SPHERES WITH MAXIMUM

INNER DIAMETER

by

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1. Introduction

Let X be a real normed linear space, and let $\Sigma(X)$ be its unit ball, with the boundary $\partial\Sigma(X)$. If $\dim X \geqq 2$, δ_X denotes the inner metric of $\partial\Sigma(X)$ induced by the norm (cf. [1; Section 3]). If no confusion is likely, we write Σ, $\partial\Sigma$, δ. In [1] we introduced and discussed parameters of X based on the metric structure of $\partial\Sigma$, namely $D(X) = \sup\{\delta(p,q) : p,q \in \partial\Sigma\}$, $M(X) = \sup\{\delta(-p,p) : p \in \partial\Sigma\}$, $m(X) = \inf\{\delta(-p,p) : p \in \partial\Sigma\}$. $D(X)$ is the <u>inner diameter</u> of $\partial\Sigma(X)$, and $2M(X)$ and $2m(X)$ are the <u>perimeter</u> and the <u>girth</u> of $\Sigma(X)$, respectively. In additional papers [2], [3], [4] we discussed the range of the girth for finite- and infinite-dimensional spaces.

In this note we propose to study the spaces X for which the inner diameter and the perimeter take their greatest possible value, namely $D(X) = M(X) = 4$. For the finite-dimensional among these spaces we obtain a complete characterization: they are precisely those spaces whose unit ball is a cylinder or the intersection of two cones.

We use the terminology, notation, and results of Sections 1-5 of [1] freely. That paper shall be referred to as S, and its contents quoted as, e.g., Theorem S.3.4, formula S(3.1).

We know that $M(X) = D(X)$ if $\dim X$ is 2 or 3, but for higher dimensions this equality is a matter of conjecture (Theorem S.5.8, Conjecture S.9.1). It might therefore be questioned whether the conditions $D(X) = 4$ and $M(X) = 4$ are, in fact, equivalent. The following proposition resolves this doubt.

1. <u>Theorem</u>. <u>Assume</u> $\dim X \geqq 2$. <u>Then</u>

$$(1.1) \qquad 2D(X) - 4 \leqq M(X) \leqq D(X) \leqq 4 .$$

<u>In particular</u>, $D(X) = 4$ <u>if and only if</u> $M(X) = 4$. <u>In that case</u>, <u>if</u> $p,q \in \partial\Sigma$ <u>and</u> $\delta(p,q) = 4$, <u>then</u> $p + q = 0$.

<u>Proof</u>. All inequalities in (1.1) except the first hold by Lemma S.5.2. If $p,q \in \partial\Sigma$, let Y be a two-dimensional subspace of X containing p,q (it is unique if $p \pm q \neq 0$). By Theorem S.3.4,(a), the first papagraph of Section 4 of S, and Theorem S.4.2,

$$\delta(-p,q) + \delta(q,p) \leqq \delta_Y(-p,q) + \delta_Y(q,p) = L(Y) \leqq 4 ;$$

therefore

(1.2) $$4 \geqq \delta(-p,p) \geqq \delta(p,q) - \delta(-p,q) \geqq 2\delta(p,q) - 4 .$$

Taking suprema for all $p,q \in \partial\Sigma$ in the second and last members of (1.2), we obtain the first inequality in (1.1). The equivalence of $D(X) = 4$ and $M(X) = 4$ follows from (1.1). If $\delta(p,q) = 4$, equality must hold throughout (1.2); therefore $\delta(-p,q) = 0$, i.e., $-p = q$.

<u>Remark</u>. The last part of the statement indicates that when $D(X) = M(X) = 4$, the suprema in the definitions of $D(X)$ and $M(X)$ are attained together and at the same points, if either is attained at all.

2. <u>Homothetic disks</u>.

The auxiliary result proved in this section is surely known. A proof without recourse to a euclidean metric or to the calculus would be desirable.

2. <u>Lemma</u>. *Let K_1, K_2 be compact convex sets in a two-dimensional real vector space, with O as a common interior point. For each ray from O, assume that K_1 and K_2 have a pair of parallel supporting lines at the*

respective boundary points on the ray. _Then there exists a number_ $\rho > 0$ _such that_ $K_2 = \rho K_1$.

Proof. Choose a euclidean metric and a system of polar co-ordinates in the given plane, and let $r = f_i(\theta) > 0$, $0 \leqq \theta < 2\pi$ be the equation of the boundary of K_i , $i = 1,2$. For all θ with a countable set of exceptions, both K_1 and K_2 have unique supporting lines at the boundary points corresponding to θ . Since these lines are parallel, $f_1'(\theta)/f_1(\theta) = f_2'(\theta)/f_2(\theta)$ for all θ with the same set of exceptions (here primes indicate differentiation with respect to θ). Since K_i is convex, f_i is absolutely continuous, $i = 1,2$; integration therefore yields $f_1(\theta)/f_1(0) = f_2(\theta)/f_2(0)$ for all θ . The conclusion holds with $\rho = f_2(0)/f_1(0)$.

