OSCILLATION AND NONOSCILLATION THEOREMS FOR SUBLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

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1. Consider the following second order nonlinear equation

(1) $$y'' + q(x)\operatorname{sgn} y\, |y|^{\alpha} = 0, \quad x > 0,$$

where $q(x)$ is continuous and nonnegative for $x > 0$ and α is a positive constant. Equation (1) may be conveniently classified as superlinear or sublinear according to whether the constant α is greater than or less than 1. We are here interested in the oscillatory behavior of solutions of (1), and in particular, in the extension of results concerning equation (1) to the more general equation

(2) $$y'' + yF(y^2,x) = 0,$$

where $yF(y^2,x)$ is continuous for $x > 0$ and $|y| < \infty$, and $F(t,x)$ is nonnegative for $t \geq 0$ and $x > 0$. Accordingly, we say that equation (2) is <u>superlinear</u> if $F(t,x)$ satisfies

(3) $$F(t_1,x) \leq F(t_2,x), \qquad t_1 \leq t_2,$$

for all x, and it is <u>sublinear</u> if $F(t,x)$ satisfies

(4) $$F(t_1,x) \geq F(t_2,x), \qquad t_1 \leq t_2,$$

for all x.

Results on the oscillatory behavior of solutions of (2) are of two types, namely, (i) sufficient conditions for all solutions to be oscillatory and for the converse, the existence of a nonoscillatory solution, and (ii) sufficient conditions for all solutions to be nonoscillatory and for the converse, the existence of an oscillatory sllution. Here a solution always means a nontrivial solution and it is called oscillatory if it has arbitrarily large zeros. On the other hand, a solution is called nonoscillatory if it is not oscillatory, i.e. if it is of one sign for all large t. In this paper, we consider both the superlinear and sublinear equations with regard to necessary and sufficient conditions for oscillation and sufficient conditions for nonoscillation. For results concerning sufficient conditions for the existence of an oscillatory solution, we refer the reader to our earlier work [3] and [4]. The results in this paper differ from those of [3] and [4] in that we consider only integral conditions rather than monotonicity conditions concerning $F(t,x)$. Although we treat both the superlinear and sublinear equations here, most of the new results are concerned with the sublinear case. Our approach to this problem follows from the study of duality between superlinear and sublinear equations initiated in our latest work [4]. The main results presented below may be considered as genuine extensions of results for equation (1) to the more general equation (2), but our emphasis will be on the duality exhibited between solutions of superlinear and sublinear equations. Such an investigation has not been made even for the simpler equation (1).

The integral conditons of concern to equation (1) are

(5) $$\int^{\infty} xq(x)\,dx = \infty$$

and

(6) $$\int^{\infty} x^{\alpha} q(x)\,dx = \infty .$$

As far as necessary and sufficient conditions for the oscillation of all solutions of (1) are concerned, we have the following two interesting results:

THEOREM A. (Atkinson[1]) *Let* $\alpha > 1$. *All solutions of* (1) *are oscillatory if and only if* (5) *holds*.

THEOREM B (Belohorec [2]) *Let* $0 < \alpha < 1$. *All solutions of* (1) *are oscillatory if and only if* (6) *holds*.

Implicit in the proofs of Theorems A and B are the following alternative results:

THEOREM A_1. Let $\alpha > 1$. *Equation* (1) *has a bounded nonoscillatory solution if and only if*

(7) $$\int^{\infty} xq(x)\,dx < \infty .$$

THEOREM B_1. *Let* $0 < \alpha < 1$. *Equation* (1) *has an unbounded asymptotically linear solution if and only if*

(8) $$\int^{\infty} x^{\alpha} q(x)\,dx < \infty .$$

Here by an asymptotically linear solution $y(x)$, we mean a solution for which there exist constants a,b not both zero such that

$$\lim_{x\to\infty} \frac{y(x)}{ax+b} = 1. \tag{9}$$

Theorem A_1 follows from a more general result of Nehari [9] and Theorem B_1 will follow from Theorem 2 of this paper. Combining Theorems A and A_1, we obtain

THEOREM A_2. _Let_ $\alpha > 1$. _The following statements are equivalent_:

(i) _Equation_ (1) _has a bounded asymptotically linear solution,_

(ii) _Equation_ (1) _has a nonoscillatory solution,_

(iii) $\int^{\infty} xq(x)\,dx < \infty$.

