, replete subcategory of</u> $\mathbb{A}^p$, <u>and every object</u> (A, ω_A) <u>of</u> $\mathbb{A}^p$, <u>for</u> $A \in \mathrm{Ob}\,\mathbb{A}$, <u>is an object of</u> $\mathbb{B}$.

<u>Proof</u>. Assume first 6.3.1. In 6.3.2.(i), let $\mathrm{id}\,A : (A, \bigcap x_i) \longrightarrow (A, \hat{x})$ be the reflection. Then $\mathrm{id}\,A : (A, \hat{x}) \longrightarrow (A, x_i)$ for all $i \in I$, and thus $\bigcap x_i \leq \hat{x} \leq x_i$ for all $i \in I$. But then $\hat{x} = \bigcap x_i$, and $(A, \bigcap x_i) \in \mathrm{Ob}\,\mathbb{B}$. In 6.3.2.(ii), let $\mathrm{id}\,A : (A, f^p\, y) \longrightarrow (A, \hat{x})$ be the reflection. Then $f : (A, \hat{x}) \longrightarrow (B, y)$. Thus $f^p\, y \leq \hat{x} \leq f^p\, y$, and $(A, f^p\, y) \in \mathrm{Ob}\,\mathbb{B}$.

Assume now 6.3.2. Let $p'\,A = \{x \in p\,A : (A,x) \in \mathrm{Ob}\,\mathbb{B}\}$ for $A \in \mathrm{Ob}\,\mathbb{A}$, and let $\iota_A : p'\,A \longrightarrow p\,A$ be the inclusion map. By (i), $p'\,A$ is a complete lattice and ι_A preserves infima. By (ii), $f^p : p\,B \longrightarrow p\,A$ induces a map $f^{p'} : p'\,B \longrightarrow p'\,A$ for $f : A \longrightarrow B$ in $\mathbb{A}$, with $\iota_A\, f^{p'} = f^p\, \iota_B$. The maps $f^{p'}$ clearly preserve infima. Thus our data define a top category $\mathbb{A}^{p'}$ over and a top functor $I : \mathbb{A}^{p'} \longrightarrow \mathbb{A}^p$ with structure maps ι_A . $\mathbb{A}^{p'}$ clearly is the full subcategory $\mathbb{B}$, and I the embedding functor.

Assume now 6.3.3. $\mathbb{B}$ is full by Lemma 6.4 below and reflective by 6.2. Put $\mathbb{B} = \mathbb{A}^{p'}$ and let $I : \mathbb{A}^{p'} \longrightarrow \mathbb{A}^p$ be the embedding functor, with structure maps ι_A for $A \in \mathrm{Ob}\,\mathbb{A}$. If z is the greatest element of $p'\,A$, then $I\,(A,z) = (A, \omega_A)$ since ι_A preserves the infimum of the empty family. If $u : (A,x) \longrightarrow I\,(B,y)$ is an isomorphism of $\mathbb{A}^p$, then $x = u^p\, \iota_B\, y = \iota_A\, u^{p'}\, y$ (see 7.2 below). Thus $(A,x) = I\,(A, u^{p'}\, y)$, and 6.3.4 is valid.

Assume now 6.3.4. Let $u : (A,x) \longrightarrow (C,z)$ be a reflection for $\mathbb{B}$. Since $(A, \omega_A) \in \mathrm{Ob}\,\mathbb{B}$, we have $h\,u = \mathrm{id}\,A$ for a map $h : (C,z) \longrightarrow (A, \omega_A)$ in $\mathbb{B}$. Now $u : (A, \omega_A) \longrightarrow (C, \omega_C)$ is in $\mathbb{B}$, and $(\mathrm{id}\,C)\,u = u\,h\,u : (A,x) \longrightarrow (C, \omega_C)$. Since $u : (A,x) \longrightarrow (C,z)$ is a reflection, $u\,h = \mathrm{id}\,C$ follows.

Now $h : (C,z) \longrightarrow (A, u^p z)$ is an isomorphism of $\mathbb{A}^p$ (see again 7.2). Since $\mathbb{B}$ is replete, $\mathrm{id}\, A = h\, u : (A,x) \to (A, u^p z)$ is a reflection for $\mathbb{B}$ ▯

6.4. LEMMA. <u>A top functor</u> $T : \mathbb{A}^p \to \mathbb{B}^q$ <u>is full</u>.

<u>Proof</u>. Let S be the cotop left adjoint of T. If $f : T(A,x) \longrightarrow T(A,y)$ in $\mathbb{A}^q$, then $f : S\,T(A,x) \longrightarrow S\,T(A,y)$ in $\mathbb{A}^p$ by applying S. Applying T to this, we get $f : T(A,x) \longrightarrow T(B,y)$ back, by 6.2▯

6.5. COROLLARY. <u>Every small or large intersection of top subcategories of a top category</u> $\mathbb{A}^p$ <u>is a top subcategory of</u> $\mathbb{A}^p$.

<u>Proof</u>. If all subcategories $\mathbb{B}_i$ satisfy 6.3.2, then their intersection $\bigcap \mathbb{B}_i$ also satisfies 6.3.2, even if the family is large▯

6.6. PROPOSITION. <u>Let</u> $\mathbb{A}^p$ <u>be a top category over a category</u> $\mathbb{A}$ <u>with products. If</u> $\mathbb{K}$ <u>is a class of objects of</u> $\mathbb{A}^p$ <u>which is closed under products, then the objects</u> $(A, f^p y)$ <u>of</u> $\mathbb{A}^p$, <u>for</u> $f : A \longrightarrow B$ <u>in</u> $\mathbb{A}$ <u>and</u> $(B,y) \in \mathbb{K}$, <u>are the objects of a top subcategory of</u> $\mathbb{A}^p$.

<u>Proof</u>. 6.3.2.(ii) is obvious in this situation; we verify 6.3.2.(i). Let $x_i = f_i^{\,p}\, y_i$ for $f_i : A \longrightarrow B_i$ and $(B_i, y_i) \in \mathbb{K}$. Let (B,y) in $\mathbb{K}$ be a product of the (B_i, y_i) with projections $\pi_i : (B,y) \longrightarrow (B_i, y_i)$. By 5.3 and its proof, $B = \prod B_i$ in $\mathbb{A}$ with projections π_i, and $y = \bigcap \pi_i^{\,p}\, y_i$. Thus there is $f : A \longrightarrow B$ in $\mathbb{A}$ with $\pi_i\, f = f_i$, for all $i \in I$, and then

$$f^p\, y \;=\; f^p \left(\bigcap (\pi_i^{\,p}\, y_i)\right) \;=\; \bigcap \left((\pi_i\, f)^p\, y_i\right) \;=\; \bigcap x_i \;.$$

This proves 6.3.2.(i) for our class▯

6.7. THEOREM. <u>Let</u> $T : \mathbb{A}^p \to \mathbb{B}^q$ <u>be a top functor over</u> $\mathbb{A}$ <u>and</u> $S : \mathbb{A}^q \to \mathbb{A}^p$ <u>the cotop left adjoint of</u> T . <u>Then</u> T <u>maps</u> $\mathbb{A}^p$ <u>onto a top subcategory</u> $T(\mathbb{A}^p)$ <u>of</u> $\mathbb{A}^q$, <u>and</u> S <u>maps</u> $\mathbb{A}^q$ <u>onto a cotop subcategory of</u> $\mathbb{A}^p$. <u>If</u> $I : T(\mathbb{A}^p) \to \mathbb{A}^q$ <u>is the top embedding functor and</u> $J : \mathbb{A}^q \to T(\mathbb{A}^p)$ <u>the cotop reflector, then</u> $J\,I = \mathrm{Id}\, T(\mathbb{A}^p)$ <u>and</u> $I\,J = T\,S$.

<u>Proof</u>. $T(\mathbb{A}^p)$ is a full subcategory of $\mathbb{A}^q$ by 6.4. The front adjunction $\mathrm{id}\,A : (A,x) \to T\,S\,(A,x)$ for an object (A,x) of $\mathbb{A}^q$ clearly is a reflection for $T(\mathbb{A}^p)$. Thus 6.3.1 applies, and $T(\mathbb{A}^p)$ is a top subcategory of $\mathbb{A}^q$. Dually, $S(\mathbb{A}^q)$ is a cotop subcategory of $\mathbb{A}^p$.

Every object (A,x) of $\mathbb{A}^q$ has a unique reflection $\mathrm{id}\,A : (A,x) \to I\,J\,(A,x)$ for I . This is $\mathrm{id}\,A : (A,x) \to T\,S\,(A,x)$, and thus $I\,J = T\,S$ for objects. As both functors lift $\mathrm{Id}\,\mathbb{A}$, we have $I\,J = T\,S$ for morphisms too. Since $I\,J\,I = I$ by 6.2 and I is injective, we have $J\,I = \mathrm{Id}\,T(\mathbb{A}^p)$ []

6.7. EXAMPLES. In general topology, a structure of one kind on a set E often induces a structure of another less rich kind. For example, every topology induces an underlying convergence structure, every uniform structure induces an underlying topology, and every proximity induces an underlying closure structure. In such a situation, we have top categories ENS^p and ENS^q , and a map $\tau_E : p\,E \to q\,E$ for every set. Typically, the maps τ_E preserve infima and satisfy $\tau_E\, f^p = f^q\, \tau_F$ for every mapping $f : E \to F$. Thus they are the structure maps of a top functor $T : \mathrm{ENS}^p \to \mathrm{ENS}^q$. The top subcatectory $T(\mathrm{ENS}^p)$ of ENS^q , and the cotop subcategory $S(\mathrm{ENS}^q)$ of "fine" spaces in ENS^p , for the cotop left adjoint S of T , are of interest in many cases.