3. Poles.

We now return to the study of a normed space X with $\dim X \geqq 2$. If $D(X) = M(X) = 4$ and we are interested in the pairs of points where the supremum in the definition of $D(X)$ is attained--if any exist--the Remark to Theorem 1 indicates that we need only consider antipodes. We define a _pole_ of X (or of Σ , or of $\partial\Sigma$) to be a point $u \in \partial\Sigma$ such that $\delta(-u,u) = 4$. While poles can exist only in spaces X with $D(X) = M(X) = 4$, such spaces need not, in general, have poles (see Example 4.3); but they must if $\dim X < \infty$, by Lemma S.5.1,(b).

If u is a pole of X, it is also a pole of every subspace Y of X that contains u, since $4 = \delta(-u,u) \leqq \delta_Y(-u,u) \leqq 4$. If, in particular, $\dim Y = 2$, then $L(Y) = 4$ and $\Sigma(Y)$ is a parallelogram, by S(4.2) and Theorem S.4.2. This observation shows, incidentally, that if u is a pole there actually exists a curve of length 4 (indeed a plane one) from $-u$ to u in $\partial\Sigma$.

The next proposition gives an interim account of the location of poles. If $u \in \partial\Sigma$, we denote by $\underset{\sim}{P}_u$ the set of all two-dimensional subspaces of X that contain u.

3. <u>Lemma</u>. <u>Assume</u> $\dim X \geqq 3$. <u>A pole</u> u <u>of</u> X <u>is either a vertex of the parallelogram</u> $\Sigma(Y)$ <u>for every</u> $Y \in \underset{\sim}{P}_u$ <u>or the midpoint of a side of</u> $\Sigma(Y)$ <u>for every</u> $Y \in \underset{\sim}{P}_u$.

<u>Proof</u>. 1. Any two distinct subspaces in $\underset{\sim}{P}_u$ are contained in a single three-dimensional subspace that contains u and therefore has u as a pole. Replacing X by this subspace, if necessary, we shall therefore assume without loss that $\dim X = 3$.

2. If there exist distinct $Y, Z \in \underset{\sim}{P}_u$ such that u is interior both to a side of $\Sigma(Y)$ and to a side of $\Sigma(Z)$, these sides span the unique supporting plane of Σ at u. If, on the other hand, u is a vertex of $\Sigma(Y)$ for some $Y \in \underset{\sim}{P}_u$, the two sides of $\Sigma(Y)$ that meet at u must belong to distinct supporting planes of Σ at u.

We claim that u cannot be interior to a side of $\Sigma(Y)$ for exactly one $Y \in \underset{\sim}{P}_u$. For assume this to be the case; then every supporting plane of Σ at u must contain this side; since the assumption and the preceding paragraph rule out a unique supporting plane of Σ at u, there are exactly two, say $u + V$ and $u + W$, with $V \neq W$.

For each $Z \in \underset{\sim}{P}_u$ distinct from Y, the sides of $\Sigma(Z)$ meeting at the vertex u must lie in $u + V$ and $u + W$; therefore another vertex must lie on the straight line $(u + V) \cap -(u + W)$; conversely, if p is any point on this line and Z is the subspace spanned by u and p, we have $Z \neq Y$, and therefore p is a vertex of $\Sigma(Z)$. Thus every point on this line lies in Σ, which is absurd. Our claim is established.

We conclude from this part of the proof that u is either a vertex of $\Sigma(Y)$ for every $Y \in \underset{\sim}{P}_u$ or else an interior point of a side of $\Sigma(Y)$ for every $Y \in \underset{\sim}{P}_u$. It remains to prove that in the latter case u is the midpoint of each such side.

3. Assume, then, that u is interior to a side of $\Sigma(Y)$ for every $Y \in \underset{\sim}{P}_u$, and let W be the two-dimensional subspace such that $u + W$ is the unique supporting place of Σ at u. Let us look at the "lid" of Σ: we have $\Sigma \cap (u+W) = u + K$, were K is a compact convex set in W with 0 in its interior; let ∂K be its boundary in W. Similarly for the "bottom": $\Sigma \cap (-u + W) = -u - K$.