On the other hand, Theorems B and B_1 give

THEOREM B_2. _Let_ $0 < \alpha < 1$. _The following statements are equivalent_:

(i) _Equation_ (1) _has an unbounded asymptotically linear solution,_

(ii) _Equation_ (1) _has a nonoscillatory solution,_

(iii) $\int^{\infty} x^{\alpha} q(x)\,dx < \infty$.

Thus from the oscillation and nonoscillation point of view, the following properties may be considered as duals to one another:

(a) the superlinear equation has an asymptotically constant solution, (b) the sublinear equation has an unbounded asymptotically linear solution. Similarly, the integral conditions (7) and (8) become dual to each other. Upon examining Theorem B_1, we find that the conclusion stated is in fact true even for $\alpha > 1$, a result due to Nehari [9] who proved it for the more general equation (2). Now further manifestation of the duality just described led us to conjecture that Theorem A_1 remains valid for $0 < \alpha < 1$, which is indeed the case (Corollary 1).

The search for sufficient conditions for nonoscillation of (1) offers further evidence of the usefulness of the concept of duality between results for superlinear and sublinear equations. Let us consider the following result

THEOREM C (Atkinson [1]) *Let* $\alpha > 1$ *and let* $q(x)$ *be nonincreasing. Then* (8) *is sufficient for all solutions of* (1) *to be nonoscillatory.*

The dual role played by conditions (7) and (8) necessitates the following:

THEOREM D (Heidel [6]) *Let* $0 < \alpha < 1$ *and let* $q(x)$ *be nonincreasing. Then* (7) *is sufficient for all solutions of* (1) *to be nonoscillatory.*

Further evidence of this duality between properties of solutions of superlinear and sublinear equations may be found in [4].

The extensions of the above mentioned results to the more general equation (2) are the main results of this paper. Although the techniques involved in such extensions are in general rather intricate, the results are quite easy to describe. The generalization of the integral conditions (7) and (8) are

$$(10) \qquad \int^{\infty} xF(c^2,x)\,dx < \infty, \quad c > 0,$$

and

$$(11) \qquad \int^{\infty} xF(c^2x^2,x)\,dx < \infty, \quad c > 0.$$

Theorem A_1 has been generalized by Nehari [9] to equation (2) with the condition that (10) hold for some $c > 0$ replacing condition (7). In [9], Nehari also has introduced the following stronger notion of superlinearity:

$$(12) \qquad t_2^{-\varepsilon} F(t_2,x) \geq t_1^{-\varepsilon} F(t_1,x), \quad t_2 \geq t_1.$$

Under this stronger assumption, we have established the generalization of Theorem A_2 to equation (1) in [4] as follows:

THEOREM E. Let $F(t,x)$ satisfy, for some $\varepsilon > 0$, condition (12). The following statements are equivalent:

(i) Equation (2) has a bounded asymptotically linear solution,

(ii) Equation (2) has a nonoscillatory solution,

(iii) For some $c > 0$, (10) holds.

We introduce, as a dual condition to (12), the "stronger" sublinearity conditon: for some $\varepsilon > 0$, $F(t,x)$ satisfies

$$(13) \qquad t_2^{\varepsilon} F(t_2,x) \leq t_1^{\varepsilon} F(t_1,x), \quad t_2 > t_1.$$

The corresponding result for equation (1) when $F(t,x)$ satisfies the stronger sublinearity condition (13) is given as Theorem 2 in the next section. The condition that $q(x)$ be nonincreasing takes the form

$$F(t,x_1) \geq F(t,x_2), \quad x_1 \leq x_2 \tag{14}$$

for all $t > 0$. Theorem C has been generalized to equation (2) again by Nehari [9]. The desired generalization of Theorem D is given below as Theorem 3. The main results concerning oscillation and nonoscillation of solutions of (2) may be summarized in the following table:

$F(t,x)$	(10) $\int^{\infty} xF(c^2,x)\,dx < \infty$	(11) $\int^{\infty} xF(cx^2,x)\,dx < \infty$
(3)	Necessary and sufficient for existence of a bounded nonoscillatory solution. (Nehari [9])	Necessary and sufficient for existence of an unbounded nonoscillatory solution. (Nehari [9])
(4)	Necessary and sufficient for existence of a bounded nonoscillatory solution (Theorem 1)	Necessary and sufficient for existence of an unbounded asymptotically linear solution (Theorem 2)
(12)	Necessary and sufficient for existence of a nonoscillatory solution (Coffman and Wong [4])	(11) ⇒ (10)
(13)	(10) ⇒ (11)	Necessary and sufficient for existence of a nonoscillatory solution (Theorem 2)
(4) ; (14)		Sufficient for nonoscillation (Nehari [9])
(4), (14)	Sufficient for nonoscillation (Theorem 3).	

In section 3, we consider further improvements of Theorems C and D wherein we relax the monotonicity condition on $q(x)$. In particular, it will be shown that the condition

(15) $$\int^{\infty} \frac{q_+'(x)}{q(x)}\, dx < \infty ,$$

where $q_+'(x) = \max(0, q'(x))$, is sufficient for the validity of Theorems C and D. These results are also given for the more general equation (2).

2. In this section, we state and prove three theorems concerning the sublinear equation (2) and which constitute the main results of this paper. The first is the sublinear analogue of a well known result of Nehari [9,Theorem I].

THEOREM 1. _Let_ $F(t,x)$ _satisfy_ (4). _Then equation_ (2) _has a bounded nonoscillatory solution if and only if for some_ $c > 0$, (10) _holds_.

PROOF. Let $y(x)$ be a bounded nonoscillatory of (2) and suppose that for $x > x_o \geq 0$,

(16) $$\frac{c}{2} < y(x) < c.$$

Integrating (2) from 0 to x twice, we obtain

(17) $$y(x) = y(0) + xy'(x) + \int_0^x sy(s)F(y^2(s),s)ds.$$

It is easy to see that $y'(x)$ is positive and non-increasing for $x \geq x_o$, thus

(18) $$y(x) - y(x_o) = \int_{x_o}^{x} y'(s)ds \geq (x-x_o)y'(x).$$

Since $y(x)$ is nondecreasing and bounded it follows from (18) that $xy'(x)$ remains bounded as x tends to infinity, which in turn implies by (17) that

(19) $$\int_0^\infty sy(s)F(y^2(s),s)ds < \infty.$$

Using condition (4) and (16), we obtain from (19)

$$\frac{c}{2}\int_{x_o}^\infty sF(c^2,s)ds \leq \int_{x_o}^\infty sy(s)F(y^2(s),s)ds < \infty,$$

thus proving the necessity of the condition.

To prove sufficiency, we suppose that condition (10) holds for some positive constant c, and we construct a solution $y(x)$ of (2) which satisfies

$$\lim_{x \to \infty} y(x) = c',$$

where c' is to be chosen later. Let $c' > c$ and define $y_n(x)$ inductively by

$$y_o(x) = c'$$

$$y_n(x) = c' - \int_x^\infty (s-x)\, y_{n-1}(s) F(y_{n-1}^2(s), s)\, ds \tag{20}$$

