CAUCHY TRIPLES*

by

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Introductory

The present report is a first write-up of a new theory. The theory may not yet be in its final form, but there are enough results to justify a belief that it will be useful. Due to the preliminary nature of this report, no references are given in the text, and some concepts related to Cauchy triples are used without explanation or with an explanation long after their first use. The reader is asked to bear with this: rewriting these notes would have delayed their appearance unduly. A list of 19 references is appended to the report, and a preprint survey on top categories is planned.

The theory of Cauchy triples had its origin in the observation that many completion and compactification constructions in general topology follow a rather rigid pattern. An early example of this pattern is the construction of the completion of a uniform space in the first edition of Bourbaki's Topologie générale, chap. II. (Later editions use a different construction which is based on special properties of uniform spaces.) Another early example is the construction of the Stone-Čech compactification of a topological space E via a compact, but not Hausdorff, space E^* of ultrafilters. Frink's construction of Wallman type compactifications [7] and the discussion of completions and compactifications for Cauchy spaces in J.F. Ramaley's thesis (see [15]) made the pattern clear. Once the pattern was clear, it was easily seen that it fits into the categorical theory of triples in a special way. This helped to illuminate the pattern further. It showed in particular that the precompletions used in all earlier examples and recognized in [15] are of purely categorical nature.

The results of [15] made it clear that regularity in topology is related to Cauchy triples and hence to completions. The exact formulation of the connection between regularity and Cauchy triples required the concept of a continuous relation for a general topological situation. Our continuity concept for relations is purely categorical and seems to work rather nicely. As a new application of our general theory of regularity, we discuss regular uniform convergence spaces. A satisfactory theory of such spaces has been an open problem for some time.

One aim of a topological completion theory is to obtain a universal separated regular completion for every space. Such a completion need not be an embedding; it may be a non-trivial problem to determine those spaces for which the completion is an embedding. For a Cauchy triple, a separated regular complete space has a unique algebra structure, given by filter convergence, and every map of separated regular complete spaces is a morphism of algebras for the triple. For sufficiently nice categories, the existence of universal separated completions follows from general category theory. Unfortunately, some important categories, such as convergence spaces, closure spaces, and uniform convergence spaces, refuse to be sufficiently nice. For these categories, completions may exist, but they have to be constructed.

As presented in this report, the theory of Cauchy triples does not include the Wallman type compactifications of Frink [7] and others. It can easily be extended to include such compactifications, but the present state of the theory of Wallman compactifications does not encourage categorical considerations.

A satisfactory completion or compactification theory for uniform or topological algebras has yet to be obtained. It is no problem to carry algebraic operations from elements of a set to subsets of the set and hence to filters on the set. However, formal laws easily manage to get lost in the process, and compression is usually not a homomorphism of algebras. For example, if a binary operation + is carried to subsets of and filters on a set E in the obvious way, and if a set E^* of Cauchy filters is given which is closed for + , then

$$X^* + Y^* \subset (X + Y)^*$$

for subsets X , Y of E , and the inclusion can be proper. Thus

$$(\Phi + \Psi)_* \leq \Phi_* + \Psi_*$$

for filters Φ , Ψ on E^* , and the inequality may be strict.

Nets instead of filters may be - horribili dictu - a solution. Nets over an algebra, with a fixed directed set as domain, certainly obey all formal laws of the algebra. However, compression may be a problem. There is e.g. no natural compression from Cauchy sequences of Cauchy sequences on a metric space to Cauchy sequences on the space.

All Cauchy triples considered in this report have remarkable formal analogies. They may be instances of a structure which is richer than a Cauchy triple, or it may be that these analogies represent properties built into the definition but not yet obtained. This is at present a wide open problem.

There are other open questions, but let the list presented here suffice.

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C A U C H Y   T R I P L E S

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1. FILTER FUNCTORS. There is a contravariant as well as a covariant filter functor on ENS , and the two functors are related by a Galois correspondence, as follows.

For a set A , denote by $\mathbb{F}A$ the set of all filters on A , including the improper or null filter, generated by the empty set and consisting of all subsets of A . We put $\varphi \leq \varphi_1$ if $\varphi_1 \subset \varphi$. With this relation, $\mathbb{F}A$ is a complete lattice, with the null filter as finest filter on A and the trivial filter $\{A\}$ as the coarsest filter. The supremum $\bigcup \varphi_i$ of a family $(\varphi_i)_{i \in I}$ of filters on A is the set intersection of all φ_i , considered as sets of subsets of A . More to the point, $\bigcup \varphi_i$ consists of all sets $\bigcup X_i$ with $X_i \in \varphi_i$ for all $i \in I$.

For a mapping $f : A \longrightarrow B$ and filters φ on A and ψ on B , let $f_*(\varphi)$ denote the filter on B generated by the sets $f(X)$ with $X \in \varphi$, and $f^*(\psi)$ the filter on A generated by the sets $f^{-1}(Y)$ with $Y \in \psi$. The filter $f_*(\varphi)$ is improper only if φ is, but if f is not surjective, then $f^*(\psi)$ may well be the null filter on A for proper ψ .

1.1. PROPOSITION. *If* $f : A \to B$ *in* ENS *and* φ *is a filter on* A , *then* $Y \in f_*(\varphi) \iff f^{-1}(Y) \in \varphi$, *for* $Y \subset B$.

<u>Proof</u>. If $Y \in f_*(\varphi)$, then $f(X) \subset Y$, hence $X \subset f^{-1}(Y)$, for some $X \in \varphi$, and thus $f^{-1}(Y) \in \varphi$. Conversely, if $f^{-1}(Y) \in \varphi$, then $f(f^{-1}(Y))$ is in $f_*(\varphi)$ and $f(f^{-1}(Y)) \subset Y$. Thus $Y \in f_*(\varphi)$ $\square$

1.2. PROPOSITION. <u>Let</u> $f : A \longrightarrow B$ <u>in</u> ENS. <u>If</u> φ <u>and</u> ψ <u>are filters on</u> A <u>and</u> B <u>respectively, then</u> $f_*(\varphi) \leq \psi \iff \varphi \leq f^*(\psi)$.

<u>Proof</u>. By 1.1 and the definitions, both statements are equivalent to the implication $Y \in \psi \implies f^{-1}(Y) \in \varphi$, for all $Y \subset B$ $\square$

It is clear from the definitions that we obtain a covariant functor $\mathbb{F}_*$ and a contravariant functor $\mathbb{F}^*$ on ENS by putting $\mathbb{F}_* f = f_* : \mathbb{F}A \longrightarrow \mathbb{F}B$ and $\mathbb{F}^* f = f^* : \mathbb{F}B \longrightarrow \mathbb{F}A$ for every $f : A \longrightarrow B$ in ENS. If we regard the complete lattices $\mathbb{F}A$ as complete categories, then the maps f_* and f^* themselves are functors. 1.2 then states that f_* is left adjoint to f^* for every $f \in$ ENS. It follows immediately that f_* preserves suprema and f^* infima.

2. CAUCHY TRIPLES. We consider a concrete category $\mathbb{C}$, i.e. one with a faithful functor $U : \mathbb{C} \rightarrow$ ENS.

2.1. DEFINITION. A <u>Cauchy triple on</u> $\mathbb{C}$ consists of the following.

2.1.1. There is a functor $R : \mathbb{C} \rightarrow \mathbb{C}$ such that $U\,R\,A$ is a set of filters on $U\,A$ for every $A \in \mathrm{Ob}\,\mathbb{C}$, the mapping $(U\,f)_*$ maps $U\,R\,A$ into $U\,R\,B$ for every $f : A \rightarrow B$ in $\mathbb{C}$, and $U\,R\,f$ is the restriction of $(U\,f)_*$ to $U\,R\,A$ and $U\,R\,B$.

2.1.2. For $A \in \mathrm{Ob}\,\mathbb{C}$, the filter $\dot{x}$ on $U\,A$ is in $U\,R\,A$ for all $x \in U\,A$, and $(U\,j_A)(x) = \dot{x}$ defines a morphism $j_A : A \longrightarrow R\,A$ in $\mathbb{C}$.

2.1.3. For every $A \in \mathrm{Ob}\,\mathbb{C}$, there is a morphism $k_A : R\,R\,A \longrightarrow R\,A$ in $\mathbb{C}$ obtained as follows. We put

$$X^* = \{\varphi \in U\,R\,A : X \in \varphi\} \quad \text{and} \quad \Phi_* = \{X \subset U\,A : X^* \in \Phi\} ,$$

for all $X \subset U\,A$ and all filters Φ on $U\,R\,A$. We require that $\Phi_* \in U\,R\,A$ for $\Phi \in U\,R\,R\,A$, and that $(U\,k_A)(\Phi) = \Phi_*$ for $\Phi \in U\,R\,R\,A$.

One sees easily that

$$(X \cap Y)^* = X^* \cap Y^* \quad \text{and} \quad \dot{x} \in X^* \iff x \in X ,$$

for subsets X, Y of $U\,A$ and $x \subset U\,A$. This shows that Φ_* is a filter on $U\,A$ for a filter Φ on $U\,R\,A$, and that Φ_* is proper if Φ is.

2.2. THEOREM. <u>The data of</u> 2.1 <u>define a functor</u> $R : \mathbb{C} \to \mathbb{C}$ <u>and natural transformations</u> $j : \mathrm{Id}\,\mathbb{C} \longrightarrow R$ <u>and</u> $k : R\,R \longrightarrow R$ <u>which form a triple over</u> $\mathbb{C}$.

<u>Proof</u>. This is an exercise in filter algebra. We must verify the formulas:

$$(R\,f)\,j_A = j_B\,f , \quad (R\,f)\,k_A = k_B\,(R\,R\,f) ,$$

$$k_A\,j_{RA} = \mathrm{id}\,R\,A = k_A\,(R\,j_A) , \quad k_A\,k_{RA} = k_A\,(R\,k_A) ,$$

for $f : A \longrightarrow B$ in $\mathbb{C}$ and $A \in \mathrm{Ob}\,\mathbb{C}$. At the set level, they are formulas 2.3.1, 2.3.3, 2.3.5, and 2.3.7, and faithfulness of U lifts these to $\mathbb{C}$ ▯

2.3. Consider $A \in \mathrm{Ob}\,\mathbb{C}$ and $f : A \longrightarrow B$ in $\mathbb{C}$. We write f_* for the restriction $U\,R\,f$ of the filter mapping $(U\,f)_*$.