Let a ray from 0 in W be given, and let p and q be its intersections with ∂K and $-\partial K$, respectively. Let $Y \in \underset{\sim}{P}_u$ be the subspace spanned by u and the given ray; $\Sigma(Y)$ is then the parallelogram with vertices $\pm(u+p)$, $\pm(-u+q)$. Consider a supporting plane of Σ that contains the side $u + p, -u + q$ of this parallelogram; its (parallel) intersections with $u + W$ and $-u + W$ are supporting lines of the "lid" $u + K$ at $u + p$ and of the "bottom" $-u - K$ at $-u + q$, respectively. Therefore K and $-K$ have parallel supporting lines at p and q, respectively. Since the ray was arbitrary, we conclude that $-K = \rho K$ for some $\rho > 0$; but then $K = -\rho K = \rho^2 K$, so that $\rho = 1$ and $-K = K$. It follows that, in the preceding construction, $q = p$, and the vertices of $\Sigma(Y)$ are $\pm(u + p)$,

$\pm(-u + p)$; thus u is the midpoint of the side $u - p$, $u + p$. Since every $Y \in \underset{\sim}{P}_u$ is spanned by u and a ray from O in W, this completes the proof.

4. Cylinders and Rhombuses.

We can now characterize all spaces possessing poles; this includes, as noted, all finite-dimensional spaces X with $D(X) = M(X) = 4$.

Assume that $\dim X \geqq 2$, and consider $u \in \partial\Sigma$. X is cylindrical at u if there exists a normed space Z and a congruence (bijective linear isometry) $S : R \oplus Z \to X$ such that $S(1 \oplus O) = u$; X is rhombic at u if $\Sigma = (u - C) \cap (-u + C)$, where C is a closed convex cone with apex O and u in its interior. If there exists a $u \in \partial\Sigma$ such that X is cylindrical [rhombic] at u, Σ is said to be a cylinder [a (solid) rhombus]. "Cylinder" thus appears in its usual meaning; "solid rhombus" as a term for the intersection of two "opposite" cones has the authority of Archimedes (cf. Περὶ σφαίρας καὶ κυλίνδρου α', 18), although his cones were three-dimensional and circular, but not necessarily congruent. Σ may well be both a cylinder and a rhombus, as, e.g., when it is a parallelotope.

4.1. Theorem. <u>Assume</u> $\dim X \geqq 3$. <u>Then</u> $u \in \partial\Sigma$ <u>is a pole of</u> X <u>if and only if</u> X <u>is either cylindrical or rhombic at</u> u .

<u>Proof</u>. 1. If X is cylindrical at u we may, replacing X by a congruent space if necessary, assume that $X = R \oplus Z$ for some normed space Z and that $u = 1 \oplus 0$. The proof of Lemma S.5.10 then shows that $\delta(-1 \oplus 0\ ,\ 1 \oplus 0) = 4$, so that $u = 1 \oplus 0$ is a pole.

Assume X is rhombic at u , with $\Sigma = (u - C) \cap (-u + C)$, and let ∂C be the boundary of C . Let c be any curve from $-u$ to u in $\partial\Sigma \subset (u - \partial C) \cup (-u + \partial C)$. Since u is interior to C , hence to $-u + C$, c must intersect the "edge" $(u - \partial C) \cap (-u + \partial C)$; if p is a point in the intersection and Y is the subspace spanned by u and p , $\Sigma(Y)$ is obviously a parallelogram with vertices $\pm u$, $\pm p$. Therefore $\ell(c) \geqq \|p + u\| + \|u - p\| = \delta_Y(-u,u) = L(Y) = 4$; since c was arbitrary, $\delta(-u,u) \geqq 4$, and u is a pole.

2. For the converse, assume that u is a pole and apply Lemma 3 . If u is a vertex of $\Sigma(Y)$ for every $Y \in \underset{\sim}{P}_u$, let $u - C$ be the supporting cone of Σ at u ; C is convex. For every $Y \in \underset{\sim}{P}_u$, $(u - C) \cap Y$ is the supporting cone of $\Sigma(Y)$ at u , i.e., the angle between

the sides of $\Sigma(Y)$ meeting at u; therefore $(u - C) \cap (-u + C) \cap Y = \Sigma(Y)$. Thus

$$\Sigma = \bigcup\{\Sigma(Y) : Y \in \underset{\sim}{P}_u\} = (u - C) \cap (-u + C) \cap \bigcup\{Y : Y \in \underset{\sim}{P}_u\}$$
$$= (u - C) \cap (-u + C) ,$$

and X is rhombic at u.

If, on the other hand, u is the midpoint of a side of $\Sigma(Y)$ for every $Y \in \underset{\sim}{P}_u$, let $u + Z$ be the (obviously unique) supporting hyperplane of Σ at u. We claim that the mapping $\lambda \oplus z \mapsto \lambda u + z : R \oplus Z \to X$ is a congruence, so that X is cylindrical at $1u + 0 = u$. Indeed, the mapping is linear and bijective (since $u \notin Z$). For given $\lambda \in R$, $z \in Z$, let $Y \in \underset{\sim}{P}_u$ be a two-dimensional subspace containing u and z (unique if $z \neq 0$). Then $\Sigma(Y)$ is a parallelogram and the midpoints of its sides are $\pm u$ and, say, $\pm z_0$, where $z = \|z\| z_0$. Then $\|\lambda u + z\| = \|\lambda u + \|z\| z_0\|_Y$ $= \max\{|\lambda|, \|z\|\} = \|\lambda \oplus z\|_{R \oplus Z}$; thus the mapping is also isometric, and our claim is proved.