If x_1 is chosen so that

$$c' \int_{x_1}^\infty sF(c^2, s)\, ds < c' - c,$$

then using (4) and (20), we get for $x \geq x_1$,

$$y_1(x) \leq c',$$

and

$$y_1(x) \geq c' - c' \int_{x_1}^\infty sF(c^2, s)\, ds > c$$

Thus, inductively we obtain for $x \geq x_1$ and $n = 1,2,\ldots,$

$$y_n(x) \leq c',$$

and

$$y_n(x) \geq c.$$

Using the above, we find for $x \geq x_1$ and all $n = 1,2,\ldots$ that

$$y_n'(x) = \int_x^\infty y_{n-1}(s) F(y_{n-1}^2(s), s)\, ds$$

(21)

$$\leq c' \int_x^\infty F(c^2, s)\, ds \leq c' - c.$$

The above estimates for y_n and (21) imply that $\{y_n(x)\}$ forms a uniformly bounded and equicontinuous family, hence it follows from the Arzela-Ascoli theorem that there exists a subsequence $\{y_{n_k}(x)\}$, uniformly convergent on every compact subinterval of $[x_1,\infty)$. Now a standard argument, see for example [10; Theorem 3], yields a function $y(x)$ satisfying

$$y(x) = c' - \int_x^{\infty} (s-x)\,y(s)\,F(y^2(s),s)\,ds,$$

which is the desired bounded non-oscillatory solution of (2).

As a corollary to the above result, we obtain

Corollary 1. *Let* $0 < \alpha < 1$. *Equation* (1) *has a bounded nonoscillatory solution if and only if* (7) *holds*.

We next prove the desired extension of Theorem B_1.

THEOREM 2. *Let* F *satisfy condition* (4). *Then equation* (2) *has an unbounded asymptotically linear solution if and only if* (11) *holds for some* $c > 0$. *If* F *satisfies the stronger condition* (13), *then* (11) *is also necessary for the existence of any nonoscillatory solution*.

PROOF. We first show that (11) is sufficient for the existence of an unbounded asymptotically linear solution. Choose x_o such that

$$\int_{x_o}^{\infty} xF(c^2x^2,x)\,dx < \frac{1}{4}\,. \tag{22}$$

Let $c' > c$, then $cx < c'(x - x_o)$ for $x > \frac{c'x_o}{c'-c} = x_1$.

Thus, in view of (4),

$$\int_{x_o}^{\infty} (x - x_o)F(c'^2(x - x_o)^2,x)dx$$

$$\leq \int_{x_o}^{x_1} (x - x_o)F(c'^2(x - x_o)^2,x)dx + \int_{x_1}^{\infty} xF(c^2x^2,x)dx$$

Since $x_1 = \dfrac{c'x_o}{c'-c}$ tends to x_o as c' tends to infinity, it follows that

(23) $$\int_{x_o}^{\infty}(x - x_o)F(c'^2(x - x_o)^2,x)dx < \frac{1}{4}$$

for sufficiently large c'. Let $y(x)$ be a solution of (2) satisfying

(24) $$y(x_o) = 0, \qquad y'(x_o) = 2c'$$

where x_o and c' are chosen so that (23) holds. We claim that $y'(x) > c'$ for $x \geq x_o$. Suppose that there exists $x_2 > x_o$ such that $y'(x) > c'$ for $x_o \leq x \leq x_2$, and $y'(x_2) = c'$. Integrating (2) and using the initial conditions (24), we have

(25) $$y'(x) = 2c' - \int_{x_o}^{x} y(s)F(y^2(s),s)ds$$

for $x_o \leq x \leq x_2$. From (25), we know that $y'(x) \leq 2c'$, and since $y(x_o) = 0$, it follows from $y'(x) > c'$ that

(26) $$c'(x - x_o) \leq y(x) \leq 2c'(x - x_o), \quad x_o \leq x \leq x_2.$$

Using (26) in (25), we obtain

(27) $$y'(x) \geq 2c'[1 - \int_{x_o}^{x} (s - x_o)F(c'^2(s - x_o)^2,s)ds].$$

Hence (23) and (27) together imply $y'(x) \geq \frac{3}{2} c'$, and in particular, $y'(x_2) \geq \frac{3}{2} c'$ contradicting our assumption. Thus, for all $x \geq x_o$, we have

$$c' < y'(x) \leq 2c'. \tag{28}$$

Since $y(x) \geq 0$ on $[x_o, \infty)$, the integrand in (25) is non-negative, therefore, $\lim_{x\to\infty} y'(x)$ exists, and in view of (28) it must be finite and positive. Clearly this implies that y is unbounded and asymptotically linear.