2.3.1. If $x \in U\,A$ and $y \in U\,B$, then $x = (U\,f)(y) \iff \dot{x} = f_*(\dot{y})$.

2.3.2. If $Y \subset U\,B$, then $(f_*)^{-1}(Y^*) = ((U\,f)^{-1}(Y))^*$.

2.3.3. If $\Phi \in U\,R\,R\,A$, then $((R\,f)_*(\Phi))_* = f_*(\Phi_*)$.

2.3.4. If $X \subset U\,A$, then $X = (U\,j)^{-1}(X^*)$.

2.3.5. If $\varphi \in U\,R\,A$, then $((U\,j_{RA})(\varphi))_* = \varphi = ((j_A)_*(\varphi))_*$.

2.3.6. If $X \subset U\,A$, then $(U\,k_A)^{-1}(X^*) = (X^*)^*$.

2.3.7. If $\mathcal{F} \in U\,R\,R\,R\,A$, then $((k_A)_*(\mathcal{F}))_* = (\mathcal{F}_*)_*$.

Proof. All proofs are straightforward; we prove 2.3.3, 2.3.6 and 2.3.7 as examples. For 2.3.3, we note that

$$Y \in ((R\,f)_*(\Phi))_* \iff Y^* \in (R\,f)_*(\Phi) \iff (f_*)^{-1}(Y^*) \in \Phi ,$$

and
$$Y \in f_*(\Phi_*) \iff (U\,f)^{-1}(Y) \in \Phi_* \iff ((U\,f)^{-1}(Y))^* \in \Phi ,$$

for $Y \subset U\,B$, and we use 2.3.2. For 2.3.6, we note that

$$\begin{aligned} \Phi \in (U\,k_A)^{-1}(X^*) &\iff \Phi_* \in X^* \iff X \in \Phi_* \\ &\iff X^* \in \Phi \iff \Phi \in (X^*)^* , \end{aligned}$$

for $\Phi \in U\,R\,R\,A$. For 2.3.7, we note that

$$X \in ((k_A)_*(\mathcal{F}))_* \iff X^* \in (k_A)_*(\mathcal{F}) \iff (U\,k_A)^{-1}(X^*) \in \mathcal{F} ,$$

and
$$X \in (\mathcal{F}_*)_* \iff X^* \in \mathcal{F}_* \iff (X^*)^* \in \mathcal{F} ,$$

for $X \subset U\,A$, and we use 2.3.6 ▯

2.4. PROPOSITION. $\Phi_* = \inf_{P \in \Phi} \sup_{\varphi \in P} \varphi$ <u>for a filter</u> Φ <u>on</u> $U\,R\,A$.

Proof. Put $P_* = \sup \varphi$ for all $\varphi \in P$. Then $X \in P_*$ if and only if $X \in \varphi$ for all $\varphi \in P$, and thus if and only if $P \subset X^*$. The filter $\inf P_*$, for $P \in \Phi$, thus is generated by the sets $X \subset U\,A$ with $P \subset X^*$ for some $P \in \Phi$. These are exactly the sets $X \subset U\,A$ with $X^* \in \Phi$ which form the filter Φ_* ▯

3. CAUCHY TRIPLES WITH CONVERGENCE. A Cauchy triple with convergence is one over a category with convergence which satisfies some additional conditions.

3.1. DEFINITION. We call a concrete category $\mathbb{C}$, with "forgetful" functor $U : \mathbb{C} \longrightarrow ENS$, a <u>category with convergence</u> if a convergence structure q_A on the set $U\,A$ is associated with every object A of $\mathbb{C}$ in a functorial manner, i.e. if $f : A \longrightarrow B$ in $\mathbb{C}$, then $U\,f : (U\,A, q_A) \longrightarrow (U\,B, q_B)$ is a continuous map of convergence spaces. In other words, the functor $U : \mathbb{C} \longrightarrow ENS$ factors through the forgetful functor from convergence spaces to sets.

We denote by CONV the category of convergence spaces. It is of course a category with convergence. The categories studied in general topology don't have to be, but always are, categories with convergence.

3.2. DEFINITION. A Cauchy triple on a category $\mathbb{C}$ with convergence is called a <u>Cauchy triple with convergence</u> if the following conditions are satisfied, for every object A of $\mathbb{C}$.

3.2.1. If $\varphi\, q_A\, x$ and $\Phi_* \leq \varphi$, then $\Phi\, q_{RA}\, \dot{x}$.

3.2.2. If $\Phi \in U\,R\,R\,A$, then $\Phi\, q_{RA}\, \Phi_*$.

3.2.3. If $\Phi\, q_{RA}\, \varphi$ and $\varphi\, q_A\, x$, then $\Phi_*\, q_A\, x$.

Here of course $x \in U\,A$, and φ and Φ are filters on $U\,A$ and on $U\,R\,A$.

If a Cauchy triple (R, j, k) with convergence is given, then we call an object A of $\mathbb{C}$ <u>separated</u>, with respect to the given Cauchy triple, if every Cauchy filter of A, i.e. every $\varphi \in U\,R\,A$, converges for q_A to at most one point $x \in U\,A$, and we call A <u>complete</u> if every Cauchy filter converges to at least one point. We shall call A <u>regular</u> if q_A is continuous in some sense which we specify later.

3.3. THEOREM. <u>If</u> (A,u) <u>is an algebra for a Cauchy triple</u> (R, j, k) <u>with convergence, then</u> $\varphi\, q_A\, x$ <u>for every</u> $\varphi \in U\,R\,A$ <u>and</u> $x = (U\,u)(\varphi)$.

<u>Proof</u>. We have $u\, j_A = \mathrm{id}\, A$ and hence $(R\, u)(R\, j_A) = \mathrm{id}\, R\, A = k_A\, (R\, j_A)$. Thus $(U\,R\,u)(\Phi) = \varphi = \Phi_*$ for $\Phi = (U\,R\,j_A)(\varphi)$. We have $\Phi\, q_{RA}\, \varphi$ by 3.2.2, and $\varphi\, q_A\, x$ follows since $U\,u : (U\,R\,A,\, q_{RA}) \longrightarrow (U\,A,\, q_A)$ in CONV ▯

3.4. REMARKS. We guarantee by 3.2.2 that $R\,A$ is complete for every object A of $\mathcal{C}$, but in general $R\,A$ will be neither separated nor regular. A is separated and complete if and only if convergence of filters induces a mapping from $U\,R\,A$ to $U\,A$. Regularity means in this situation that the mapping can be lifted to a morphism $u : R\,A \longrightarrow A$, and we shall see that (A,u) always is an algebra for (R, j, k) in this situation.

4. EXAMPLES. We discuss Cauchy triples for topological spaces, for convergence spaces, and for categories in between.

4.1. TOPOLOGICAL SPACES. We denote by TOP the category of topological spaces. We assume that a set A^* of filters on the underlying set $U\,A$ is assigned to every topological space A so that the set theoretic parts of 2.1.1 and 2.1.2 are satisfied. We define a topology on A^* by using all sets X^* , with X open for A , as a basis of open sets. Since $(X \cap Y)^* = X^* \cap Y^*$ for subsets X , Y of $U\,A$, this works. By 2.3.2, the mapping $f_* : A^* \longrightarrow B^*$ is continuous for $f : A \longrightarrow B$ continuous, and thus the functor R is defined. By 2.3.4, $(U\,j_A)(x) = \dot{x}$ defines a continuous mapping $j_A : A \longrightarrow R\,A$, and 2.1.2 is satisfied. Having $R\,A$, we have $(R\,A)^* = U\,R\,R\,A$, and we require that $\Phi_* \in U\,R\,A$ for $\Phi \in U\,R\,R\,A$. This defines k_A at the set level, and

k_A is continuous by 2.3.6. The conditions of 3.2 are also easily verified.

There are at least five examples, probably more. The five examples are: all filters; all ultrafilters; all convergent filters; all convergent ultrafilters; all filters $\dot{x}$. This refers of course to the elements of $U\,R\,A$. In the last example, j_A is a homeomorphism for every space A , and so is k_A . Even so, regularity for this example has some meaning.

The implication $\Phi \in U\,R\,R\,A \implies \Phi_* \in U\,R\,A$ is always the least easy to verify. In the ultrafilter examples, it follows from the following lemma.

4.2. LEMMA. <u>If</u> $U\,R\,A$ <u>consists of ultrafilters on</u> $U\,A$ <u>and</u> Φ <u>is an ultrafilter on</u> $U\,R\,A$, <u>then</u> Φ_* <u>is an ultrafilter on</u> $U\,A$.

<u>Proof</u>. Since $X \cup Y \in \varphi \iff X \in \varphi$ or $Y \in \varphi$ for an ultrafilter φ , we have $(X \cup Y)^* = X^* \cup Y^*$ if $U\,R\,A$ consists of ultrafilters, for all subsets X , Y of $U\,A$. Now if Φ is an ultrafilter, then

$$X \in \Phi_* \text{ or } Y \in \Phi_* \iff X^* \in \Phi \text{ or } Y^* \in \Phi \iff X^* \cup Y^* \in \Phi$$
$$\iff (X \cup Y)^* \in \Phi \iff X \in \Phi_* \text{ or } Y \in \Phi_* ,$$

and this shows that Φ_* is an ultrafilter $\square$

4.3. CONVERGENCE SPACES. We assume again that a set A^* of filters on the set $U\,A$ is given for every space A , so that the set theoretic parts of 2.1.1 and 2.1.2 are satisfied. We put on $A^* = U\,R\,A$ the finest convergence structure which satisfies 3.2.1 and 3.2.2, by putting

$$\Phi\ q_{RA}\ \dot{x} \iff \Phi_*\ q_A\ x \quad \text{and} \quad \Phi\ q_{RA}\ \varphi \iff \Phi_* \leq \varphi ,$$

for a filter Φ on $U\,R\,A$, and for $x \in U\,A$ and $\varphi \in U\,R\,A \setminus (U\,j_A)(U\,A)$.

Now 2.3.1 and 2.3.3, the latter applied to filters on U R A, guarantee continuity of f_* for $f : A \longrightarrow B$ in CONV, and 2.3.5 guarantees the first Fréchet axiom and continuity of j_A. The second Fréchet axiom follows immediately from the definition of q_{RA}. We assume the set theoretic part of 2.1.3, and it remains to prove continuity of k_A. Thus assume $\mathcal{F}\ q_{RRA}\ \Phi$; we must show that $(k_A)_*(\mathcal{F})$ converges to Φ_* for q_{RA}. Since validity of $\Psi\ q_{RA}\ \varphi$ depends only on Ψ_* for given φ, it is sufficient to prove $\mathcal{F}_*\ q_{RA}\ \Phi_*$ and apply 2.3.7. We have $\Phi\ q_{RA}\ \Phi_*$ by 3.2.2 and $\mathcal{F}\ q_{RRA}\ \Phi$ by assumption, and hence $\mathcal{F}_*\ q_{RA}\ \Phi_*$ by 3.2.3, applied to R A.

The five examples given for topological spaces in 4.1 apply also for convergence spaces, and again there are others.

4.4. LIMIT SPACES AND CLOSURE SPACES. There are several interesting categories "between" TOP and CONV. We consider only limit spaces and closure or neighborhood spaces.

In a limit space, we require a third axiom:

If $\varphi\ q_A\ x$ and $\psi\ q_A\ x$, then $(\varphi \cup \psi)\ q_A\ x$;

in addition to the two Fréchet axioms. In a neighborhood space, we require:

For every $x \in U\ A$, there is a filter N_x on U A so that $\varphi\ q_A\ x \Longleftrightarrow \varphi \leq N_x$, for every filter φ on U A.

It follows easily from 4.5 below that R A is a limit space if A is one, and that R A is a neighborhood space if A is one. Thus the given Cauchy triples on CONV induce Cauchy triples for limit spaces and for neighborhood spaces.