Top subcategories occur quite often, not only in general topology but also elsewhere. For instance, the top category ENS^e of equivalence relations (3.6) is a top subcategory of the category of pre-ordered sets, and pre-ordered sets form a top subcategory of the category of reflective relations.

In general topology, coreflective subcategories are cotop if replete (i.e. closed under isomorphisms). We shall consider this later. Reflective full subcategories are usually not top, but they have simple top hulls by 6.6.

7. EPIREFLECTIVE SUBCATEGORIES. The present state of this aspect of the theory of top categories is not very satisfactory. Basic results were obtained by Kennison, generalized by the author and further refined by Shukla. Herrlich obtained a much more general theory, but further generalizations seem possible. In this survey, we present only the ~~more~~ easily accessible basic results. Before we can discuss these, we need some lemmas concerning monos and epis, extremal monos and epis, and isomorphisms. We have already used some of these repeatedly.

We recall that a mono m in a category $\mathbb{A}$ is called extremal if $m = g\,f$ in $\mathbb{A}$ and f epi always implies that f is isomorphic in $\mathbb{A}$. Extremal epis are defined dually.

7.1. PROPOSITION. *A morphism $u : (A,x) \longrightarrow (B,y)$ of a top category $\mathbb{A}^p$ is a monomorphism of $\mathbb{A}^p$ if and only if u is a monomorphism of $\mathbb{A}$, and $u : (A,x) \longrightarrow (B,y)$ is an extremal monomorphism of $\mathbb{A}^p$ if and only if u is an extremal monomorphism of $\mathbb{A}$ and $x = u^p\,y$.*

7.2. PROPOSITION. *A morphism $u : (A,x) \longrightarrow (B,y)$ of a top category $\mathbb{A}^p$ is isomorphic in $\mathbb{A}^p$ if and only if u is isomorphic in $\mathbb{A}$ and $x = u^p\,y$.*

Proof. The faithful functor $P : \mathbb{A}^p \longrightarrow \mathbb{A}$ reflects monomorphisms, and thus

$u : (A,x) \longrightarrow (B,y)$ is mono if $u : A \longrightarrow B$ is. On the other hand, if $u f = u g$ in $\mathbb{A}$ for f, g from C to A, then $u f = u g$ in $\mathbb{A}^p$ for f, g from (C, α_C) to (A,x). Thus u is mono in $\mathbb{A}$ if u is mono in $\mathbb{A}^p$.

If $u : (A,x) \longrightarrow (B,y)$ is iso in $\mathbb{A}^p$ with inverse $v : (B,y) \longrightarrow (A,x)$, then $x \leq u^p y$ and $y \leq v^p x$, and $v = u^{-1}$ in $\mathbb{A}$. But then $v^p = (u^p)^{-1}$ in ORD, and $u^p y \leq u^p v^p x = x$. Thus $x = u^p y$. Conversely, if $v = u^{-1}$ in $\mathbb{A}$ and $x = u^p y$, then $y = v^p x$, and thus $u : (A,x) \longrightarrow (B,y)$ is iso in $\mathbb{A}^p$ with inverse $v : (B,y) \longrightarrow (A,x)$.

Consider now $u : (A,x) \longrightarrow (B,y)$ in $\mathbb{A}^p$. If $x < u^p y$, then

$$(A,x) \xrightarrow{\text{id } A} (A, u^p y) \xrightarrow{u} (B,y)$$

is a factorization of u in $\mathbb{A}^p$ with the first factor epi, but not iso. For $u = g f$ in $\mathbb{A}$ with f not iso, we have a factorization

$$(A,x) \xrightarrow{f} (C,\ g^p y) \xrightarrow{g} (B,y)$$

in $\mathbb{A}^p$ with the first factor not iso (by 7.2 which we have proved). Thus assume that u is an extremal mono in $\mathbb{A}$ and $x = u^p y$. Consider a factorization

$$(A,x) \xrightarrow{f} (C,z) \xrightarrow{g} (B,y)$$

with an epi first factor. By the dual of the first part of 7.1, f is epi in and hence iso. Let $h = f^{-1}$. Then $g = u h$, and $g^p y = h^p x$ follows. From the first factor, we have $x \leq f^p z$ and thus $z \leq h^p x = g^p y$. From the second factor, $z \leq g^p y$. Thus $z = g^p y$, and $f^p z = u^p y = x$. This shows that the first factor is isomorphic in $\mathbb{A}^p$ []

7.3. DEFINITION. A full subcategory $\mathcal{B}$ of a category $\mathbb{A}$ is called <u>epi-reflective</u> if every $A \in \text{Ob}\mathbb{A}$ admits a reflection $\rho_A : A \longrightarrow R\,A$ for $\mathcal{B}$ with

ρ_A epimorphic in $\mathbb{A}$. We call $\mathbb{B}$ <u>productive</u> if every family $(B_i)_{i \in I}$ of objects of $\mathbb{B}$ has a product B in $\mathbb{A}$ such that $B \in \mathrm{Ob}\,\mathbb{B}$, and we call $\mathbb{B}$ <u>hereditary</u> if for every extremal monomorphism $m : A \longrightarrow B$ in $\mathbb{A}$ with $B \in \mathrm{Ob}\,\mathbb{B}$ there is an isomorphism $u : A \longrightarrow C$ in $\mathbb{A}$ with $C \in \mathrm{Ob}\,\mathbb{B}$. We call $\mathbb{A}$ <u>factored over</u> $\mathbb{B}$ if every morphism $f : A \longrightarrow B$ in $\mathbb{A}$ with $B \in \mathrm{Ob}\,\mathbb{B}$ has a factorization $f = m\,e$ with e epimorphic and m an extremal monomorphism of $\mathbb{A}$. We say that $\mathbb{A}$ is <u>colocally small over</u> $\mathbb{B}$ if for every object A of $\mathbb{A}$ there is a (small) family of epimorphisms $e_i : A \longrightarrow B_i$ with codomains $B_i \in \mathrm{Ob}\,\mathbb{B}$ such that every epimorphism $e : A \longrightarrow B$ with $B \in \mathrm{Ob}\,\mathbb{B}$ is of the form $e = u\,e_i$ with $u : B_i \longrightarrow B$ isomorphic, for at least one $e_i : A \longrightarrow B$.

7.4. THEOREM. <u>Let $\mathbb{B}$ be a full subcategory of a category $\mathbb{A}$. If $\mathbb{B}$ is epireflective, then $\mathbb{B}$ is hereditary. Conversely, if $\mathbb{B}$ is hereditary and productive, and if $\mathbb{A}$ is factored over $\mathbb{B}$ and colocally small over $\mathbb{B}$, then $\mathbb{B}$ is epireflective.</u>

<u>Proof</u>. The proof follows a standard pattern due to Bourbaki; we omit some details. If $\mathbb{B}$ is epireflective, and if $m : A \longrightarrow B$ is an extremal mono with $B \in \mathrm{Ob}\,\mathbb{B}$ and $\rho_A : A \longrightarrow R\,A$ a reflection for $\mathbb{B}$, then $m = f\,\rho_A$ in $\mathbb{A}$ for a morphism $f : R\,A \longrightarrow B$. Since ρ_A is epimorphic, ρ_A is isomorphic. Thus $\mathbb{B}$ is hereditary.

For the converse, let $A \in \mathrm{Ob}\,\mathbb{B}$, **with a "representative family" of epis** $e_i : A \longrightarrow B_i$ with $B_i \in \mathrm{Ob}\,\mathbb{B}$. Form the product B in $\mathbb{A}$ of the B_i, with projections $\pi_i : B \longrightarrow B_i$ and with $B \in \mathrm{Ob}\,\mathbb{B}$. Each e_i has a factorization

$$A \xrightarrow{\rho_A} R\,A \xrightarrow{m_A} B \xrightarrow{\pi_i} B_i$$

in $\mathbb{A}$ with ρ_A epi, m extremal mono, $R\,A \in \mathrm{Ob}\,\mathbb{B}$, and with ρ_A and m_A

independent of i. If $f : A \longrightarrow B_1$ in $\mathbb{A}$ with $B_1 \in \mathrm{Ob}\,\mathbb{B}$, then we can factor f into $A \xrightarrow{e} C \xrightarrow{m} B_1$ with m extremal mono, e epi, and $C \in \mathrm{Ob}\,\mathbb{B}$. Then e and hence f factors through one of the e_i, and hence $f = g\,\rho_A$ for some $g : R\,A \longrightarrow B_1$ in $\mathbb{B}$. Since ρ_A is epi, g is unique $\square$

7.5. REMARKS. Theorem 7.4 is a variation of a theorem of Herrlich which had been obtained for TOP by Kennison and for top categories by the author. Herrlich's version is much more complete than either of its predecessors or our present result. If $\mathbb{A}$ has products and is colocally small and factored (over $\mathbb{A}$ and hence over any full subcategory), then every top category over $\mathbb{A}$ has these properties, by 5.3, 7.1, and the definitions. In this situation, every reflective full subcategory of $\mathbb{A}$ is productive, and 7.4 becomes: A full subcategory $\mathbb{B}$ of $\mathbb{A}$ is epireflective if and only if $\mathbb{B}$ is hereditary and productive.