<u>Remark</u>. If $\dim X = 2$ and $u \in \partial\Sigma(X)$, then $\delta(-u,u) = L(X)$; thus u is a pole if and only if $L(X) = 4$, i.e., if and only if Σ is a parallelogram, regardless of whether u is a vertex, or a midpoint of a side, or neither.

4.2. <u>Corollary</u>. <u>If</u> $2 \leqq \dim X < \infty$, <u>then</u> $D(X) = M(X) = 4$ <u>if and only if</u> $\Sigma(X)$ <u>is a cylinder or a rhombus</u>.

<u>Proof</u>. As noted at the beginning of Section 3, $D(X) = M(X) = 4$ is equivalent, for finite-dimensional X, to the existence of a pole. For $\dim X \geqq 3$ the conclusion then follows from Theorem 4.1. If $\dim X = 2$, a pole exists if and only if Σ is a parallelogram (Remark to Theorem 4.1); but Σ is a cylinder or a rhombus (in fact both) in exactly the same case.

4.3. <u>Example</u>. We construct a space X without poles but satisfying $D(X) = M(X) = 4$; its dimension will be countably infinite, but similar examples can be constructed with any prescribed infinite dimension. Let X be the space of all sequences $x = (x_i)$ of real numbers with only finitely many non-zero terms, with the termwise algebraic operations. The set $\Sigma = \{x \in X : \sum_{i=1}^{\infty} |x_i|^{i+1} \leqq 1\}$ is non-empty, convex, balanced, radial, and radially closed, so that X is a normed space with unit ball Σ. Since Σ is strictly convex, it is neither a cylinder nor a rhombus; thus X has no poles. It remains to show that $D(X) = 4$.

Set $e_n = (\delta_{ni})$, $n = 1,2,\ldots,$ the usual basis vectors. Let Y_n be the subspace of X defined by $Y_n = \{x \in X : x_n = 0\}$, $n = 1,2,\ldots$. We consider the obvious isomorphisms $T_n : R \oplus Y_n \to X$ defined by $T_n(\lambda \oplus y) = \lambda e_n + y$; the inverses $T_n^{-1} : X \to R \oplus Y_n$ are given by $T_n^{-1} x = x_n \oplus (x - x_n e_n)$.

Let n be fixed for the time being. Assume $\lambda \oplus y \in \Sigma(R \oplus Y_n)$, i.e., $|\lambda| \leqq 1$, $y_n = 0$, $\sum_1^\infty |y_i|^{i+1} \leqq 1$. Set $x = \alpha_n T_n(\lambda \oplus y)$, where $\alpha_n = n^{-n^{-1}} < 1$. Then

$$\sum_{i=1}^{\infty} |x_i|^{i+1} = |\alpha_n \lambda|^{n+1} + \sum_{i=1}^{\infty} |\alpha_n y_i|^{i+1}$$

$$\leqq \alpha_n^{n+1} + \alpha_n^2 < \alpha_n^{n+1} + \alpha_n = n^{-n^{-1}}(n^{-1} + 1)$$

$$= (n^{-1}(1 + n^{-1})^n)^{n^{-1}} < (n^{-1} e)^{n^{-1}} \leqq 1 ,$$

provided $n \geqq 3$. With this proviso, we conclude that $x = \alpha_n T_n(\lambda \oplus y) \in \Sigma(X)$, so that

$$\|T_n\| \leqq n^{n^{-1}} , \quad n \geqq 3 . \tag{4.1}$$

On the other hand, if $x \in \Sigma(X)$, then obviously $|x_n| \leqq 1$ and $x - x_n e_n \in Y_n \cap \Sigma(X) = \Sigma(Y_n)$, so that

$T_n^{-1} x \in \Sigma(R \oplus Y_n)$. Therefore

(4.2) $$\|T_n^{-1}\| \leqq 1 , \quad n \geqq 1 .$$

We now make use of Sections 6 and 7 of S. Combining (4.1), (4.2), we find $\Delta(X, R \oplus Y_n) \leqq \log \|T_n\| \, \|T_n^{-1}\| \leqq n^{-1} \log n$, $n \geqq 3$. By S(7.3) and Lemma S.5.10, $|4 - D(X)| = |D(R \oplus Y_n) - D(X)| \leqq 6(n^{n^{-1}} - 1) \to 0$ as $n \to \infty$. Therefore $D(X) = 4$.

The same conclusions hold for the completion of this space X, a separable Banach space.

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