A topological space can be regarded as a closure space, with additional properties. If we form R A in CONV for a topological space A, we obtain in general a neighborhood structure q_{RA} which is not topological. If we form R A in TOP , using 4.1 instead of 4.3, we obtain a coarser convergence structure. One sees easily that this is the finest topological convergence structure on $A^* = U\,R\,A$ which is coarser than q_{RA} .

4.5. If $U\,A$ and a set $U\,R\,A$ of filters on $U\,A$ are given, then filter algebra imposes the following definition. If φ is a filter on $U\,A$, then we denote by φ^* the filter on $U\,R\,A$ generated by all sets X^* with $X \in \varphi$. We note two properties of this filter φ^* and a consequence.

4.5.1. If φ is a filter on $U\,A$, then $(\varphi^*)_* = \varphi$.

4.5.2. For filters φ on $U\,A$ and Φ on $U\,R\,A$, we have

$$\Phi \leq \varphi^* \Longleftrightarrow \Phi_* \leq \varphi \ .$$

4.5.3. For families $(\varphi_i)_{i\in I}$ and $(\Phi^j)_{j\in J}$ of filters on $U\,A$ and $U\,R\,A$ respectively, we have

$$(\bigcap \varphi_i)^* = \bigcap (\varphi_i^*) \quad \text{and} \quad (\bigcup \Psi^j)_* = \bigcup (\Psi^j{}_*) \ .$$

<u>Proof</u>. Since $X^* \subset Y^* \Longleftrightarrow X \subset Y$ for subsets X , Y of $U\,A$, we have $X^* \in \varphi^* \Longleftrightarrow X \in \varphi$, and 4.5.1 follows. Now both sides of 4.5.2 are equivalent to $X \in \varphi \Longrightarrow X^* \in \Phi$, for all $X \subset U\,A$. Finally, 4.5.3 is a general property of Galois correspondences, and we have one by 4.5.2 ▯

5. UNIFORM CONVERGENCE SPACES. We modify the definition of a uniform convergence space and we construct Cauchy triples of Cauchy filters.

5.1. DEFINITION. A <u>uniform convergence structure</u> on a set E is a set $\mathcal{U}$ of filters on $E \times E$ which satisfies the following four axioms.

5.1.1. If $x \in E$, then $\dot{x} \times \dot{x} = (x,x)^{\cdot}$ is in $\mathcal{U}$.

5.1.2. If $\Phi \in \mathcal{U}$ and $\Psi \leq \Phi$, then $\Psi \in \mathcal{U}$.

5.1.3. If $\Phi \in \mathcal{U}$, then $\Phi^{-1} \in \mathcal{U}$.

5.1.4. If $\Phi \in \mathcal{U}$ and $\Psi \in \mathcal{U}$, then $\Phi \circ \Psi \in \mathcal{U}$.

A <u>uniform limit structure</u> is a uniform convergence structure which satisfies:

5.1.5. If $\Phi \in \mathcal{U}$ and $\Psi \in \mathcal{U}$, then $\Phi \cup \Psi \in \mathcal{U}$.

A <u>uniform structure</u> is a principal uniform convergence structure, i.e. $\Phi \in \mathcal{U} \Longleftrightarrow \Phi \leq \Phi_u$ for a filter Φ_u which satisfies the Bourbaki axioms.

Uniform convergence spaces lead to uniformly continuous functions in the usual way, and the category UNCV of uniform convergence spaces is obtained.

5.2. Add to a uniform convergence structure all filters $\Phi \leq \Delta$, where Δ is the principal filter on $E \times E$ based on the diagonal of $E \times E$, and a uniform convergence structure $\mathcal{U}_d$ with $\Delta \in \mathcal{U}_d$ is obtained, the finest one coarser than $\mathcal{U}$. One sees easily that $\mathcal{U}$ and $\mathcal{U}_d$ have the same Cauchy filters, and that $(E,\mathcal{U}) \longmapsto (E,\mathcal{U}_d)$ defines a functor, the reflector from UNCV to the reflective subcategory of spaces $(E,\mathcal{U})$ with $\Delta \in \mathcal{U}$.

For uniform limit structures, $\mathcal{U}_d$ consists of all filters $\Phi \leq \Delta \cup \Psi$, for some $\Psi \in \mathcal{U}$. The remarks made above also apply.

This shows that replacing $\Delta \in \mathcal{U}$ by 5.1.1 does not change the theory of uniform convergence or limit spaces very much. In view of existing examples, 5.1.1 seems to be the more natural axiom, and it certainly makes the construction of uniform convergence spaces much easier.

5.3. Assume again that the set-theoretic part of 2.1.1 and 2.1.2 is satisfied, for all spaces $A = (U\,A, \mathcal{U}_A)$ and all uniformly continuous maps in UNCV.

We put $A^* = U\,R\,A$ for a space A, and we put

$$V^* = \{(\varphi,\psi) \in A^* \times A^* : V \in \varphi \times \psi\}$$

for $V \subset U\,A \times U\,A$. We define the <u>compression</u> $\mathcal{F}_*$ of a filter $\mathcal{F}$ on $A^* \times A^*$ by putting

$$\mathcal{F}_* = \{V \subset U\,A \times U\,A : V^* \in \mathcal{F}\},$$

and we put $R\,A = (A^*, \mathcal{U}^*)$ with

$$\mathcal{F} \in \mathcal{U}^* \iff \mathcal{F}_* \in \mathcal{U}$$

for a filter $\mathcal{F}$ on $A^* \times A^*$.

We have used the same notations for filters on $U\,A \times U\,A$ and on $A^* \times A^*$ as for filters on $U\,A$ and on A^*. This will not lead to confusion; the context always makes it clear what is meant. We must verify that we have indeed set the stage for Cauchy triples on UNCV. This calls for more filter algebra.

5.4. We have again

$$(V \cap W)^* = V^* \cap W^* \quad \text{and} \quad (\dot{x},\dot{y}) \in V^* \iff (x,y) \in V,$$

for subsets V, W and a point (x,y) of $U\,A \times U\,A$. Proposition 2.4 and its proof, and the considerations of 4.5, also carry over as is (i.e. only with the obvious changes) to the new situation. In addition, we note the following.

5.4.1. $(\dot{\varphi} \times \dot{\psi})_* = \varphi \times \psi$ for filters φ and ψ on $U\,A$.

5.4.2. $(V^{-1})^* = (V^*)^{-1}$ for $V \subset U\,A \times U\,A$.

5.4.3. $(\mathcal{F}^{-1})_* = (\mathcal{F}_*)^{-1}$ for a filter $\mathcal{F}$ on $A^* \times A^*$.

5.4.4. $V^* \circ W^* \subset (V \circ W)^*$ for subsets V, W of $U\,A \times U\,A$.

5.4.5. $(\Phi \circ \Psi)_* \leq \Phi_* \circ \Psi_*$ for filters Φ, Ψ on $A^* \times A^*$.

5.4.6. $((U\, f \times U\, f)^{-1}(V))^* = (f_* \times f_*)^{-1}(V^*)$ for $f : A \longrightarrow B$ in UNCV and $V \subset U\, B \times U\, B$.

5.4.7. $((f_* \times f_*)_*(\mathcal{F}))_* = (U\, f \times U\, f)_*(\mathcal{F}_*)$ for $f : A \longrightarrow B$ in UNCV and a filter $\mathcal{F}$ on $A^* \times A^*$.

5.4.8. $(U\, j_A \times U\, j_A)^{-1}(V^*) = V$ for $V \subset U\, A \times U\, A$.

5.4.9. $((U\, j_A \times U\, j_A)(\Phi))_* = \Phi$ for a filter Φ on $U\, A \times U\, A$.

5.4.10. $(X \times Y)^* = X^* \times Y^*$ for subsets X , Y of $U\, A$.

5.4.11. $(\varphi \times \psi)^* = \varphi^* \times \psi^*$ for filters φ , ψ on $U\, A$.

5.4.12. $V^* \in \Phi \times \Psi \implies V \circ V^{-1} \circ V \in \Phi_* \times \Psi_*$, for $V \subset U\, A \times U\, A$ and filters Φ, Ψ on A^* .

5.4.13. $(\Phi \times \Psi)_* \leq \Phi_* \times \Psi_* \leq (\Phi \times \Psi)_* \circ (\Psi \times \Phi)_* \circ (\Phi \times \Psi)_*$ for filters Φ, Ψ on A^* .

5.4.14. $(U\, k_A \times U\, k_A)^{-1}(V^*) \subset (V^*)^* \subset (U\, k_A \times U\, k_A)((V \circ V^{-1} \circ V)^*)$ for $V \subset U\, A \times U\, A$.

5.4.15. $(\mathcal{R}_*)_* \leq ((U\, k_A \times U\, k_A)_*(\mathcal{R}))_* \leq (\mathcal{R}_*)_* \circ ((\mathcal{R}_*)_*)^{-1} \circ (\mathcal{R}_*)_*$ for a filter $\mathcal{R}$ on $A^{**} \times A^{**}$, where $A^{**} = U\, R\, R\, A$.

<u>Proof</u>. The proofs are mostly straightforward, and often similar to those of analogous formulas in 2.3. We prove 5.4.4, 5.4.5, and the last four statements.

If $(\varphi,\psi) \in V^*$ and $(\psi,\chi) \in W^*$, then $V \circ W$ is in the filter

$$\varphi \times \chi = (\varphi \times \psi) \circ (\psi \times \chi) \quad .$$

This proves 5.4.4. In 5.4.5, $\Phi_* \circ \Psi_*$ is generated by the sets $V \circ W$ with

$V^* \in \Phi$ and $W^* \in \Psi$. For these sets, $(V \circ W)^* \in \Phi \circ \Psi$ by 5.4.4, and thus $V \circ W \in (\Phi \circ \Psi)_*$. This proves 5.4.5.

For 5.4.12, let $P \times Q \subset V^*$ with $P \in \Phi$ and $Q \in \Psi$. Fix $\varphi_1 \in P$ and $\psi_1 \in Q$, and fix $X_1 \in \varphi_1$ and $Y_1 \in \psi_1$ with $X_1 \times Y_1 \subset V$. All of this can be done by $V^* \in \Phi \times \Psi$ and definitions. For every $\varphi \in P$ there is $X_\varphi \in \varphi$ and $Y'_\varphi \in \psi_1$ such that $X_\varphi \times Y'_\varphi \subset V$, and for every $\psi \in Q$ there is $Y_\psi \in \psi$ and $X'_\psi \in \varphi_1$ such that $X'_\psi \times Y_\psi \subset V$. Put $X = \bigcup X_\varphi$ and $Y = \bigcup Y_\psi$, for all $\varphi \in P$ and $\psi \in Q$. Then $X \in \varphi$ and $Y \in \psi$ for all such φ and ψ, i.e. $P \subset X^*$ and $Q \subset Y^*$. Thus $X \in \Phi_*$ and $Y \in \Psi_*$. If $(x,y) \in X \times Y$, let $x \in X_\varphi$ and $y \in Y_\psi$, and let $x' \in X'_\psi \cap X_1$ and $y' \in Y'_\varphi \cap Y_1$. Then (x,y'), (x',y'), (x',y) are in V, and thus $(x,y) \in V \circ V^{-1} \circ V$. This shows that $X \times Y \subset V \circ V^{-1} \circ V$ and proves 5.4.12.