Kennison, Ehrbar, Shukla, and possibly others, have generalized 7.4 by considering different factorizations in $\mathbb{A}$, through images or coimages of some kind. In all of these generalizations, it seems important that either e is always epi or m always mono in an "admissible factorization" $f = m\,e$ in $\mathbb{A}$. Further research in this direction seems indicated.

7.6. EXAMPLES. Every top category over ENS is factored in the sense of 7.3 and colocally small. Thus 7.4 applies. Examples of epireflective subcategories of TOP are T_i spaces for $i = 0, 1, 2, 3, 3\frac{1}{2}$, and regular and completely regular spaces. Normal spaces fail the test: the product of normal spaces is not necessarily normal. There are of course many reflective subcategories of top categories which are not epireflective: the usual separated completions and compactifications furnish examples.

8. POINT SEPARATORS. We discuss in this section a categorical theory of point separation axioms. The theory is not restricted to top categories; it is valid for any concrete category, and we present it in this general context.

8.1. DEFINITIONS. We recall that a <u>concrete category</u> $\mathbb{A}$ is one with a faithful "underlying set" functor $U : \mathbb{A} \longrightarrow ENS$. If $\mathbb{A}$ is concrete, then every top category over $\mathbb{A}$ is concrete. We denote by ENS^r the top category of reflective relations and by $R : ENS^r \longrightarrow ENS$ its projection functor. We say that a full subcategory $\mathbb{B}$ of a concrete category $\mathbb{A}$ is <u>injective</u> if for every morphism $m : A \longrightarrow B$ of $\mathbb{A}$ with $U\,m$ injective and $B \in \mathrm{Ob}\,\mathbb{B}$, we have m in $\mathbb{B}$ and thus $A \in \mathrm{Ob}\,\mathbb{B}$. An injective subcategory is always replete.

We recall that the faithful functor $U : \mathbb{A} \longrightarrow ENS$ always reflects monomorphisms. Thus m is monomorphic in $\mathbb{A}$ if $U\,m$ is injective. If U has a left adjoint, then U preserves monomorphisms, and products. It follows that an injective subcategory of $\mathbb{A}$ is always hereditary.

8.2. DEFINITION. Let $\mathbb{A}$ be a concrete category. A functor $S : \mathbb{A} \longrightarrow ENS^r$ such that $R\,S = U$ is called a <u>point separator</u> on $\mathbb{A}$. If $S : \mathbb{A} \longrightarrow ENS^r$ is a point separator, then we write $S\,A = (U\,A,\ s_A)$ for $A \in \mathrm{Ob}\,\mathbb{A}$. We say that A is <u>separated</u> for S if $s_A = I_{UA}$, the identity relation on $U\,A$, and we denote by $\mathrm{sep}\,S$ the full subcategory of $\mathbb{A}$ with separated objects of $\mathbb{A}$ as its objects. We note that $S\,f = U\,f : (U\,A,\ s_A) \longrightarrow (U\,B,\ s_B)$ for a morphism $f : A \longrightarrow B$ of $\mathbb{A}$ and a point separator S on $\mathbb{A}$. Thus S is determined by the relations s_A , and these must satisfy $s_A \leq (U\,f)^r\,s_B$ for $f : A \longrightarrow B$ in $\mathbb{A}$.

8.3. PROPOSITION. <u>If</u> $S : \mathbb{A} \longrightarrow ENS^r$ <u>is a point separator, then</u> $\mathrm{sep}\,S$

<u>is injective.</u> <u>If</u> $\mathbb{A}$ <u>has products and</u> U <u>preserves products, then</u> sep S <u>is productive.</u>

<u>Proof.</u> If $u : E \longrightarrow F$ is injective, then $u^r\, I_F = I_E$. If $E = \prod E_i$ in ENS with projections π_i , then $\bigcap (\pi_i{}^r\, I_{E_i}) = I_E$ in r E , These properties of reflective relations are easily verified; 8.3 is based on them.

If $f : A \longrightarrow B$ in $\mathbb{A}$ with U f injective, then $s_A \leq (U\, f)^r\, s_B = I_{UA}$ if B is separated, and thus $s_A = I_{UA}$. If $A = \prod A_i$ in $\mathbb{A}$ with projections π_i and all A_i separated, then $s_A \leq \bigcap((U\, \pi_i)^r\, I_{UA_i})$ in r U A , and $s_A = I_{UA}$ follows if U preserves products▯

8.4. COROLLARY. <u>If</u> $\mathbb{A}$ <u>is factored and colocally small, and if</u> $U : \mathbb{A} \longrightarrow$ ENS <u>preserves products and monomorphisms, then</u> sep S <u>is an epireflective subcategory of</u> $\mathbb{A}$ <u>for every point separator</u> S <u>on</u> $\mathbb{A}$.

<u>Proof.</u> This follows immediately from 8.3 and 7.4▯

8.5. DEFINITION. We say that a point separator S' on $\mathbb{A}$ is <u>finer</u> than a point separator S , and we write $S' \leq S$, if $\mathrm{id}\, U\, A : S'\, A \rightarrow S\, A$ in ENS^r for every object A of $\mathbb{A}$, or in other words if always $s'_A \leq s_A$ in r U A . If $S' \leq S$, then sep S clearly is a subcategory of sep S' .

Different point separators may produce the same subcategory sep S . We say that a point separator S on $\mathbb{A}$ is <u>coarse</u> if $S' \leq S$ for every point separator S' on $\mathbb{A}$ such that sep S is a subcategory of sep S' .

8.6. THEOREM. <u>If</u> $\mathcal{B}$ <u>is a reflective and injective full subcategory of a concrete category</u> $\mathbb{A}$, <u>then</u> $\mathcal{B}$ = sep S <u>for a unique coarse point separator</u> S <u>on</u> $\mathbb{A}$.

Proof. We write f for $U f$ in ENS or ENS^r if $f \in \mathbb{A}$. We put $s_A = h^r I_{UC}$ in $r U A$ if $h : A \longrightarrow C$ is a reflection for $\mathbb{B}$. Since h is determined by A up to an isomorphism $u : C \longrightarrow C'$ in $\mathbb{B}$, this determines s_A uniquely. $s_A = I_{UA}$ only if h is injective. But then $A \in \mathrm{Ob}\, \mathbb{B}$. Conversely, if $A \in \mathrm{Ob}\, \mathbb{B}$, then h is isomorphic, and $s_A = I_{UA}$.

If $f : A \longrightarrow A'$ in $\mathbb{A}$, and if $h : A \longrightarrow C$ and $k : A' \longrightarrow C'$ are reflections for $\mathbb{B}$, then $k f = g h$ for a morphism $g : C \longrightarrow C'$. But then

$$s_A = h^r I_{UC} \leq h^r g^r I_{UC'} = f^r k^r I_{UC'} = f^r s_{A'}$$

in $r U A$. Thus the relations s_A define a point separator $S : \mathbb{A} \longrightarrow ENS^r$, and we have already seen that $\mathbb{B} = \mathrm{sep}\, S$.

If $\mathbb{B}$ is a subcategory of $\mathrm{sep}\, S'$ for a point separator S' on $\mathbb{A}$ and $h : A \longrightarrow C$ a reflection for $\mathbb{B}$, then $s'_A \leq h^r s'_C = h^r I_{UC} = s_A$. Thus S is coarse, and hence uniquely determined by $\mathbb{B}$ []

8.7. REMARKS. We may call a point separator S strict if all relations s_A are in fact equivalence relations. Since sets with equivalence relations form a top subcategory of ENS^r, one sees easily that for every point separator S there is a finest strict point separator $S_1 \geq S$, with $\mathrm{sep}\, S_1 = \mathrm{sep}\, S$. Thus coarse point separators are strict. Strict point separators over TOP were studied by Sharpe, Beattie and Marsden, who obtained 8.3 and the first part of 8.8 below.

We need the following definitions. We call $f \in \mathbb{A}$ a P-**quotient map** for a functor $P : \mathbb{A} \longrightarrow \mathbb{C}$ if f is P-opfibred (1.1) and $P f$ epimorphic in $\mathbb{C}$. For a top category (ENS^P, P) over ENS, this coincides with the usual concept of a quotient map. A morphism $f : A \longrightarrow B$ of a category $\mathbb{A}$ is called

<u>reflective</u> over a full subcategory $\mathbb{B}$ of $\mathbb{A}$ if for every morphism $g : A \longrightarrow C$ in $\mathbb{A}$ with $C \in \mathrm{Ob}\,\mathbb{B}$ there is a unique morphism $h : B \longrightarrow C$ in $\mathbb{A}$ such that $g = h\,f$. This includes all reflections for $\mathbb{B}$ and all isomorphisms of $\mathbb{A}$, and reflective morphisms over $\mathbb{B}$ form a subcategory of $\mathbb{A}$.