Next we suppose that there exists an unbounded asymptotically linear solution $y(x)$ of (2), i.e. a solution y such that

$$\lim_{x\to\infty} \frac{y(x)}{x} = c > 0. \tag{29}$$

From (29), it follows that there exists $x_o > 0$ such that for $x \geq x_o$

$$\frac{c}{2} < \frac{y(x)}{x} < 2c. \tag{30}$$

Integrating (2), we have

$$y'(x) = y'(x_o) - \int_{x_o}^{x} y(s)F(y^2(s),s)\,ds. \tag{31}$$

We claim that $y'(x) \geq 0$ for all $x \geq x_o$. Suppose that there exists $x_1 > x_o$ such that $y'(x_1) < 0$. By (2) and (30), we have $y''(x) \leq 0$, thus $y'(x) \leq y'(x_1)$ and $y(x) \leq y(x_1) + y'(x_1)(x - x_1)$ for $x \geq x_1$. Letting $x \to \infty$, we obtain a contradiction to (30). Now, $y'(x) \geq 0$ and (31) give

$$\int_{x_o}^{\infty} y(x)F(y^2(x),x)\,dx < \infty \tag{32}$$

Using (30) in the above, we obtain

$$\frac{c}{2} \int_{x_o}^{\infty} xF(4c^2x^2,x)dx \leq \int_{x_o}^{\infty} y(x)F(y^2(x),x)dx < \infty$$

proving (10).

Finally, we assume $F(t,x)$ satisfies the stronger condition (13) and wish to show that (10) is also necessary for the existence of any nonoscillatory solution. Let y be a nonoscillatory solution of (2) and let $y(x) > 0$ for $x > x_o$. Observe that

$$y(x) = y(x_o) + \int_{x_o}^{x} y'(s)ds \tag{32}$$

$$\geq \int_{x_o}^{x} y'(s)ds \geq y'(x)(x - x_o)$$

On the other hand, we have

$$y(x) \leq y(x_o) + y'(x_o)(x - x_o). \tag{33}$$

From (32) and (33), it follows that there exist constants c_1 and x_1 such that for $x \geq x_1 > x_o$.

$$0 < y(x) \leq c_1 x. \tag{34}$$

Using (34), we obtain from the given hypothesis that

$$y^{2\varepsilon}(x)F(y^2(x),x) \geq (c_1x)^{2\varepsilon}F(c_1^2x^2,x) \tag{35}$$

for all $x \geq x_1$. Consider

$$-(y'^{2\varepsilon}(x))' = -2\varepsilon(y'(x))^{2\varepsilon-1}y''(x) \tag{36}$$

$$= 2\varepsilon(y'(x))^{2\varepsilon-1}y(x)F(y^2(x),x)$$

Using (35) in (36), we obtain

(37) $-(y'^{2\varepsilon}(x))' \geq 2\varepsilon(y'(x))^{2\varepsilon-1}(y(x))^{1-2\varepsilon}(c_1x)^{2\varepsilon}F(c_1^2x^2,x).$

Choose $x_1 \geq x_o$ so that $x - x_o \geq \frac{x}{2}$ for all $x \geq x_1$. We may now use (32) to estimate (37) as follows

(38) $-(y'^{2\varepsilon}(x))' \geq 2\varepsilon(y'(x))^{2\varepsilon-1}(y'(x)\frac{x}{2})^{1-2\varepsilon}(c_1x)^{2\varepsilon}F(c_1^2x^2,x).$

$$\geq (2c_1)^{2\varepsilon} \, xF(c_1^2x^2,x)$$

Integrating (38) from x_1 to X, we have

(39) $y'^{2\varepsilon}(x_2) - y'^{2\varepsilon}(X) \geq (2c_1)^{2\varepsilon} \int_{x_1}^{X} xF(c_1^2x^2,x)dx.$

Since $y'(x) > 0$ for all large x, (10) follows immediately from (39).