In 5.4.13, $\Phi_* \times \Psi_*$ is generated by sets $X \times Y$ with $X^* \in \Phi$ and $Y^* \in \Psi$, and hence $(X \times Y)^* = X^* \times Y^*$ in $\Phi \times \Psi$. Thus $X \times Y \in (\Phi \times \Psi)_*$ for these sets. This verifies the first inequality. The filter at right is generated by sets $V \circ V^{-1} \circ V$ with $V \in (\Phi \circ \Psi)_*$. These sets are in $\Phi_* \times \Psi_*$ by 5.4.12, and thus the second inequality holds.

In 5.4.14, we have $(\Phi, \Psi) \in (U\, k_A \times U\, k_A)^{-1}(V^*) \iff V \in \Phi_* \times \Psi_*$. In this situation, $V \in (\Phi \times \Psi)_*$ by 5.4.12, and thus $(\Phi, \Psi) \in (V^*)^*$. For $(\Phi, \Psi) \in (V^*)^*$, we have $V^* \in \Phi \times \Psi$, and then $(\Phi_*, \Psi_*) \in (V \circ V^{-1} \circ V)^*$ by 5.4.12 and definitions. The second inclusion of 5.4.14 follows.

For 5.4.15, we note that $V \in (\mathfrak{K}_*)_* \iff (V^*)^* \in \mathfrak{K}$, and that

$$V \in ((U\, k_A \times U\, k_A)_*(\mathfrak{K}))_* \iff (U\, k_A \times U\, k_A)^{-1}(V^*) \in \mathfrak{K} \;;$$

compare the proof of 2.3.7. The first inequality now follows immediately from the first inclusion in 5.4.14. The filter at right is generated by the sets

$V \circ V^{-1} \circ V$ with $(V^*)^* \in \mathcal{R}$, and the second inclusion of 5.4.14 and the displayed statement above show that these sets are in the middle filter. This proves the second inequality of 5.4.15 $\square$

6. PRECOMPLETE UNIFORM CONVERGENCE SPACES. We continue to use the assumptions and notations of 5.3. We note from 5.4.1 that the space $R\,A = (A^*, \mathcal{U}^*)$, for a uniform convergence space $A = (U\,A, \mathcal{U})$, satisfies 5.1.1 if and only if $\varphi \times \varphi \in \mathcal{U}$ for every $\varphi \in A^*$, i.e. if and only if A^* consists of Cauchy filters of A. This leads to the following definition.

6.1. DEFINITION. A uniform convergence space A is called <u>precomplete</u>, with respect to the data of 5.3, if every filter $\varphi \in A^*$ is a Cauchy filter of A. The full subcategory of UNCV with precomplete spaces as its objects is called the <u>category of precomplete uniform convergence spaces</u>, for the Cauchy triple given by 5.3. We abbreviate it by UNCV_{pc}.

We shall also assume the set theoretic part of 2.1.3 for precomplete spaces, by requiring that $\Phi_* \in A^*$ for every filter Φ in $A^{**} = (R\,A)^*$ if A is a precomplete space.

6.2. THEOREM. <u>The data of</u> 5.3 <u>and</u> 6.1 <u>define a Cauchy triple</u> (R, j, k) <u>with convergence on the corresponding category</u> UNCV_{pc}.

<u>Proof</u>. The restriction to precomplete spaces guarantees 5.1.1 for $\mathcal{U}^*$. 5.1.2 is obvious from the definition, and 5.1.3 and 5.1.4 follow from 5.4.3 and 5.4.5. Thus $R\,A$ is a uniform convergence space if A is precomplete. 5.4.7 and the definition of $\mathcal{U}^*$ show that $f_* = U\,R\,f$ for a uniformly continuous mapping $R\,f : R\,A \longrightarrow R\,B$ if $f : A \longrightarrow B$ in UNCV_{pc}. Thus R is a functor.

If $\Phi \in A^{**}$, then $\Phi_* \times \Phi_* \in \mathcal{U}$ since A is precomplete and $\Phi_* \in A^*$. Then $\Phi \times \Phi \in \mathcal{U}^*$ by 5.4.13. Thus R A is also precomplete, and R is an endofunctor of $UNCV_{pc}$. Now j_A and k_A are uniformly continuous, by 5.4.9 and 5.4.15, and we have a Cauchy triple (R, j, k) on $UNCV_{pc}$. In our situation, $\varphi\, q_A\, x$ means $\varphi \times \dot{x} \in \mathcal{U}$. Using this and 5.4.13, and 5.1.4 for 3.2.3, we see that (R, j, k) is a Cauchy triple with convergence$\square$

6.3. **PROPOSITION.** <u>The full subcategory</u> $UNCV_{pc}$ <u>of</u> UNCV <u>defined by the data of</u> 5.3 <u>is a top subcategory of</u> UNCV <u>over</u> ENS .

<u>Proof</u>. For each set E , there is a complete lattice S E of uniform convergence structures on E , and every mapping $f : E \longrightarrow F$ induces an inverse image structure map $f^S : S\,F \longrightarrow S\,E$. We must show that the intersection of precomplete structures in S E is precomplete, and that $f^S(\mathcal{U})$ in S E is precomplete if $\mathcal{U}$ in S F is.

Let $A_i = (E, \mathcal{U}_i)$, for $i \in I$, be precomplete spaces over the same set E , and let $A = (E, \bigcap \mathcal{U}_i)$. Put $A^* = U\,R\,A$ and $A_i^* = U\,R\,A_i$. For every $i \in I$, $\mathrm{id}\,E : A \longrightarrow A_i$ is uniformly continuous. The induced mapping $(\mathrm{id}\,E)_* : A^* \longrightarrow A_i^*$ is a restriction of $\mathrm{id}\,\mathbb{F}\,E$, and thus $A^* \subset A_i^*$ for every $i \in I$. Thus if $\varphi \in A^*$, then $\varphi \times \varphi \in \mathcal{U}_i$ for every $i \in I$, as all spaces A_i are precomplete. But then $\varphi \times \varphi \in \bigcap \mathcal{U}_i$, and A is precomplete.

If $f : E \longrightarrow F$ in ENS and $\mathcal{U} \in S\,F$, then $f^S(\mathcal{U})$ consists of all filters Φ on $E \times E$ with $(f \times f)_*(\Phi)$ in $\mathcal{U}$. Put $A = (E, f^S(\mathcal{U}))$ and $B = (F, \mathcal{U})$. The filter mapping f_* maps A^* into B^*. If $\varphi \in A^*$, then $(f \times f)_*(\varphi \times \varphi) = f_*(\varphi) \times f_*(\varphi)$ is in $\mathcal{U}$ if B is precomplete. But then $\varphi \times \varphi \in f^S(\mathcal{U})$, and A is precomplete$\square$

6.4. EXAMPLES. The following six examples satisfy all conditions. For a uniform convergence space A , the set A^* could be the set of all Cauchy filters of A , or the set of all convergent filters, or the set of all ultrafilters on U A , or the set of all Cauchy ultrafilters, or the set of all convergent ultrafilters, or the set of all filters $\dot{x}$, $x \in U\,A$. In five of these six examples, all uniform convergence spaces are precomplete. In the example of all ultrafilters, the precomplete spaces are the precompact or totally bounded spaces.

The statements in 4.5, applied to filters on $U\,A \times U\,A$ and the construction in 5.3, show that $\mathcal{U}^*$ is a uniform limit structure or a uniform structure if $\mathcal{U}$ is a uniform limit structure or uniform structure respectively, and precomplete. Thus every Cauchy triple for uniform convergence spaces, of the kind considered from 5.3 on, induces Cauchy triples for precomplete uniform limit spaces and precomplete uniform spaces, by restriction of R , j , k .

7. CONTINUOUS RELATIONS. We consider the following general situation. A commutative diagram

$$\begin{array}{ccc} \mathcal{C}^t & \xrightarrow{\Upsilon} & \mathrm{ENS}^s \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{U} & \mathrm{ENS} \end{array}$$

is given in which $\mathcal{C}^t$ and ENS^s are top categories over $\mathcal{C}$ and ENS respectively, the functor U is faithful, and the vertical arrows are the projection functors. In this situation, the upper horizontal arrow is a "structure functor", with $\Upsilon(A,x) = (U\,A, \upsilon_A\,x)$ and $\Upsilon f = U\,f : (U\,A, \upsilon_A\,x) \longrightarrow (U\,B, \upsilon_B\,y)$ for an object (A,x) and a map $f : (A,x) \longrightarrow (B,y)$ of $\mathcal{C}^t$. The functor Υ is given by its structure maps $\upsilon_A : T\,A \longrightarrow S\,U\,A$, one for each $A \in \mathrm{Ob}\,\mathcal{C}$, from the set $T\,A$ of t-structures of A to the set $S\,U\,A$ of s-structures of $U\,A$.

7.1. We make the following assumptions.

7.1.1. $\mathcal{C}$ has finite products and pullbacks. The functor U preserves finite products and pullbacks, and reflects isomorphisms.

7.1.2. $\mathcal{C}$ has $\mathcal{J}$-images for a class $\mathcal{J}$ of morphisms of $\mathcal{C}$ such that $U\,j$ is injective, and $U\,p$ surjective, whenever (p,j) is a $\mathcal{J}$-image in $\mathcal{C}$.

7.1.3. For every $A \in \mathrm{Ob}\,\mathcal{C}$, $T\,A$ is a subset of $S\,U\,A$, closed under intersections in $S\,U\,A$, and $\upsilon_A : T\,A \longrightarrow S\,U\,A$ is the inclusion mapping. For every $f : A \longrightarrow B$ in $\mathcal{C}$, we have $(U\,f)^s\,\upsilon_B = \upsilon_A\,f^t$.

These assumptions usually are satisfied if $\mathcal{C}$ is a category of algebras and $\mathcal{C}^t$ a category of topological algebras, constructed over $\mathcal{C}$ and ENS^s.

7.2. It follows from 7.1.2 and the last part of 7.1.1 that $U\,j$ is injective for every $j \in \mathcal{J}$, and that (p,j) is a $\mathcal{J}$-image of f in $\mathcal{C}$ whenever $f = j\,p$ in $\mathcal{C}$ with $j \in \mathcal{J}$ and $U\,p$ surjective.

Call j and j' in $\mathcal{J}$ equivalent if $j' = j\,h$ for an isomorphism h in $\mathcal{C}$. A subobject of an object A of $\mathcal{C}$ is given by an equivalence class of morphisms in $\mathcal{J}$ with codomain A. We put $u' \leq u$ for subobjects u, u' of A, represented by j, j' in $\mathcal{J}$, if $j' = j\,h$ for some $h \in \mathcal{C}$.