8.8. PROPOSITION. <u>Let</u> S <u>be a point separator on</u> $\mathbb{A}$, <u>and let</u> $f : A \longrightarrow B$ <u>in</u> $\mathbb{A}$. <u>If</u> f <u>is a</u> U-<u>quotient map and</u> $s_A = (U\,f)^r\, s_B$, <u>then</u> f <u>is reflective over</u> sep S. <u>Conversely, if</u> S <u>is coarse and</u> sep S <u>reflective in</u> $\mathbb{A}$, <u>and if</u> f <u>is reflective for</u> sep S, <u>then</u> $s_A = (U\,f)^r\, s_B$.

<u>Proof</u>. Write again f for U f if $f \in \mathbb{A}$. For the first part, let $g : A \longrightarrow C$ with $C \in \mathrm{Ob}$ sep S. Then $f^r\, I_{UB} \leq f^r\, s_B = s_A \leq g^r\, s_C = I_{UC}$ in r U A. This means that $f(x) = f(y) \implies g(x) = g(y)$ for all x, y in U A. As f is surjective, it follows that $g = h\,f$ for a unique $h : U\,B \longrightarrow U\,C$. As f is U-opfibred, h lifts to a unique morphism $h : B \longrightarrow C$. Thus f is reflective over $\mathbb{B}$.

For the second part, let $h : B \longrightarrow C$ be the reflection for sep S. Then $h\,f : A \longrightarrow C$ is a reflection for sep S as f is reflective. Thus

$$s_A = (h\,f)^r\, I_{UC} = f^r\, h^r\, I_{UC} = f^r\, s_B ,$$

by the proof of 8.6, as required▯

8.9. EXAMPLES. For a topological space A and points x, y of A, put $x\, s_A\, y$ if x and y have the same neighborhoods in A, and put $x\, s'_A\, y$ if x is in the closure of $\{y\}$ for A. This defines point separators S and S' on TOP, with S coarse and S' not even strict. Objects of sep S are T_0 spaces, and objects of sep S' are T_1 spaces. T_2 spaces also are obtained from a point separator. Aull has given a long list of point separation axioms and references to yet more axioms. One example for which we do not have a convenient point separator are the spaces for which every compact set is closed.

Regular and completely regular spaces do not define injective subcategories of TOP . Thus T_3 and $T_{3\frac{1}{2}}$ are not point separation axioms.

9. TOP CATEGORIES OF ALGEBRAS. For every category $\mathbb{A}$ of algebras, a category $\mathbb{A}^t$ of topological algebras over $\mathbb{A}$ can be constructed, and $\mathbb{A}^t$ turns out to be a top category over $\mathbb{A}$. We generalize this construction in an appropriate categorical setting.

9.1. OPERATIONS. Let $U : \mathbb{A} \rightarrow \mathbb{C}$ be a functor. We call <u>operation over</u> U or by abuse of language operation in $\mathbb{A}$ a triple (D, ω, R) consisting of two endofunctors D and R of $\mathbb{C}$ and a natural transformation $\omega : D\,U \longrightarrow R\,U$. This is not the most general concept of an operation. For instance, the "relational systems" of universal algebra, and filter convergence in topology, are excluded. A categorical definition of operations which includes these cases has been given, but the theory of this section has not been generalized to this.

Operations in algebra are included in the definition given above. An n-ary operation in a category $\mathbb{A}$ of algebras, with underlying set functor $U : \mathbb{A} \longrightarrow$ ENS , associates with every algebra $A \in \mathrm{Ob}\,\mathbb{A}$ a mapping $\omega_A : (U\,A)^n \longrightarrow U\,A$, and every morphism $f : A \longrightarrow B$ of $\mathbb{A}$ satisfies $f\,\omega_A = \omega_B\,f^n$. Thus ω is a natural transformation $\omega : D\,U \longrightarrow R\,U$, where $D\,f = f^n : E^n \longrightarrow F^n$ for a mapping $f : E \longrightarrow F$, and $R = \mathrm{Id}\,\mathrm{ENS}$.

9.2. ASSUMPTIONS. We are concerned with the following situation. A functor $U : \mathbb{A} \to \mathbb{C}$, a family $\Omega = (D^i, \omega^i, R^i)_{i \in I}$ of operations over U, and a top category $\mathbb{C}^q$ over $\mathbb{C}$ are given. Moreover, all functors D^i and R^i can be lifted to endofunctors Δ^i and P^i on $\mathbb{C}^q$, with structure maps δ^i_C and ρ^i_C respectively. We wish to lift $\mathbb{A}$ and U to a top category $\mathbb{A}^p$ and a functor $Y : \mathbb{A}^p \to \mathbb{C}^q$ such that every operation in Ω can be lifted to an operation $(\Delta^i, \tilde{\omega}^i, P^i)$ over Y. The following diagram illustrates this.

$$\begin{array}{ccc} \mathbb{A}^p & \xrightarrow{\;Y\;} & \mathbb{C}^q \\ P\downarrow & & \downarrow Q \\ \mathbb{A} & \xrightarrow{\;U\;} & \mathbb{C} \end{array}$$

We make $\mathbb{A}^p$ and Y unique up to isomorphism by requiring a pullback property.

9.3. THEOREM. Add to the stated assumptions the condition that P^i is taut over R^i for every $i \in I$. Then a top category $\mathbb{A}^p$ over $\mathbb{A}$ and a functor $Y : \mathbb{A}^p \to \mathbb{C}^q$ which lifts U exist, with the desired property and with the following pullback property. If $U F = Q \Phi$ for functors $F : \mathbb{X} \to \mathbb{A}$ and $\Phi : \mathbb{X} \to \mathbb{C}^q$, and if a family $(\Delta^i, \chi^i, P^i)_{i \in I}$ of operations over Y exists such that $Q\chi^i = \omega^i F$ for all $i \in I$, then $F = P\,T$ and $\Phi = Y\,T$ for a unique functor $T : \mathbb{X} \to \mathbb{A}^p$. These conditions determine $\mathbb{A}^p$ and Y up to an isomorphism of top categories which lifts $\mathrm{Id}\,\mathbb{A}$. **Moreover,** Y is taut over U and has injective structure maps ν_A.

Proof. For $A \in \mathrm{Ob}\,\mathbb{A}$, let $p\,A$ be the set of all $x \in q\,U\,A$ such that $\omega^i_A : \Delta^i(U\,A, x) \to P^i(U\,A, x)$ in $\mathbb{C}^q$, i.e. $\delta^i_{UA}\,x \le (\omega^i_A)^q \rho^i_{UA}\,x$, for all $i \in I$, and let $\nu_A : p\,A \to q\,U\,A$ be the inclusion map. We claim that this does the job.

If $A \in \mathrm{Ob}\,\mathbb{A}$ and $(x_k)_{k \in K}$ is a family of elements of $p\,A$, then

$$\delta^i_{UA}\left(\bigcap x_k\right) \leq \bigcap\left(\delta^i_{UA}\, x_k\right) \leq \bigcap\left((\omega^i_A)^q \rho^i_{UA}\, x_k\right) = (\omega^i_A)^q \rho^i_{UA}\left(\bigcap x_k\right)$$

for all $i \in I$, since the maps ρ^i_{UA} preserve infima. Thus $\bigcap x_k$, taken in $q\,U\,A$, is in $p\,A$. This shows that $p\,A$ is a complete lattice, and that υ_A preserves infima.

Similarly, if $f : A \longrightarrow B$ in $\mathbb{A}$ and $y \in p\,B$, then

$$\delta^i_{UA}\,(U\,f)^q\,y \leq (D\,U\,f)^q \delta^i_{UB}\,y \leq (D\,U\,f)^q\,(\omega^i_B)^q \rho^i_{UB}\,y$$

$$= (\omega^i_A)^q\,(R\,U\,f)^q \rho^i_{UB}\,y = (\omega^i_A)^q \rho^i_{UA}\,(U\,f)^q\,y\ ,$$

for $i \in I$, by (4.2.1) for Δ^i, naturality of ω^i, and tautness of P^i. Thus $(U\,f)^q$ maps $p\,B$ into $p\,A$, and $\upsilon_A\,f^p = (U\,f)^q\,\upsilon_B$ for a unique map $f^p : p\,B \longrightarrow p\,A$ in ORD.

The **maps** f^p clearly preserve infima, and the sets $p\,A$ and maps f^p define a contravariant functor $p : \mathbb{A}^{op} \longrightarrow \mathrm{ORD}$. Thus $\mathbb{A}^p$ is defined, and the maps υ_A are the structure maps of a functor $Y : \mathbb{A}^p \longrightarrow \mathbb{C}^q$ which lifts U tautly. Moreover, $\omega^i_A : \Delta^i\,Y(A,x) \longrightarrow P^i\,Y(A,x)$ in $\mathbb{C}^q$ for every object (A,x) of $\mathbb{A}^p$ by our construction, and thus the operations in Ω can be lifted to operations $(\Delta^i, \tilde{\omega}^i, P^i)$ over Y, as desired.

It remains to verify the pullback property. If F and Φ are given, then we can put $\Phi\,X = (U\,F\,X, \varphi_X)$, with $\varphi_X \in q\,U\,F\,X$, for $X \in \mathrm{Ob}\,\mathbb{X}$, and then $\varphi_X \leq (U\,F\,h)^q \varphi_Y$ for $h : X \longrightarrow Y$ in $\mathbb{X}$. The χ^i must satisfy

$$\chi^i_X = \omega^i_{FX} : \Delta^i\,(U\,F\,X, \varphi_X) \longrightarrow P^i\,(U\,F\,X, \varphi_X)$$

for $X \in \mathrm{Ob}\,\mathbb{X}$. Thus $\varphi_X \in p\,F\,X$ for all $X \in \mathrm{Ob}\,\mathbb{X}$. The only way to define T such that $F = P\,T$ and $\Phi = Y\,T$ is to put

$$T\,h = F\,h : (F\,X, \varphi_X) \longrightarrow (F\,Y, \varphi_Y)$$

for $h : X \longrightarrow Y$ in $\mathbb{X}$. Since $\varphi_X \in p\,F\,X$, this maps $\mathrm{Ob}\,\mathbb{X}$ into $\mathrm{Ob}\,\mathbb{A}^p$. We have $\varphi_X \leq (U\,F\,h)^q \varphi_Y = (F\,h)^p \varphi_X$, and thus $T\,h : T\,X \longrightarrow T\,Y$ in $\mathbb{A}^p$. Since $P\,T$ is a functor and P faithful, it follows that T is a functor.

The pullback property implies that $\mathbb{A}^p$ and Y are defined up to an isomorphism over $\mathrm{Id}\,\mathbb{A}$, and we have already proved the last part of the Theorem▯

9.4. COROLLARY. <u>If</u> $U : \mathbb{A} \longrightarrow \mathbb{C}$ <u>in</u> 9.3 <u>has a left adjoint</u> $F : \mathbb{C} \longrightarrow \mathbb{A}$, <u>then</u> F <u>can be lifted cotautly to a left adjoint</u> $\Phi : \mathbb{C}^q \longrightarrow \mathbb{A}^p$ <u>of</u> Y.

<u>Proof.</u> This follows immediately from 5.4▯

In 9.1, we call $\mathbb{A}$ <u>operational</u> over $\mathbb{C}$, for a family Ω of operations over U, if U is faithful, and for every morphism $f_1 : U\,A \longrightarrow U\,B$ in $\mathbb{C}$ such that $\omega_B\,(D\,f_1) = (R\,f_1)\,\omega_A$ for all operations in Ω, there is a (unique) morphism $f : A \longrightarrow B$ in $\mathbb{A}$ such that $f_1 = U\,A$.

9.5. PROPOSITION. <u>If</u> $\mathbb{A}$ <u>in</u> 9.3 <u>is operational over</u> $\mathbb{C}$, <u>for the given family</u> Ω <u>of operations, then</u> $\mathbb{A}^p$ <u>is operational over</u> $\mathbb{C}^q$ <u>for the family of all lifted operations</u> $(\Delta^i, \tilde{\omega}^i, \rho^i)$.

<u>Proof.</u> Since P is faithful, Y is faithful if U is. If $f_1 : (U\,A, x) \longrightarrow (U\,B, y)$ in $\mathbb{C}^q$, with $x \in p\,A$ and $y \in p\,B$, is homomorphic for the lifted operations, then $f_1 : U\,A \longrightarrow U\,B$ in $\mathbb{C}$ is homomorphic for the operations in Ω. If $f_1 = U\,f$ for $f : A \longrightarrow B$ in $\mathbb{A}$, then $x \leq f^p\,y = (U\,f)^q\,y$ in $p\,A$, and thus $f_1 = Y f$ for $f : (A,x) \longrightarrow (B,y)$ in $\mathbb{C}^q$. Combining these statements, one sees easily that $\mathbb{A}^p$ is operational if $\mathbb{A}$ is▯

9.6. APPLICATIONS. The familiar categories of Algebra are operational over ENS for operations $(D^n, \omega, \mathrm{Id}\,\mathrm{ENS})$ of finite arity n, where $D^n\,f = f^n$:

$E^n \longrightarrow F^n$ for a mapping $f : E \longrightarrow F$. Another operational category over ENS is the category of compact Hausdorff spaces, with a single operation (D,ω,R) , where D is the (covariant) ultrafilter functor on ENS , R = Id ENS , and ω_A is the convergence of ultrafilters for a compact Hausdorff space A .

If ENS^q is a top category over ENS , we lift Id ENS to Id ENS^q . This is of course taut over Id ENS , as required for a range functor by 9.3. Lifting D^n is easy: let $\Delta^n(E,x)$ be E^n provided with the product structure (see 5.3) of n copies of (E,x) . The usual categories of topological algebras result from 9.3 and these functors Δ^n if ENS^q = TOP . For other top categories ENS^q , we get categories of uniform algebras, of convergence algebras, of proximity algebras, and so on.

Another way of lifting the functors D^n is described in 9.7 below. This leads to categories of topological algebras with separately continuous operations.

If ENS^q is a top category with filter convergence, i.e. with a top functor Γ: $ENS^q \longrightarrow$ CONV to the category of convergence spaces, and with a suitable notion of regularity, then the ultrafilter functor D : ENS $\longrightarrow$ ENS can be lifted. A top category over compact Hausdorff spaces results. We have not investigated this queer animal.

By 5.3, all categorical limits and colimits can be lifted from a category $\mathbb{A}$ of algebras to a category $\mathbb{A}^p$ of algebras with some kind of topological structure. This is well known for limits, but seems to be not quite so well known for colimits. 9.4 was first discovered for topological groups by Samuel; the general result seems to be new.

The definition of algebras by operations is by now old-fashioned. However, it is much easier to lift operations than triples.

The following special case may be of interest. If $\mathbb{A}$ in 9.3 is a category of algebras and $\mathcal{C}^q = ENS^e$, the category of equivalence relations (3.6), then an object of $\mathbb{A}^p$ is an algebra $A \in Ob\,\mathbb{A}$ with a congruence relation.

9.7. SEPARATE CONTINUITY. Let $E = \prod E_i$ be the product of a family $(E_i)_{i\in I}$ of sets, with projections $\pi_i : E \longrightarrow E_i$. We call a mapping $h : E_i \longrightarrow E$ an <u>injection</u> of E_i into E if $\pi_i h = id\, E_i$, and $\pi_j h : E_i \longrightarrow E_j$ is a constant map for all $j \neq i$. This requires that either all sets E_i are non-empty, or all sets E_i empty.

If a top category ENS^q over ENS is given, and if $((E_i,x_i))_{i\in I}$ is a family of objects of ENS^q, then we call <u>weak product</u> of the (E_i,x_i) the object (E,x) of ENS^q with $E = \prod E_i$ and $x = \bigcup h_q x_i$, for all $i \in I$ and all injections $h : E_i \longrightarrow E$. If a family of maps $f_i : (E_i,x_i) \longrightarrow (E_i,y_i)$ is given, and if (E,x) and (F,y) are weak products, then we note that $\prod f_i : (E,x) \longrightarrow (F,y)$ in ENS^q. For if $h : E_i \longrightarrow E$ is an injection, then one sees easily that $(\prod f_i)\, h = k\, f_i$ for an injection $k : F_i \longrightarrow F$. Thus

$$f_q\, x = \bigcup (f_q\, h_q\, x_i) \leq \bigcup (k_q\, (f_i)_q\, x_i) \leq \bigcup (k_q\, y_i) = y ,$$

for $f = \prod f_i$ and all possible injections $h : E_i \longrightarrow E$ and $k : F_i \longrightarrow F$.

In particular, if $(E^n, \delta^n_E\, x)$ is the weak product of n copies of (E,x), then $f^n : (E^n,\delta^n_E\, x) \longrightarrow (F^n, \delta^n_E\, x)$ in ENS^q if $f : (E,x) \longrightarrow (E,y)$. This provides an other lifting of the functors D^n. A map $u : (E^n, \delta^n_E\, x) \longrightarrow (E,x)$ is called <u>separately continuous</u>. Thus this lifting of the functors D^n provides us with categories of <u>topological algebras with separately continuous operations</u>. 9.4 and the lifting of limits and colimits from $\mathbb{A}$ to $\mathbb{A}^p$ remain valid for these categories.

10. COTOP SUBCATEGORIES. Gleason pointed out first that TOP has non-trivial coreflective subcategories. Kennison proved that every non-trivial full replete coreflective subcategory of TOP is, in our language, a cotop subcategory. Herrlich pointed out that this is due to the fact that every object of TOP is initial (the empty space) or a generator. We provide in this section a generalization, due to Shukla, of Gleason's construction of coreflective subcategories of TOP, and we generalize Kennison's theorem.