7.3. A relation $u : A \longrightarrow B$ over $\mathcal{C}$ is a triple (A,u,B) consisting of two objects A, B of $\mathcal{C}$ and a subobject u of $A \times B$. If two morphisms

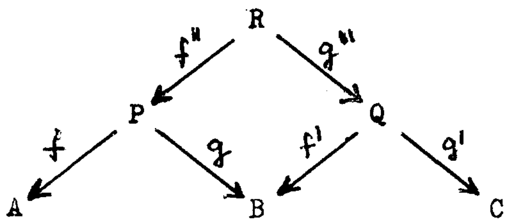
$$A \xleftarrow{\;f\;} P \xrightarrow{\;g\;} B$$

of $\mathcal{C}$ are given, then the image of the induced morphism $\{f,g\} : P \longrightarrow A \times B$

determines a relation $u : A \longrightarrow B$ which we denote by $[f,g]$. Every relation over $\mathcal{C}$ can be represented in this form, usually in many ways.

Composition of relations is defined by means of diagrams

(7.3.1)



in which the square is a pullback. In this situation, we put

(7.3.2) $$[f',g'] [f,g] = [f f", g' g"] .$$

One must show of course, and it follows from our assumptions, that this does not depend on the particular representations $[f,g]$ and $[f',g']$ of the factors.

Under our assumptions, objects of $\mathcal{C}$ and relations over $\mathcal{C}$ are the objects and morphisms of a category $\mathrm{Rel}\,\mathcal{C}$. $[\mathrm{id}\,A, \mathrm{id}\,A]$ is the identity relation on an object A of $\mathcal{C}$, and $I\,A = A$, $I\,f = [\mathrm{id}\,A, f]$, for an object A and a morphism $f : A \longrightarrow B$ of $\mathcal{C}$, defines an embedding functor $I : \mathcal{C} \longrightarrow \mathrm{Rel}\,\mathcal{C}$. Relations over ENS are relations in the usual sense, and the functor $U : \mathcal{C} \longrightarrow$ ENS preserves relations and their composition. In other words, U can be extended to a faithful functor from $\mathrm{Rel}\,\mathcal{C}$ to Rel ENS .

7.4. DEFINITION. Let $(A,x) \longrightarrow (B,y)$ be objects of $\mathcal{C}^t$. We say that a relation $u : A \longrightarrow B$ over $\mathcal{C}$ is a <u>continuous relation</u> $u : (A,x) \longrightarrow (B,y)$ over $\mathcal{C}^t$ if for every pair of morphisms $A \xleftarrow{f} P \xrightarrow{g} B$ of $\mathcal{C}$ such that $[f,g] \leqslant u$, and for every structure $p \in T\,P$ of the common domain P of f and g such that $f : (P,p) \longrightarrow (A,x)$ in $\mathcal{C}^t$, we also have $g : (P,p) \longrightarrow (B,y)$ in $\mathcal{C}^t$.

Since $f : (P,p) \longrightarrow (A,x)$ in $\mathcal{C}^t$ if and only if $p \leqslant f^t(x)$ in $T\,P$, and $g : (P,p) \longrightarrow (B,y)$ if and only if $g_t(p) \leqslant y$ in $T\,B$, we have $u : (A,x) \longrightarrow (B,y)$ for a relation $u : A \longrightarrow B$ over $\mathcal{C}$ if and only if

(7.4.1) $$g_t(f^t(x)) \leqslant y$$

whenever $[f,g] \leqslant u$ in $\mathrm{Rel}\,\mathcal{C}$.

7.5. PROPOSITION. The mapping $g_t\, f^t : T\,A \longrightarrow T\,B$ is the same for all representations $u = [f,g]$ of a relation $u : A \longrightarrow B$ over $\mathcal{C}$, and $u : (A,x) \longrightarrow (B,y)$ over $\mathcal{C}^t$ if and only if (7.4.1) holds for one such representation.

Proof. Let $j : P_1 \longrightarrow A \times B$ in $\mathfrak{J}$ be a representative of u, and put $j = \{f_1, g_1\}$. If $[f,g] \leqslant u$, then $\{f,g\} = j\,h$, and thus $f = f_1\,h$ and $g = g_1\,h$, for a morphism h in $\mathcal{C}$. It follows that

$$g_t\, f^t = (g_1)_t\, h_t\, h^t\, (f_1)^t .$$

Since always $h_t(h^t(p)) \leqslant p$, $(g_1)_t((f_1)^t(x)) \leqslant y$ implies $g_t(f^t(x)) \leqslant y$.

If $[f,g] = u$, then h is surjective. Thus $(U\,h)^s$ and the restriction h^t of $(U\,h)^s$ are injective. We have $h^t\, h_t\, h^t = h^t$ in any case, and with h^t injective, we have $h_t\, h^t = \mathrm{id}\,(T\,P_1)$. Thus $g_t\, f^t = (g_1)_t\,(f_1)^t$ ▯

7.6. THEOREM. A morphism $f : A \longrightarrow B$ of $\mathcal{C}$ is continuous from (A,x) to (B,y) in $\mathcal{C}^t$ if and only if $[\mathrm{id}\,A,\, f] : (A,x) \longrightarrow (B,y)$ over $\mathcal{C}^t$. If $u : (A,x) \longrightarrow (B,y)$ and $v : (B,y) \longrightarrow (C,z)$ are continuous relations over $\mathcal{C}^t$, then $v\,u : (A,x) \longrightarrow (C,z)$ is continuous.

Proof. (7.4.1) for $[\mathrm{id}\,A,\, f]$ is $f_t(x) \leqslant y$, and this is also the requirement for $f : (A,x) \longrightarrow (B,y)$. With 7.5, this proves the first part.

For the second part, we consider a diagram (7.3.1). Put

$$p = f^t(x) \quad , \quad q = (f')^t(y) \quad , \quad r = (f'')^t(p) \quad .$$

Then $g_t(p) \leq y$ and $(g')_t(q) \leq z$ if u and v are continuous. It follows that $g\, f'' = f'\, g'' : (R,r) \longrightarrow (B,y)$ in $\mathcal{C}^t$. Since $q = (f')_t(y)$, this implies $g'' : (R,r) \longrightarrow (Q,q)$. Thus $(g'')_t(r) \leq q$. Now

$$(g'\, g'')_t((f\, f'')^t(x)) = (g'\, g'')_t(r) \leq (g')_t(q) \leq z \quad ,$$

and $v\, u = [f\, f'', g'\, g'']$ is continuous by 7.5▯

Theorem 7.6 shows that continuous relations over $\mathcal{C}^t$ form a category into which $\mathcal{C}^t$ is embedded, by a functor which lifts the functor I of 7.3.

8. SEPARATED SPACES. We define separation for a top category ENS^s over ENS on which a Cauchy triple (R, j, k) with convergence is defined.

8.1. We recall that a top category ENS^s over ENS is obtained as follows. A complete lattice $S\,E$ of structures on E is assigned to every set E, and mappings $f_s : S\,E \rightarrow S\,F$ and $f^s : S\,F \longrightarrow S\,E$ are assigned to every mapping $f : E \longrightarrow F$. Both assignments are functorial, one covariant, the other contravariant, and $f_s(u) \leq v \iff u \leq f^s(v)$ for all $u \in S\,E$ and $v \in S\,F$ if $f : E \longrightarrow F$. It follows that the maps f_s preserve suprema and the maps f^s infima. Objects of ENS^s are pairs (E,u) of a set E and some $u \in S\,E$, and a map $f : (E,u) \longrightarrow (F,v)$ is a mapping $f : E \longrightarrow F$ such that $u \leq f^s(v)$. Composition in ENS^s is composition of mappings.

The category CONV of convergence spaces, which we also denote by ENS^q, is an example. Other examples are topological spaces, uniform spaces, uniform convergence spaces, limit spaces, and neighborhood or closure spaces.

8.2. Let ENS^s and ENS^r be top categories over ENS . If a mapping $\tau_E : R\,E \longrightarrow S\,E$ which preserves infima is assigned to every set E , and if $f^s \tau_F = \tau_E f^r$ for every mapping $f : E \longrightarrow F$, then

$$T\,(E,u) \;=\; (E,\tau_E\, u) \quad , \qquad T\,f \;=\; f : (E,\tau_E\, u) \qquad (F,\tau_F\, v) \quad ,$$

for every object (E,u) and morphism $f : (E,u) \longrightarrow (F,v)$ of ENS^r , defines a functor $T : ENS^r \longrightarrow ENS^s$ which we call a <u>top functor</u>.

Among all functors $T : ENS^r \longrightarrow ENS^s$, the top functors are characterized by the fact that they lift Id ENS and have left adjoints which also lift Id ENS . In fact, if (E,u) is an object of ENS^s and if

$$\sigma_E\, u \;=\; \inf \{ v \in R\,E : u \leq \tau_E\, v \} \;,$$

then $\mathrm{id}\, E : (E,u) \longrightarrow T\,(E,\sigma_E\, u)$ is a universal morphism for a top functor T given by structure maps τ_E . We do not go further into this.

If in particular $R\,E$ is a subset of $S\,E$, closed under infima in $S\,E$, for every set E , and if f^s maps $R\,F$ into $R\,E$ for every mapping $f : E \longrightarrow F$, then we obtain a top category ENS^r and a top functor $T : ENS^r \longrightarrow ENS^s$ by letting $f^r : R\,F \longrightarrow R\,E$ be the restriction of f^s for every mapping $f : E \longrightarrow F$, and $\tau_E : R\,E \longrightarrow S\,E$ the inclusion mapping for every set E . In this situation, ENS^r is a full subcategory of ENS^s which we call a <u>top subcategory</u> of ENS^s , and T is the inclusion functor.

8.3. DEFINITION. We assume from now on that a top category ENS^s with a top functor $ENS^s \longrightarrow ENS^q$ to convergence spaces, and a Cauchy triple (R, j, k) with convergence, over ENS^s , are given. We put $R\,(E,u) = (E^*,u^*)$ for an object (E,u) of ENS^s , and we denote by $q_E(u)$ the convergence structure on E obtained by applying the given top functor to (E,u) . We say that (E,u)

is <u>separated</u> for the triple (R, j, k) if every filter $\varphi \in E^*$ converges to at most one point $x \in E$, i.e. if the restriction of $q_E(u)$ to E^* is a functional relation from E^* to E. We do not require that every filter $\varphi \in E^*$ converges. If this is the case for a separated space (E,u), then convergence of filters induces a mapping from E^* to E.

8.4. PROPOSITION. <u>The product of separated spaces is separated. If</u> $f : E \longrightarrow F$ <u>is injective and</u> (F,v) <u>a separated space, and if</u> $u \leqslant f^s(v)$ <u>in</u> $S\,E$, <u>then</u> (E,u) <u>is a separated space.</u>

<u>Proof.</u> Let first $\prod(E_i,u_i) = (E,u)$, with sets of Cauchy filters E_i^* and E^*, with convergence relations q_i and q, and with projections π_i. If $\varphi\, q\, x$, with $\varphi \in E^*$, then $(\pi_i)_*(\varphi)$ is in E_i^* and converges for q_i to $\pi_i(x)$, for every i. If the spaces (E_i,u_i) are separated, this determines the points $\pi_i(x)$ and hence x uniquely, and (E,u) is separated.