10.1. THEOREM. A subcategory $\mathbb{B}$ of a top category $\mathbb{A}^p$ is a cotop subcategory of $\mathbb{A}^p$ if and only if there is a functor $T : \mathbb{A}^p \longrightarrow \mathbb{A}^p$ with the following properties.

10.1.1. T lifts id $\mathbb{A}$.

10.1.2. $\mathrm{id}\, A : T\,(A,x) \longrightarrow (A,x)$ in $\mathbb{A}^p$ for every object (A,x) of $\mathbb{A}^p$.

10.1.3. $T\,u = u$ for $u \in \mathbb{A}^p$ if and only if $u \in \mathbb{B}$.

Proof. If $\mathbb{B}$ is cotop, and if $I : \mathbb{B} \longrightarrow \mathbb{A}^p$ is the embedding functor and $J : \mathbb{A}^p \longrightarrow \mathbb{B}$ the top right adjoint of I, then $T = I\,J$ satisfies all conditions. Conversely, assume that T exists, with structure maps $\tau_A : p\,A \longrightarrow p\,A$ for $A \in \mathrm{Ob}\,\mathbb{A}$. Then $\tau_A\, x \leq x$ for $x \in p\,A$ by 10.1.2, and $\mathbb{B}$ is clearly a full subcategory of $\mathbb{A}^p$, with $(A,x) \in \mathrm{Ob}\,\mathbb{B}$ if and only if $(A,x) \in \mathrm{Ob}\,\mathbb{A}^p$ and $\tau_A\, x = x$. We show that $\mathbb{B}$ satisfies the dual conditions of 6.3.2.

If $(x_i)_{i \in I}$ is a family of elements of $p\,A$ such that $\tau_A\, x_i = x_i$ for all $i \in I$, then $x_i \leq \tau_A(\bigcup x_i) \leq \bigcup x_i$ for $i \in I$. Thus $\tau_A(\bigcup x_i) = \bigcup x_i$ and the dual of 6.3.2.(i) is valid for $\mathbb{B}$. If $f : A \longrightarrow B$ in $\mathbb{A}$ and $x \in p\,A$, then $f_p \tau_A\, x \leq \tau_B\, f_p\, x$ by the dual of (4.2.1). If $\tau_A\, x = x$, it follows that $f_p\, x \leq \tau_B\, f_p\, x \leq f_p\, x$, and thus the dual of 6.3.2.(ii) is satisfied∎

10.2. We recall some definitions. An object C of a category $\mathbb{A}$ is called <u>terminal</u> if for every $A \in \mathrm{Ob}\,\mathbb{A}$ there is exactly one morphism $f : A \to C$ in $\mathbb{A}$. Dually, C is called <u>initial</u> if for every $A \in \mathrm{Ob}\,\mathbb{A}$ there is exactly one $f : C \to A$ in $\mathbb{A}$. We call $C \in \mathrm{Ob}\,\mathbb{A}$ a <u>generator</u> of $\mathbb{A}$ if for every pair of morphisms $f : A \to B$ and $g : A \to B$ of $\mathbb{A}$ such that $f \neq g$ there is a morphism $u : C \to A$ such that $f\,u \neq g\,u$. A <u>bimorphism</u> of $\mathbb{A}$ is a morphism of $\mathbb{A}$ which is both epimorphic and monomorphic, and $\mathbb{A}$ is called <u>balanced</u> if the isomorphisms of $\mathbb{A}$ are the only bimorphisms. A full subcategory $\mathbb{B}$ of $\mathbb{A}$ is called <u>epireflective</u> (or monoreflective or bireflective) if $\mathbb{B}$ is reflective with epimorphic (or monomorphic or bimorphic reflections). <u>Monocoreflective</u> and allied terms are defined in the same way. We state and prove two results of Herrlich for coreflective full subcategories.

10.3. PROPOSITION. <u>If a full subcategory $\mathbb{B}$ of $\mathbb{A}$ is epicoreflective, then $\mathbb{B}$ is monocoreflective and hence bicoreflective.</u>

<u>Proof</u>. Consider a diagram

$$B' \xrightarrow{u'} A' \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{u} A ,$$

where u and u' are coreflections. If $u\,f = u\,g$, then $u\,(f\,u') = u\,(g\,u')$ with $f\,u'$ and $g\,u'$ in $\mathbb{B}$. But then $f\,u' = g\,u'$, and $f = g$ follows if u' is epimorphic $\square$

10.4. PROPOSITION. <u>If a full subcategory $\mathbb{B}$ of $\mathbb{A}$ is coreflective and there is a generator of $\mathbb{A}$ in $\mathrm{Ob}\,\mathbb{B}$, then $\mathbb{B}$ is bicoreflective.</u>

<u>Proof</u>. By 10.3, it is sufficient to show that every coreflection for $\mathbb{B}$ is

epimorphic in $\mathbb{A}$. Thus let $u : B \to A$ be a coreflection and $f u = g u$. If $h : C \to A$ with $C \in \mathrm{Ob}\, \mathbb{B}$, then $h = u h'$ for a unique $h' \in \mathbb{B}$, and it follows that $f h = g h$. If C is a generator of $\mathbb{A}$, then $f = g$ follows[]

The following result provides generators for top categories over ENS .

10.5. PROPOSITION. <u>If</u> ENS^p <u>is a top category over</u> ENS <u>such that</u> $p S$ <u>is a singleton if</u> S <u>is a singleton, then every object</u> (C,z) <u>of</u> ENS^p <u>with</u> C <u>non-empty is a generator of</u> ENS^p .

<u>Proof</u>. Let $f : (A,x) \to (B,y)$ and $g : (A,x) \to (B,y)$ with $f \neq g$, and let (S,s) be a singleton with its unique structure. There is a mapping $h : S \to A$ such that $f h \neq g h$, and then $f h k \neq g h k$ for the unique mapping $k : B \to S$. We have necessarily $\omega_S = s = h^p x$. Thus $k^p s = \omega_C$, and $h k : (C,z) \to (A,x)$ in ENS^p with $f h k \neq g h k$.

10.6. COROLLARY. <u>If</u> ENS^p <u>is a top category over</u> ENS <u>such that</u> $p S$ <u>is a singleton if</u> S <u>is a singleton, and if</u> $\mathbb{B}$ <u>is a full replete coreflective subcategory of</u> ENS^p <u>which has an object</u> (C,z) <u>with</u> C <u>non-empty, then</u> $\mathbb{B}$ <u>is a cotop subcategory of</u> ENS^p .

<u>Proof</u>. By 10.5 and 10.4, every coreflection $u : (B,y) \to (A,x)$ for $\mathbb{B}$ is bimorphic in ENS^p , and then u is bijective by 7.1. It follows that $u : (B,y) \to (A, u_p y)$ is an isomorphism of $\mathbb{B}$, by the dual of 7.2 and the fact that $\mathbb{B}$ is replete. But then $u^{-1} u = \mathrm{id}\, A : (A, u_p y) \to (A,y)$ is also a coreflection for $\mathbb{B}$, and $\mathbb{B}$ is cotop by the dual of 6.3[]

10.7. EXAMPLES AND REMARKS. The condition that $p S$ is a singleton for a singleton S is satisfied by many categories ENS^p of interest in general topology. Topological spaces, uniform spaces, convergence spaces, uniform convergence spaces, proximity spaces are examples. The following example shows that 10.6 is not valid without this condition.

Let ENS^p be the category of pairs of sets (3.7), and let $\mathbb{B}$ be the full subcategory with pairs (B,B) as objects. We claim that $\mathbb{B}$ is monocoreflective, but not epicoreflective, and hence not a cotop subcategory of ENS^p . Indeed, let $X \subset A$, and let $j : (X,X) \longrightarrow (A,X)$ be the inclusion map. This is monomorphic, but not epimorphic if $X \neq A$. If $f : (B,B) \longrightarrow (A,X)$, then f maps B into X , and thus $f = j\,f'$ for a unique map $f' : (B,B) \longrightarrow (X,X)$. Thus $j : (X,X) \longrightarrow (A,X)$ is a coreflection for $\mathbb{B}$.

ENS is of course not the only category with the properties needed for the proofs of 10.5 and 10.6. For example, every full subcategory of ENS with a singleton among its objects qualifies.

11. IMAGES AND RELATIONS. We present in this section the categorical background for a theory of continuous relations. Images in the sense of this section were investigated by H. Ehrbar and the author jointly, and relations by A. Klein and the author independently.

11.1. IMAGES. Let $\mathbb{A}$ be a category and $\mathbb{J}$ a class of morphisms of $\mathbb{A}$. We call $\mathbb{J}$-<u>image</u> of $f \in \mathbb{A}$ a pair (p,j) of morphisms of $\mathbb{A}$ such that $j\,p = f$ in $\mathbb{A}$ and $j \in \mathbb{J}$, and whenever $v\,f = g\,u$ in $\mathbb{A}$ with $g \in \mathbb{J}$, then $u = x\,p$ and $g\,x = v\,j$ in $\mathbb{A}$ for a unique $x \in \mathbb{A}$. Note that no conditions are put on $\mathbb{J}$ in this definition. In particular, we do not require that $\mathbb{J}$ consists of monomorphisms of $\mathbb{A}$. We say that $\mathbb{A}$ <u>has</u> $\mathbb{J}$-<u>images</u> if every $f \in \mathbb{A}$ has one. In the following, we usually omit the prefix $\mathbb{J}$.