In the second part, let E^* and F^* be the sets of Cauchy filters, and q and q' the convergence relations. If $\varphi \in E^*$ and $\varphi\, q\, x$, then $f_*(\varphi) \in F^*$ and $f_*(\varphi)\, q'\, f(x)$. If (F,v) is separated and f injective, this determines $f(x)$ and then x uniquely, and thus (E,u) is separated[]

8.5. **THEOREM.** <u>If</u> ENS^s <u>satisfies the conditions of</u> 8.3 <u>and</u> ENS^r <u>is a top subcategory of</u> ENS^s, <u>then the separated spaces in</u> Ob ENS^r <u>are the objects of an epireflective full subcategory of</u> ENS^s.

<u>Proof.</u> We sketch the construction which follows a standard pattern. Let K be the class of all separated spaces in Ob ENS^r. For every object (E,u) of ENS^s, there is a family of surjective maps $g_i : (E,u) \longrightarrow (E_i,u_i)$ in ENS^s

with $(E_i, u_i) \in K$ such that every surjective $g : (E,u) \longrightarrow (F,v)$ in ENS^S with $(F,v) \in K$ is up to an isomorphism $(E_i, u_i) \longrightarrow (F,v)$ one of the g_i. If $f : (E,u) \longrightarrow (F,v)$ with $(F,v) \in K$, let $f = h\, g$ with $g : E \longrightarrow E'$ surjective and h injective. If $v' = h^S(v)$, then $g : (E,u) \longrightarrow (E',v')$ in ENS^S with $(F',v') \in K$. Thus f factors through one of the g_i. Now form the product (E',u') of the (E_i, u_i) and the map $g' : (E,u) \longrightarrow (E',u')$ with projections g_i. Put $g' = h''\, g''$ with $g'' : E \longrightarrow E''$ surjective and h'' injective. If $u'' = (h'')^S(u')$, then $g'' : (E,u) \longrightarrow (E'',u'')$ in ENS^S with $(E'',u'') \in K$, and every map $f : (E,u) \longrightarrow (F,v)$ in ENS^S with $(F,v) \in K$ factors through this one. Since g'' is surjective, $g'' : (E,u) \longrightarrow (E'',u'')$ is epimorphic in ENS^S, and the factorization is unique $\square$

9. REGULAR SPACES. We continue to use the assumptions of 8.3.

9.1. DEFINITION. A space $A = (E,u)$ in Ob ENS^S is called <u>regular</u>, for the Cauchy triple (R, j, k), if the convergence relation $q_A : E^* \longrightarrow E$, induced by $q_E(u)$ for $R\, A = (E^*, u^*)$, is continuous from (E^*, u^*) to (E,u).

9.2. THEOREM. <u>Regular spaces in</u> Ob ENS^S <u>are the objects of a top subcategory of</u> ENS^S.

PROOF. Let $R\, E$ be the set of all $u \in S\, E$ with (E,u) regular, for a set E. Consider first a family of structures $u_i \in R\, E$ and put $u = \bigcap u_i$. Put $R\,(E,u_i) = (E_i^*, u_i^*)$ and $R\,(E,u) = (E^*, u^*)$, and put $h_i = R\,(\mathrm{id}\, E) : (E^*, u^*) \longrightarrow (E_i^*, u_i^*)$. If $f : A \longrightarrow E^*$ and $g : A \longrightarrow E$ with $f(x)\, q\, g(x)$ for every $x \in A$ and $q = q_E(u)$, we must show that $g_S(f^S(u^*)) \leqslant u$. In this situation, h_i is an inclusion mapping, and $q_i = q_E(u_i)$ an extension of q, for all i . Thus $(h_i\, f)(x)\ q_i\ g(x)$ for all i , and $g_s((h_i\, f)^s)(u_i{*}) \leq u_i$ follows since (E,u_i) is regular. But $u^* \leq (h_i)^s(u_i{*})$, and hence

$$g_s(f^s(u^*)) \leq g_s(f^s((h_i)^s(u_i{*}))) = g_s((h_i\, f)^s(u_i{*})) \leq u_i$$

for all i . Thus $g_s(f^s(u^*)) \leq u$, and (E,u) is regular.

Let now $f : E \longrightarrow F$ and $u = f^s(v)$ with $v \in R\,E$. Put $R\,(E,u) = (E^*,u^*)$ and $R\,(F,v) = (F^*,v^*)$, and put $q = q_E(u)$ and $q' = q_F(v)$. If $g : A \longrightarrow E$ and $h : A \longrightarrow F$ with $h(x)\ q\ g(x)$ for every $x \in A$, then $f_*(h(x))\ q'\ f(g(x))$ for every $x \in A$. Thus $u^* \leq (f_*)^s(v^*)$, and

$$(f\ g)_s(h^s(u^*)) \leq (f\ g)_s(h^s((f_*)^s(v^*))) = (f\ g)_s((f_*\ h)^s(v^*)) \leq v ,$$

by regularity of (F,v) . But then $g_s(h^s(u^*)) \leq f^s(v) = u$, and (E,u) is regular[]

9.3. COROLLARY. <u>If</u> ENS^r <u>is a top subcategory of</u> ENS^s , <u>then the regular and the separated regular spaces in</u> Ob ENS^r <u>are the objects of full reflective subcategories of</u> ENS^s .

<u>Proof</u>. The regular spaces in Ob ENS^r are in fact the objects of a top subcategory of ENS^s . This is reflective, and we can use (8.5) for separated regular spaces in Ob ENS^r []

9.4. TWO EXAMPLES. We consider topological spaces and convergence spaces with the Cauchy triples of 4.1 and 4.3. In either case, let (E,u) be a space and $R\,(E,u) = (E^*,u^*)$. We put $q = q_E(u)$. Then $q(X^*) = \bar{X}$, the closure of $X \subset E$ for u , if E^* consists of all filters on E , or of all ultrafilters on E , or of all convergent filters, or of all convergent ultrafilters. If φ is a filter on E , then the sets $\bar{X}$, $X \in \varphi$, form a basis of a filter on E

which we denote by $\overline{\varphi}$. We formulate T_3 as follows.

T_3 . If $x \in E$ and $\varphi q x$, then $\overline{\varphi} q x$.

A topological or convergence space usually is called regular if it satisfies T_3 . We wish to show that this is equivalent to our definition 9.1, for the Cauchy triples mentioned above.

Consider first $f : A \to E^*$ and $g : A \to E$ with $f(x)\ q\ g(x)$ for all $x \in A$, and let $q^* = q_{E^*}(u^*)$ and $q' = q_A(f^s(u^*))$. Then $q' = f^q(q^*)$, and $\sigma\ q'\ x$ for $x \in A$ and a filter σ on A if and only if $f_*(\sigma)\ q^*\ f(x)$. As $f(x)\ q\ g(x)$, this implies $\varphi\ q\ g(x)$ for $\varphi = (f(x))_*$. If $X^* \in f(x)$, then $f^{-1}(X^*) \in \sigma$, and $g(f^{-1}(X^*)) \subset \overline{X}$. Thus $g(\sigma) \leq \overline{\varphi}$, and $g(\sigma)\ q\ g(x)$ follows if (E,u) satisfies T_3 . This means that $g : (A,\ f^s(u^*)) \longrightarrow (E,u)$ is continuous, i.e that $g_s(f^s(u^*)) \leq u$.

For the converse, let A be the set of all (φ,x) in $E^* \times E$ such that $\varphi\ q\ x$, and put $f(\varphi,x) = \varphi$ and $g(\varphi,x) = x$ for $(\varphi,x) \in A$. For $X \subset E$, let $S_X = f^{-1}(X^*)$. Then $f(S_X) \subset X^*$ and $g(S_X) = \overline{X}$. For a filter φ on E , the sets S_X with $X \in \varphi$ generate a filter S_φ on A , with $f_*(S_\varphi) \leq \varphi^*$ and $g_*(S_\varphi) = \overline{\varphi}$. If $\varphi\ q\ x$, then $\varphi^*\ q^*\ \dot{x}$, and $S_\varphi\ q'\ (\dot{x},x)$ follows. If $g :$ $(A,\ f^s(u^*)) \longrightarrow (E,u)$, it follows that $g_*(S_\varphi)\ q\ x$. This is T_3 for (E,u) .

9.5. T_1 SPACES. Putting $D\,(E,u) = (E^{\cdot},\ h^s(u))$ for an object (E,u) of a top category ENS^s , where $E^{\cdot} = \{\dot{x} : x \in E\}$ and $h(x) = \dot{x}$ for $x \in E$, defines a Cauchy triple $(D,\ j,\ k)$ for every top category ENS^s over sets. For this triple, j and k are natural equivalences, and $D\,j = j\,D = k^{-1}$. We say that (E,u) is a T_1 <u>space</u> if (E,u) is separated for $(D,\ j,\ k)$.

If $q = q_E(u)$, then $\dot{x}\ q\ x$ for every $x \in E$. Thus a T_1 space is one which satisfies the following axiom.

T_1 . If x , y are in E and $\dot{x}\, q\, y$, then $x = y$.

This is of course the usual separation axiom T_1 in filter form.

Regular spaces for the triple (D, j, k) have occurred in completion constructions, but we shall not discuss them here.

As a corollary of 8.5, we have the following result.

9.6. PROPOSITION. _If_ ENS^s _is a top category over_ ENS _and_ ENS^r _a top subcategory of_ ENS^s , _then the_ T_1 _spaces in_ Ob ENS^r _are the objects of an epireflective full subcategory of_ ENS^s ▯

10. REGULAR UNIFORM CONVERGENCE SPACES. Our general theory strongly suggests the following definition.

10.1. DEFINITION. A uniform convergence space $(E,\mathcal{U})$ is called _regular_ if $(E,\mathcal{U})$ is regular for the Cauchy triple (R, j, k) 5.3, with E^* one of the following sets: all Cauchy filters on E , all Cauchy ultrafilters on E , all convergent filters on E , all convergent ultrafilters on E .

We have enumerated four triples; the preceding paragraph suggests that the same spaces are regular for each of these triples.

10.2. Let $(E,\mathcal{U})$ be a uniform convergence space, $E^* = U\,R\,(E,\mathcal{U})$, and $q = q_E(\mathcal{U})$. Then $\varphi\, q\, x$, for $x \in E$ and a filter φ on E , if and only if $\varphi \times \dot{x} \in \mathcal{U}$. For $U \subset E \times E$, we put $\bar{U} = (q \times q)(U^*)$. Thus $(x,y) \in \bar{U}$ if and only if $\varphi\, q\, x$ and $\psi\, q\, y$ for filters φ and ψ on E with $U \in \varphi \times \psi$. We call $\bar{U}$ the _uniform closure_ of U . The set $\bar{U}$ is easily seen to be the same for each of the four triples involved. This observation and 10.3 show that regularity is the same for each of the four triples.

For a filter Φ on $E \times E$, we denote by $\bar{\Phi}$ the filter on $E \times E$ with all sets $\bar{U}$, for $U \in \Phi$, as basis. We have the following result.