We say that $f \in \mathbb{A}$ is $\mathbb{J}$-strict if for $v\,f = g\,u$ in $\mathbb{A}$ with $g \in \mathbb{A}$ there always is a unique $x \in \mathbb{A}$ such that $u = x\,f$ and $g\,x = v$ in $\mathbb{A}$. If p is $\mathbb{J}$-strict and $j\,p$ defined in $\mathbb{A}$ with $j \in \mathbb{J}$, then (p,j) is an image of $j\,p$. However, p need not be $\mathbb{J}$-strict if (p,j) is a $\mathbb{J}$-image.

11.2. SUBOBJECTS. Let us call f and f' in $\mathbb{A}$ left equivalent if $f' = f\,u$ in $\mathbb{A}$ for an isomorphism u of $\mathbb{A}$. A left equivalence class of morphisms $f \in \mathbb{A}$ with codomain $A \in \mathrm{Ob}\,\mathbb{A}$ and at least one representative in $\mathbb{J}$ is called a $\mathbb{J}$-subobject of A. It will be convenient to assume, and we shall do this, that there is a class $\mathbb{J}_0$ of morphisms of $\mathbb{A}$ such that every left equivalence class with a representative in $\mathbb{J}$ has exactly one representative in $\mathbb{J}_0$. If $f \in \mathbb{A}$ has a $\mathbb{J}$-image, it follows that f has a $\mathbb{J}_0$-image (p_0, j_0) with j_0 (but possibly not p_0) uniquely determined by f. If $\mathbb{A}$ has $\mathbb{J}$-images, then every isomorphism of $\mathbb{A}$ is left equivalent to some $j \in \mathbb{A}$. Thus we may assume, without loss of generality, that all identity maps of $\mathbb{A}$ are in $\mathbb{J}_0$, and we shall do this. It follows that $f : A \longrightarrow B$ is $\mathbb{J}$-strict if and only if $(f,\ \mathrm{id}\ B)$ is the (unique) $\mathbb{J}_0$-image of f.

11.3. IMAGE FUNCTORS. $\mathbb{J}$ defines a full subcategory $\mathbb{A}^2[\mathbb{J}]$ of the morphism category $\mathbb{A}^2$ (see 1.2); we denote by I the embedding functor. One sees easily that (p,j) is a $\mathbb{J}$-image of $f : A \longrightarrow B$ in $\mathbb{A}$ if and only if $(p,\ \mathrm{id}\ B) : f \longrightarrow j$ is a reflection for $\mathbb{A}^2[\mathbb{J}]$ in $\mathbb{A}^2$. It follows that $\mathbb{A}$ has $\mathbb{J}$-images if and only if I has a left adjoint functor $\mathrm{im} : \mathbb{A}^2 \longrightarrow \mathbb{A}^2[\mathbb{J}]$ and a front adjunction $\pi : \mathrm{id}\,\mathbb{A}^2 \longrightarrow I\ \mathrm{im}$ such that $D_1\ I\ \mathrm{im} = D_1$ and $D_1 \pi = \mathrm{id}\ D_1$. We call the pair (im, π), or by abuse of language just the functor im, a $\mathbb{J}$-image functor for $\mathbb{A}$.

We make the image functor im (but not necessarily π) unique by requiring that $\mathrm{im}\, f \in \mathbb{J}_0$ for every $f \in \mathbb{J}$. This is no loss of generality.

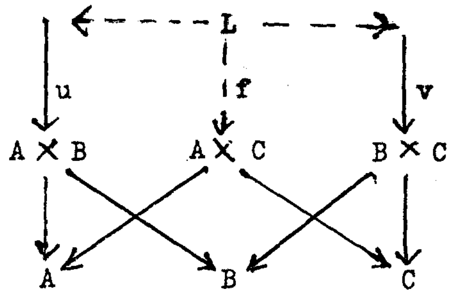
If $\mathbb{A}$ has $\mathbb{J}$-images, then the functor $D_1 I : \mathbb{A}^2[\mathbb{J}] \longrightarrow \mathbb{A}$ is an opfibration (1.1). If $\mathbb{A}$ has pullbacks and images, then $D_1 I$ is also a fibration. A P-fibred morphism of $\mathbb{A}^2[\mathbb{J}]$, for $P = D_1 I$, is called an <u>inverse image</u>, and a P-obfibred morphism a <u>direct image</u>.

11.4. RELATIONS. We assume from now on that $\mathbb{A}$ has finite limits, i.e. products and pullbacks, and $\mathbb{J}$-images. If A and B are objects of $\mathbb{A}$, then a $\mathbb{J}$-subobject of $A \times B$ is called a $\mathbb{J}$-<u>relation</u> from A to B. It will be convenient to replace the $\mathbb{J}$-subobject by its unique representative in $\mathbb{J}_0$; we shall always do so. A change in $\mathbb{J}_0$ will change this representative, but it does not change the composition of relations or anything else which we may want to do with relations.

Relations from A to B are the objects of a category Rel (A,B). Morphisms of this category are morphisms $(u,v) : j \longrightarrow j'$ in $\mathbb{C}^2[\mathbb{J}]$ with j, j' in $\mathbb{J}_0$ and $v = \mathrm{id}\,(A \times B)$. If $\mathbb{A}$ is $\mathbb{J}$-locally small, i.e. if subobjects of every $A \in \mathrm{Ob}\,\mathbb{A}$ form a set, then Rel (A,B) is small. If $\mathbb{A}$ is $\mathbb{J}$-locally small and $\mathbb{J}$ consists of monomorphisms of $\mathbb{A}$, then Rel (A,B) is an ordered set (partly ordered set if the reader prefers to say so) in which every finite family has an infimum. If $\mathbb{A}$ is not only finitely complete but complete, then Rel (A,B) is a complete lattice.

If $A \xleftarrow{f} C \xrightarrow{g} B$ in $\mathbb{A}$, then f and g induce a unique morphism $\{f,g\} : C \longrightarrow A \times B$, and $\mathrm{im}\{f,g\}$ is (a representative of) a relation from A to B. We denote this relation by $[f,g]$.

11.5. COMPOSITION OF RELATIONS. Two relations $u : A \to B$ and $v : B \to C$ over $\mathbb{A}$, i.e. subobjects u of $A \times B$ and v of $B \times C$, determine a diagram as follows (with dashed arrows omitted).



In this diagram, arrows like $A \times B \to A$ are projections. We construct a limit L of the diagram in $\mathbb{A}$, with the dashed arrows as projections, and we put

$$v \circ u = \operatorname{im} f : A \to C ,$$

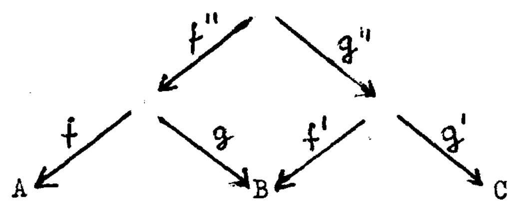
for the projection $f : L \to A \times C$ of this limit.

Composition of relations defines **composition functors**

$$\operatorname{Rel}(A,B) \times \operatorname{Rel}(B,C) \to \operatorname{Rel}(A,C) ,$$

but composition of three relations is in general not associative.

11.6. Consider a diagram in $\mathbb{A}$



in which the square is a pullback. If the functor $I\,\mathrm{im} : \mathbb{A}^2 \to \mathbb{A}^2$ preserves pullback squares, then we always have

(11.6.1) $$[f',g'] \circ [f,g] = [f\,f'', g'\,g'']$$

in this situation. It follows that composition of relations over $\mathbb{A}$ is associative if I im preserves pullbacks. In general, (1.6.1) can be proved only for the case that $\{f,g\}$ and $\{f',g'\}$ are in $\mathbb{J}$, up to left equivalence.

11.7. We define a relation rel f over $\mathbb{A}$ for $f \in \mathbb{A}$ by putting

$$\text{rel } f = [\text{id } A, f]$$

for $f : A \longrightarrow B$ in $\mathbb{A}$. A relation of this form is called functional.

If every coretraction in $\mathbb{A}$ (i.e. every $f \in \mathbb{A}$ with a left inverse) is in $\mathbb{J}$, up to left equivalence, then (11.6.1) is valid for functional relations, and rel is a functor, or more exactly a pseudofunctor, from $\mathbb{A}$ to the "bicategoroid" of relations over $\mathbb{A}$. Moreover, rel id A = [id A, id A] acts as identity relation on A, for $A \in \text{Ob}\,\mathbb{A}$, not only for functional relations but for all relations in this situations.

11.8. EXAMPLES AND REMARKS. Good behavior of relations clearly depends on two properties which images may or may not have:

(i) the functor I im : $\mathbb{A}^2 \to \mathbb{A}^2$ preserves pullback squares;

(ii) every coretraction in $\mathbb{A}$ is in $\mathbb{J}$, up to left equivalence.

In the more special situation investigated by A. Klein, (ii) is always satisfied, and (i) is equivalent to the universal validity of (11.6.1), and hence to the associative law for the composition of relations.