10.3. THEOREM. <u>A uniform convergence space</u> $(E, \mathcal{U})$ <u>is regular if and only if it satisfies the following condition.</u>

T_3^u. <u>If a filter</u> Φ <u>on</u> $E \times E$ <u>is in</u> $\mathcal{U}$, <u>then</u> $\bar{\Phi} \in \mathcal{U}$.

<u>Proof.</u> Put UNCV = ENS^u for this proof. If $f : A \longrightarrow E^*$ and $g : A \longrightarrow E$ are given so that $f(x)\ q\ g(x)$ for every $x \in A$, and if $(f \times f)(V) \subset U^*$ for $V \subset A \times A$ and $U \subset E \times E$, then clearly $(g \times g)(V) \subset \bar{U}$. If Σ is a filter on $A \times A$ and Φ one on $E \times E$, it follows that

$$(f \times f)_*(\Sigma) \leq \Phi^* \implies (g \times g)_*(\Sigma) \leq \bar{\Phi} \quad .$$

Now $\Sigma \in f^u(\mathcal{U}^*)$ if and only if $(f \times f)_*(\Sigma) \leq \Phi^*$ for some $\Phi \in \mathcal{U}$. Thus T_3^u implies that $g : (A, f^u(\mathcal{U}^*)) \longrightarrow (E, \mathcal{U})$ is continuous. This is (7.4.1): $g_u(f^u(\mathcal{U}^*)) \leq \mathcal{U}$.

For the converse, let $A = \{(\varphi, x) \in E^* \times E : \varphi\ q\ x\}$, and let $f(\varphi, x) = \varphi$ and $g(\varphi, x) = x$ for $(\varphi, x) \in A$. Put $S_U = (f \times f)^{-1}(U^*)$ for $U \subset E \times E$. Then $(f \times f)(S_U) \subset U^*$ and $(g \times g)(S_U) = \bar{U}$. This is easily verified. For a filter Φ on $E \times E$, let S_Φ be the filter on $A \times A$ with the sets S_U, $U \in \Phi$, as basis. Then $(f \times f)_*(S_\Phi) \leq \Phi^*$ and $(g \times g)_*(S_\Phi) = \bar{\Phi}$. Thus $S_\Phi \in f^u(\mathcal{U}^*)$ for $\Phi \in \mathcal{U}$. If $(E, \mathcal{U})$ is regular, then $g : (A, f^u(\mathcal{U}^*)) \longrightarrow (E, \mathcal{U})$ is uniformly continuous. Applied to the filters S_Φ for $\Phi \in \mathcal{U}$, this implies T_3^u for $(E, \mathcal{U})$ []

10.4. THEOREM. <u>Every uniform space is a regular uniform convergence space.</u>

<u>Proof.</u> For every entourage U, there are entourages V and W such that

$W \circ V \circ W^{-1} \subset U$. If $(x,y) \in \bar{V}$ with $\varphi\, q\, x$, $\psi\, q\, y$, and $V \in \varphi \times \psi$, then there are sets $X \in \varphi$ and $Y \in \psi$ such that $X \subset W(x)$ and $Y \subset W(y)$. Then $V \cap (X \times Y)$ in $\varphi \times \psi$ is not empty. Thus $(x,y) \in W \circ V \circ W^{-1}$, and we conclude that $\bar{V} \subset U$. If Φ is the filter of entourages, this shows that $\bar{\Phi} \leq \Phi$. But then the space cleary satisfies T_3^u and is regular▯

11. COMPLETE SPACES. We use again the assumptions and notations of 8.3.

11.1. DEFINITION. A space (E,u) , with $R\,(E,u) = (E^*,u^*)$, is called <u>complete</u>, for the triple (R, j, k) , if every filter $\varphi \in E^*$ converges for $q_E(u)$ to some point $x \in E$. In other words, we require $q^{-1}(E) = E^*$ for the relation $q : E^* \longrightarrow E$ induced by $q_E(u)$.

The space (E^*,u^*) always is complete, by 3.2.2. More generally, the underlying space of an algebra for the triple (R, j, k) is complete, by 3.3.

11.2. PROPOSITION. <u>If</u> (E,u) <u>is a regular space in</u> ENS^s , <u>then</u> $(E, q_E(u))$ <u>is a regular convergence space</u>.

<u>Proof</u>. Put $q = q_E(u)$ and $q^* = q_{E^*}(u^*)$. If $f : A \longrightarrow E^*$ and $g : A \longrightarrow E$ are mappings such that $f(x)\; q\; g(x)$ for all $x \in A$, then $g : (A, f^s(u^*)) \longrightarrow (E,u)$ is continuous. Since $q_A(f^s(u^*)) = f^q(q_{E^*}(u^*)) = f^q(q^*)$, the map $g : (A, f^q(q^*)) \longrightarrow (E,q)$ also is continuous. Thus (E,q) is regular▯

11.3. THEOREM. <u>A separated space</u> (E,u) <u>in</u> ENS^s <u>is complete and regular if and only if</u> (E,u) <u>is the underlying space of an algebra for</u> (R, j, k) .

<u>Proof</u>. If $q : (E^*,u^*) \longrightarrow (E,u)$ is an algebra for (R, j, k) , for a separated space (E,u) , then $q : E^* \longrightarrow E$ is filter convergence by 3.3, and

continuous as a relation by 7.6. Thus (E,q) is complete and regular.

Conversely, if (E,u) is separated, regular and complete, then filter convergence defines a map $q : (E^*,u^*) \longrightarrow (E,u)$ in ENS^S. By 3.3, this map is the only possible algebra structure of (E,u) for the triple (R, j, k). Thus we must show that $q\, j_A = id\,(E,u)$ and $q\, k_A = q\,(R\, q)$ for $A = (E,u)$.

The first of these laws requires $\dot{x}\; q\; x$ for all $x \in E$. This is always true. The second law requires that $\Phi_*\; q\; x \implies q_*(\Phi)\; q\; x$ for a filter in $U\, R\, (E^*,q^*)$ and $x \in E$. If $X \in \Phi_*$, then $q(X^*) = \bar{\bar{X}}$ is in $q_*(\Phi)$. Thus $q_*(\Phi) \leq \bar{\bar{\varphi}}$ for $\varphi = \Phi_*$, and $q\, k_A = q\,(R\, q)$ reflects T_3 for (E,q). This is valid by 11.2 and 9.4 ▯

11.4. THEOREM. <u>If</u> $f : (E,u) \longrightarrow (F,v)$ <u>is a map in</u> ENS^S <u>with</u> (F,v) <u>separated, regular, complete, then there is exactly one map</u> $g : R\,(E,u) \longrightarrow (E,v)$ <u>in</u> ENS^S <u>for which</u> $g\, j_A = f$, <u>for</u> $A = (E,u)$.

<u>Proof.</u> Put $B = (F,v)$. By 11.3, filter convergence defines an algebra $q_B : R\, B \longrightarrow B$ for the triple (R, j, k), and $q_B\,(R\, f)\, j_A = q_B\, j_B\, f = f$. Thus $g = q_B\,(R\, f)$ satisfies the conditions.

On the other hand, if $g : R\, A \longrightarrow B$ in ENS^S and $\Phi \in U\, R\, R\, A$, then $\Phi\; q\; \Phi_*$, and thus $g_*(\Phi)\; q_B\; g(\Phi_*)$. Thus $g\, k_A = q_B\, g_*$ at the set level, and $g\, k_A = q_B\,(R\, g)$ at the ENS^S level. In other words, g is a homomorphism of algebras for (R, j, k). Now

$$q_B\, R(g\, j_A) \;=\; q_B\,(R\, g)(R\, j_A) \;=\; g\, k_A\,(R\, j_A) \;=\; g \;.$$

Thus $g = q_B\,(R\, f)$ is the only map $g : R\, A \longrightarrow B$ such that $g\, j_A = f$ ▯

11.5. REMARK. By 3.3, a separated space (E,u) has at most one algebra structure for a Cauchy triple (R, j, k) with convergence, and by 11.3 such a structure exists if, and only if, (E,u) is separated and complete. The proof of 11.4 consists of two parts. We show first that every map $f : R\,A \longrightarrow B$ in ENS^s with B regular, separated, complete, is a homomorphism $f : (R\,A, k_A)$ (B, q_B) of algebras, and then we prove a general triple-theoretic statement for the case on hand: if (B, q) is a triple algebra and $f : A \longrightarrow B$ a morphism, then $f = g\, j_A$ for a unique morphism $g : (R\,A, k_A) \longrightarrow (B,q)$ of algebras.

12. COMPLETIONS. We show that separated regular complete spaces, for a given Cauchy triple (R, j, k) with convergence on ENS^s, form a reflective subcategory of ENS^s, and an epireflective subcategory of the category of separated spaces in ENS^s, provided that the latter category is nice enough to be colocally small. We use a general theorem of Herrlich for this purpose.

12.1. PROPOSITION. The following three properties of a map $f : (E,u)$ (F,v) of separated spaces in ENS^s are logically equivalent.

(i) $f : E \longrightarrow F$ in ENS is injective.

(ii) f is monomorphic in ENS^s.

(iii) f is monomorphic in the category of separated spaces of ENS^s.

Proof. (i) $\Longrightarrow$ (ii) trivially: the projection functor $ENS^s \longrightarrow ENS$, like every faithful functor, reflects monomorphisms. (ii) $\Longrightarrow$ (iii) even more trivially. We prove (iii) $\Longrightarrow$ (i) by contradiction.

Suppose $f(x) = f(y)$ but $x \neq y$. Consider a space $(\{a\}, t)$ consisting of a singleton with the finest structure. This space is necessarily separated. $g(a) = x$ and $h(a) = y$ defines maps of ENS^s from this space to (E,u) by the choice of t . We have $g \neq h$, but $f g = f h$ ▯

12.2. LEMMA. Let $f : (E,u) \longrightarrow (F,v)$ be an extremal monomorphism of separated spaces in ENS^s , for a Cauchy triple (R, j, k) with convergence. If $\varphi \in E^*$ and $f_*(\varphi)\, q_B\, y$ for $B = (F,v)$ and $y \in F$, then $y = f(x)$ for a point $x \in E$ such that $\varphi\, q_A\, x$, for $A = (E,u)$.

Proof. We put $R\,(E,u) = (E^*,u^*)$. Let F' be the set of all $y \in F$ such that $f_*(\varphi)\, q_B\, y$ for some $\varphi \in E^*$. Since $f_*(\dot{x})\, q_B\, f(x)$ for $x \in E$, we have $f(E) \subset F'$, and hence a factorization

$$(E,u) \xrightarrow{g} (F', j^s(v)) \xrightarrow{j} (F,v)$$

of f in ENS^s where $j : F' \longrightarrow F$ is the inclusion mapping. The space in the middle obviously is separated. If $h\, g = h'\, g$ for maps h , h' in ENS^s with separated codomain, then $h_*(g_*(\varphi)) = h'_*(g_*(\varphi))$ and $f_*(\varphi) = j_*(g_*(\varphi))$ for a filter $\varphi \in E^*$. If $B' = (F', j^s(v))$, then $q_{B'} = j^q(q_B)$ for the convergence relation. If $f_*(\varphi)\, q_B\, y$, it follows that $g_*(\varphi)\, q_{B'}\, y$ (note that $y = j(y)$). In this situation, $h_*(g_*(\varphi))$ converges to $h(y)$ and to $h'(y)$. Since the codomain of h and h' is separated, $h(y) = h'(y)$ follows. Thus $h = h'$, and h is epimorphic. As f is extremal, h is isomorphic in ENS^s , and a fortiori bijective. Thus $F' = f(E)$.