Examples are: sets with subsets as subobjects; groups with subgroups as subobjects; topological spaces or Hausdorff spaces with subspaces as subobjects; topological spaces or Hausdorff spaces with closed subspaces as subobjects; groups with normal subgroups as subobjects. (i) and (ii) are valid for the first four examples. $\mathbb{J}$ consists of all extremal monomorphisms of $\mathbb{A}$ in the first three examples, and for Hausdorff spaces with closed subspaces as subobjects. Hausdorff spaces with closed subspaces satisfy (ii), but not (i). In the last example, groups with normal subgroups, im = ker coker defines an image functor, but neither (i) nor (ii) is valid.

12. CONTINUOUS RELATIONS. A relation $u : A \longrightarrow B$ between topological spaces has been called continuous if $u^{-1}(Y)$ is closed in A for every closed $Y \subset B$, and open for every open Y. This works reasonably well but it can hardly be generalized to other situations. We propose in this section a general definition of continuous relations over a top category which seems to work quite well. For TOP, our definition is not equivalent, but quite close, to the definition mentioned above.

12.1. LEMMA. <u>If</u> $\mathbb{A}^p$ <u>is a top category and</u> $f : A \longrightarrow B$ <u>in</u> $\mathbb{A}$, <u>then the following statements are valid</u>.

12.1.1. $x \leq f^p y \iff f_p x \leq y$ <u>for all</u> $x \in pA$ <u>and</u> $y \in pB$.

12.1.2. $x \leq f^p f_p x$ <u>and</u> $f_p f^p y \leq y$ <u>for all</u> $x \in pA$ <u>and</u> $y \in pB$.

12.1.3. $f^p f_p f^p = f^p$ <u>and</u> $f_p f^p f_p = f_p$.

12.1.4. f_p <u>injective</u> $\iff$ f^p <u>surjective</u> $\iff$ $f^p f_p = \mathrm{id}\ pA$, <u>and</u> f_p <u>surjective</u> $\iff$ f^p <u>injective</u> $\iff$ $f_p f^p = \mathrm{id}\ pB$.

<u>Proof</u>. 12.1.1 is a restatement from 2.5, and each of the other statements follows immediately from the preceding one $\Box$

12.2. PROPOSITION. <u>If</u> $\mathbb{A}^p$ <u>is a top category over</u> $\mathbb{A}$, <u>and if a commutative square</u>

$$\begin{array}{ccc} & \xrightarrow{\;u\;} & B \\ \Big\downarrow{\scriptstyle v} & & \Big\downarrow{\scriptstyle g} \\ A & \xrightarrow{\;f\;} & \end{array}$$

in $\mathbb{A}$ *is given, then* $u_p\, v^p\, x \leq g^p\, f_p\, x$ *in* $p\,B$ *for every* $x \in p\,A$.

Proof. Using 12.1.2, we have, equivalently,

$$v^p\, x \;\leq\; v^p\, f^p\, f_p\, x \;=\; u^p\, g^p\, f_p\, x$$

for $x \in p\,A$ []

12.3. DEFINITION. Let $\mathbb{A}^p$ be a top category over $\mathbb{A}$, and let $\mathbb{A}$ have finite products and pullbacks, and $\mathbb{J}$-images for some class $\mathbb{J}$ of morphisms of $\mathbb{A}$, so that relations over $\mathbb{A}$ are defined. For objects (A,x) and (B,y) of $\mathbb{A}^p$, we define a *relation*, or *continuous relation*, $u : (A,x) \longrightarrow (B,y)$ as a relation $u : A \longrightarrow B$ with the following property. If (C,z) is an object of $\mathbb{A}^p$ and if $f : C \longrightarrow A$ and $g : C \longrightarrow B$ are morphisms of $\mathbb{A}$ such that $\{f,g\} = u\,\varphi$ for some $\varphi \in \mathbb{A}$, then $z \leq f^p\, x$ always implies $z \leq g^p\, y$.

In other words, we require that continuity of f always implies continuity of g if $\{f,g\}$ factors through the subobject u of $A \times B$ in $\mathbb{A}$.

The following added definition will be convenient. We say that $\mathbb{J}$-images are *p-taut* if $u^p\, u_p = \mathrm{id}\; p\,X$ for every $\mathbb{J}$-image (u,j) of a morphism $f : A \xrightarrow{u} X \xrightarrow{j} B$ in $\mathbb{A}$.

12.4. PROPOSITION. *If a relation* $u : A \longrightarrow B$ *over* $\mathbb{A}$ *is represented by* $\{\bar{f},\bar{g}\}$ *for* $A \xleftarrow{\bar{f}} \quad \xrightarrow{\bar{g}} B$ *in* $\mathbb{A}$, *then* $u : (A,x) \longrightarrow (B,y)$ *for* $x \in p\,A$ *and* $y \in p\,B$ *if and only if* $\bar{g}_p\, \bar{f}^p\, x \leq y$, *or* $\bar{f}^p\, x \leq \bar{g}^p\, y$.

Proof. We have $u = \{\bar{f}, \bar{g}\}$, up to left equivalence. If $\{f,g\} = u\varphi$, it follows that $f = \bar{f}\varphi$ and $g = \bar{g}\varphi$, and

$$g_p f^p x = \bar{g}_p \varphi_p \varphi^p \bar{f}^p x \leq \bar{g}_p \bar{f}^p x$$

for $x \in pA$, by 12.1.2. Thus $u : (A,x) \longrightarrow (B,y)$ if $\bar{g}_p \bar{f}^p x \leq y$ []

12.5. PROPOSITION. Let $\mathbb{A}^p$ be a top category over $\mathbb{A}$, and let $\mathbb{A}$ have p-taut images. If $A \xleftarrow{f} \cdot \xrightarrow{g} B$ in $\mathbb{A}$, then $[f,g] : (A,x) \longrightarrow (B,y)$ over $\mathbb{A}^p$, for $x \in pA$ and $y \in pB$, if and only if $g_p f^p x \leq y$.

Proof. If $[f,g]$ is represented by $j = \{\bar{f}, \bar{g}\}$ in $\mathbb{D}$, then $f = \bar{f}u$ and $g = \bar{g}u$ for an image (u,j) of $\{f,g\}$. It follows that

$$g_p f^p x = \bar{g}_p u_p u^p \bar{f}^p x = \bar{g}_p \bar{f}^p x ,$$

and now 12.5 follows immediately from 12.4[]

12.6. PROPOSITION. Let $\mathbb{A}^p$ be a top category over $\mathbb{A}$, and let $\mathbb{A}$ have p-taut images. If $u : (A,x) \longrightarrow (B,y)$ and $v : (B,y) \longrightarrow (C,z)$ are continuous relations, then $v \circ u : (A,x) \longrightarrow (C,z)$ is continuous. If $f \in \mathbb{A}$, then $f : (A,x) \longrightarrow (B,y)$ in $\mathbb{A}^p$ if and only if $[\mathrm{id}\, A, f] : (A,x) \longrightarrow (B,y)$ over $\mathbb{A}^p$.

Proof. If $\{f,g\}$ represents u and $\{f',g'\}$ represents v, then we can use (11.6.1) for the product, and we find that

$$(g' g'')_p (f f'')^p x = (g')_p (g'')_p (f'')^p f^p x$$
$$\leq (g')_p (f')^p g_p f^p x \leq (g')_p (f')^p y \leq z ,$$

with 12.2 used in the second step, if u and v are continuous. Now use 12.5. Also by 12.5, $f_p x \leq y$ is the continuity condition for $[\mathrm{id}\, f, f]$ and for f []

12.7. REMARKS. If we use the usual images in ENS , with subsets as subobjects, then relations over ENS are relations in the usual elementary sense. In this case, images are always taut. For if (u,j) is an image, then u is surjective and has a right inverse. Thus u^p has a left inverse and is injective, and $u_p u^p$ is an identity map by 12.1.4. More generally, images in $\mathbb{A}$ are always p-taut if $\mathbb{A}$ is a category of algebras, with subalgebras as subobject, and with $\mathbb{A}^p$ constructed from a top category ENS^q as in section 9.

Except for trivial situations, relations $u : (A,x) \longrightarrow (B,y)$ over a top category $\mathbb{A}^p$ are not relations over $\mathbb{A}^p$ in the sense of 11.4, and the results of section 11 apply only to the underlying relations $u : A \longrightarrow B$ over $\mathbb{A}$. Relations in the sense of 11.4 would be subspaces of product spaces, and not continuous relations in any useful meaning of this term.

Much work remains to be done on continuous relations; this section is just a start. Many questions remain. For example, we have a commutative diagram

$$\begin{array}{ccc} \mathbb{A}^p & \longrightarrow & \mathrm{Rel}\,\mathbb{A}^p \\ \downarrow{\scriptstyle p} & & \downarrow \\ \mathbb{A} & \longrightarrow & \mathrm{Rel}\,\mathbb{A} \end{array}$$

with bicategories, in the Bénabou sense, of continuous relations over $\mathbb{A}^p$ and of relations over $\mathbb{A}$ at right. What is the exact categorical nature of this diagram? This is just one of many questions.

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