Now $u = f^s(v)$ since otherwise

$$(E,u) \xrightarrow{\text{id } E} (E,f^s(v)) \xrightarrow{f} ((F,v)$$

in ENS^s , with id E epimorphic but not isomorphic, and the middle space separated. This cannot happen if f is extremal. It follows that $q_A = f^q(q_B)$, and thus $\varphi\, q_A\, x$ for $\varphi \in E^*$ if $f_*(\varphi)\, q_B\, y$ for $y = f(x)$ in F ▯

12.3. THEOREM. *For a Cauchy triple* (R, j, k) *with convergence on* ENS^s *, the class of separated regular complete spaces is closed under the formation of products and of extremal subobjects.*

Proof. Consider first $A = \prod A_i$ with projections π_i , with $A = (E,u)$ and $A_i = (E_i,u_i)$. Convergence for A_i defines an algebra $q_i : R\,A_i \longrightarrow A_i$ for the triple (R, j, k) . There is a unique map $q : R\,A \longrightarrow A$ in ENS^s such that $\pi_i\, q = q_i\,(R\pi_i)$ for all i . We have

$$\pi_i\, q\, j_A = q_i\,(R\pi_i)\, j_A = q_i\, j_{A_i}\pi_i = \pi_i ,$$

and

$$\pi_i\, q\,(R\,q) = q_i\,(R\pi_i)(R\,q) = q_i\, R(\pi_i\, q)$$

$$= q_i\, R(q_i\,(R\pi_i)) = q_i\,(R\,q_i)(R\,R\pi_i)$$

$$= q_i\, k_{A_i}\,(R\,R\pi_i) = q_i\,(R\pi_i)\, k_A = \pi_i\, q\, k_A ,$$

for all π_i . Thus $q\, j_A = \mathrm{id}\, A$ and $q\,(R\,q) = q\, k_A$, and we have an algebra $q : R\,A \longrightarrow A$. Since A is separated by 8.4, this shows that A is regular and complete, by 11.3.

Now let $A = (E,f^s(v))$ and $B = (F,v)$ for an extremal monomorphism $f : A \longrightarrow B$ of separated spaces, and put $R\,A = (E^*,u^*)$. If B is separated and complete and $\varphi \in E^*$, then $f_*(\varphi)\; q_B\; y$ for a unique $y \in F$. By 12.2, there is a unique $x \in E$ such that $\varphi\; q_A\; x$ and $y = f(x)$. Thus convergence defines a mapping $q_A : E^* \longrightarrow E$ such that $f\, q_A = q_B\, f_*$. Since q_B and f_* are continuous, we have $q_A : (E^*,u^*) \longrightarrow (E,f^s(v))$ in ENS^s . Thus A is regular and complete []

12.4. COROLLARY. *If the category of separated spaces in* ENS^s *is colocally small, then separated regular complete spaces define a reflective subcategory of*

ENS^s *and an epireflective subcategory of the category of separated spaces.*

Proof. The category of separated spaces in ENS^s is locally small by 12.2. It is reflective in ENS^s and hence complete. Thus the general theory of Herrlich applies if it is colocally small$\square$

13. AN UNPLEASANT EXAMPLE. Much of this paper is a buildup toward 12.4. We proceed to deflate 12.4 by proving the following result.

13.1. THEOREM. *The following categories are not colocally small*: *separated convergence spaces*, *separated limit spaces*, *separated closure spaces*, *separated uniform convergence spaces*, *separated uniform limit spaces*.

Proof. With obvious symbols, we have forgetful functors as follows.

$$\begin{array}{ccccc} & & UNLIM & \longrightarrow & UNCV \\ & & \downarrow & & \downarrow \\ CLOS & \longrightarrow & LIM & \longrightarrow & CONV \end{array}$$

The horizontal arrows represent full and faithful functors with left inverse left adjoints. These functors preserve separated spaces and epimorphisms of separated spaces. The vertical arrows represent functors with left adjoints which preserve separated spaces and their epimorphisms. Thus it is sufficient for the proof to show that there are separated closure spaces which can be embedded as dense subspaces into closure spaces of arbitrarily high cardinality. An example follows$\square$

13.2. We recall that a set $X \subset E$ is called *open* for a convergence space (E,q) if $\varphi\, q\, x$, for a filter φ on E and $x \in X$, always implies $X \in \varphi$. A set $X \subset E$ is called *dense* for (E,q) if $\emptyset$ is the only open set for (E,q)

which is disjoint from X .

A map $f : (E,q) \longrightarrow (F,r)$ of separated convergence spaces is epimorphic in the category of separated convergence spaces if (and in fact only if) $f(E)$ is dense for (F,r) . For consider maps g and h from (F,r) to the same separated space, and let Y be the set of all $y \in F$ such that $g(y) \neq h(y)$. If $\psi \; r \; y$ for a filter ψ on F and $y \in Y$, then the filters $g_*(\psi)$ and $h_*(\psi)$ converge to points $g(y) \neq h(y)$. A filter finer than $g_*(\psi)$ and $h_*(\psi)$ would converge to both points, and this cannot happen in a separated space. Thus $g_*(\psi) \cap h_*(\psi)$ is the null filter, and hence $g(Y') \cap h(Y") = \emptyset$ for sets Y' and $Y"$ in ψ . But then $Y' \cap Y" \subset Y$, and hence $Y \in \psi$. Thus Y is open. If $g\,f = h\,f$, then $Y \cap f(E) = \emptyset$, and if $f(E)$ is dense for (F,r) , this implies $Y = \emptyset$ and hence $g = h$.

13.3. We construct the example. We begin with an infinite discrete space (E_0,q_0) . By transfinite induction, we construct for each ordinal number $n > 0$ a space (E_n,q_n) and for each pair of ordinals $m \leq n$ a map $u_{mn} : (E_m,q_m)$ (E_n,q_n) , with the following properties.

13.3.1. $u_{nn} = id\,(E_n,q_n)$, and $u_{mp} = u_{np}\,u_{mn}$ if $m \leq n \leq p$.

13.3.2. Every map u_{mn} is injective.

13.3.3. Every space (E_n,q_n) is a closure (or neighborhood) space, and if $m \leq n$ and N_x is the neighborhood filter of $x \in E_m$ for q_m , then $(u_{mn})_*(N_x)$ is the neighborhood filter of $u_{mn}(x)$ in (E_n,q_n) .

Suppose that the spaces (E_n,q_n) and maps u_{mn} are already constructed for $m \leq n < k$. If k is a limit ordinal, put $(E_k,q_k) = \lim\,(E_n,u_n)$ in the category of convergence spaces, for the given spaces and maps. This is a directed limit, and injective maps $u_{nk} : (E_n,q_n) \longrightarrow (E_k,u_k)$ are automatically given so that $u_{nk}\, u_{mn} = u_{mk}$ if $m \leq n < k$. The maps u_{mk} are "collectively onto", and one sees easily that 13.3.3, for $n = k$, describes q_k fully. If p , q are in E_k , then $p = u_{nk}(x)$ and $q = u_{nk}(y)$ for one $n < k$ and points x , y of E_n . The neighborhood filters N_x and N_y for q_n contain disjoint sets X and Y , and then $u_{nk}(X)$ and $u_{nk}(Y)$ are in the neighborhood filters N_p and N_q for q_k . Since u_{nk} is injective, these sets are disjoint, and thus (E_k, q_k) is separated.

Let now $k = h + 1$. Let E_k consist of all filters $\dot{x}$ with $x \in E_h$ and all ultrafilters on E_h which do not converge for q_h . Put $u_{hk}(x) = \dot{x}$ for $x \in E_h$, and put $u_{mk} = u_{hk}\, u_{mh}$ if $m \quad h$. This defines the maps u_{mk} . Put $N_{\dot{x}} = (u_{hk})_*(N_x)$ for $x \in E_h$ and $N = \dot{\varphi} \cup (u_{hk})_*(\varphi)$ for $\varphi \in E_k \setminus u_{hk}(E_h)$. This defines q_k . The conditions of 13.3 are easily verified. We show that (E_k, q_k) is separated by showing that the neighborhood filters N_p and N_q of distinct points p and q of E_k contain disjoint sets. Three cases must be considered; we leave the details to the reader.

13.4. We show by transfinite induction that $u_{mn}(E_m)$ is dense for (E_n, q_n) if $m \leq n$. Thus all maps u_{mn} are epimorphic.

Suppose that $u_{mn}(E_m)$ is dense for (E_n, q_n) if $m \leq n < k$. Let $A \subset E_k$ be open and disjoint from $u_{mk}(E)$. If $u_{nk}(x) \in A$ for some $x \in E_n$, with $n < k$, then necessarily $m < n$. In this situation, $u_{nk}^{-1}(A)$ is open for (E_n, q_n) and disjoint from $u_{mn}(E_m)$, and $x \in u_{nk}^{-1}(A)$. This contradicts our assumption that $u_{mn}(E_m)$ is dense for (E_n, q_n) . If $\varphi \in A$ is not in one of the sets $u_{nk}(E_n)$ with $n < k$, then k cannot be a limit ordinal, and φ is a non-convergent ultrafilter on E_h if $k = h + 1$. But then $(u_{hk})_*(\varphi)\ q_h\ \varphi$ by our construction, and hence $A \in (u_{hk})_*(\varphi)$, and $u_{hk}^{-1}(A) \in \varphi$. Thus

${u_{hk}}^{-1}(A)$ is not empty, and we obtain the same contradiction as before for $x \in {u_{hk}}^{-1}(A)$. This shows that we must have $A = \emptyset$, as claimed.

13.5. We examine the cardinalities $|E_n|$. Let $G \; q_n \; x$ for an ultrafilter G on E_n and $x \in E_n$, and let $m \leqslant n$ be the least ordinal number such that $x = u_{mn}(z)$ for some $z \in E_m$. Then $G \leqslant (u_{mn})_*(N_z)$, and it follows that $G = (u_{mn})_*(F)$ for some ultrafilter $F \leqslant N_z$ on E_m . If $m = 0$, then $F = \dot{z}$ and $G = \dot{x}$. If $m > 0$, then m cannot be a limit ordinal. If $m = h + 1$, then z is a non-convergent ultrafilter on E_h , and $F \leqslant \dot{z} \cup (u_{hm})_*(z)$. Thus either $F = \dot{z}$ or $F \leqslant (u_{hm})_*(z)$, and then $F = (u_{hm})_*(z)$. This shows that at most two ultrafilters on E_n converge to a point of E_n . It follows that $|E_{n+1}|$ is the cardinality of the set of all ultrafilters on E_n , and thus $|E_n| < |E_{n+1}|$. We conclude that $|E_n|$ can be made arbitrarily large for n large enough, and thus our example does the job it is supposed to do.

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