Notice that for the linear equation, Theorem A_1 and Corollary 1, taken together, yield the well known theorem of Bocher, [5, Corollary 9.1, p.380] as well as its converse. Restricting our attention to equation (1), we see in Theorem A_2 the equivalence of (i) and (ii) for $0 < \alpha < 1$ and in Theorem B_2 that of (i) and (ii) for $\alpha > 1$. This observation gives an explanation of why when $\alpha \neq 1$, we can find necessary and sufficient conditions for all solutions of (1) to be oscillatory. This being that, in either case, the existence of any nonoscillatory solution implies the existence of a nonoscillatory solution of a particular type for whose existence even in the linear case one can give necessary and sufficient conditions. Finally, Theorem E shows that such an equivalence remains valid for the more general equation (2) with $F(t,x)$ satisfying the strong superlinearity condition (12) and Theorem 2 shows that the dual statement is true for equation (2) with $F(t,x)$ satisfying (13). Such nonlinearity conditions obviously exclude the linear equation in which case, when (7) holds, there exist both unbounded and bounded asymptotically linear (hence nonoscillatory) solutions. This perhaps offers an explanation of why for the linear equation there is no necessary and sufficient conditions for all solutions to be oscillatory. For an explanation of this from an entirely different point of view, see [8, Theorem I] and the remarks which follow.

We now wish to present a generalization of Theorem D to equation (2) under the sublinearity condition (4).

THEOREM 3. Let $F(t,x)$ satisfy (4) and (14). Suppose that for each $c > 0$, (10) holds, then all solutions are nonoscillatory.

PROOF. We note first that condition (4) and (14) imply left uniqueness of the zero solution of (2), (see [4], Appendix). In particular, a solution $y(x)$ satisfying nontrivial initial conditions at some point x_0 has only a finite number of zeros in any bounded interval $[x_0, x_1], x_1 > x_0$. Indeed these facts follow readily from the proof below, but to keep our discussion simple, we shall rely on the results of [4].

Assume now that (2) has an oscillatory solution $y(x)$ and let $x_1 < x_2 < \ldots$ denote the consecutive zeros of $y(x)$, then in view of the above observation,

(40) $$\lim_{k\to\infty} x_k = \infty .$$

Let $\bar{x}_k \in [x_k, x_{k+1}]$ denote the point in that interval where y' vanishes. We wish to show that $|\bar{y}(x_k)| \leq |\bar{y}(x_{k+1})|$, for $k = 1,2,\ldots.$ For a given value of k we can assume $y(\bar{x}_k) > 0$, then $y(\bar{x}_{k+1}) < 0$, and $y'(x) < 0$ in $[\bar{x}_k, \bar{x}_{k+1}]$. We have then

(41) $$(y'(x_{k+1}))^2 = -2\int_{\bar{x}}^{x_{k+1}} y'yF(y^2,x)\,dx = \int_0^{y^2(\bar{x}_k)} F(\lambda, x_1(\lambda))\,d\lambda,$$

similarly,

(42) $$(y'(x_{k+1}))^2 = 2\int_{x_{k+1}}^{\bar{x}_{k+1}} y'yF(y^2,x)\,dx = \int_0^{y^2(\bar{x}_{k+1})} F(\lambda, x_2(\lambda))\,d\lambda_3$$

where $x_1(\lambda)$ and $x_2(\lambda)$ are obtained by inverting respectively

$$y^2(x) = \lambda, \qquad \bar{x}_k \leq x \leq x_{k+1} ,$$

and

$$y^2(x) = \lambda, \qquad x_{k+1} \leq x \leq \bar{x}_{k+1} .$$

It follows from the definition of $x_1(\lambda)$ and $x_2(\lambda)$ that $x_2(\lambda) > x_{k+1}$, $0 < \lambda < y^2(\bar{x}_{k+1})$ and $x_1(\lambda) < x_{k+1}, 0 < \lambda < y^2(\bar{x}_k)$. Thus, if we assume $y^2(\bar{x}_{k+1}) < y^2(\bar{x}_k)$, then from (14) we conclude that

$$\int_0^{y^2(\bar{x}_k)} F(\lambda, x_1(\lambda))\,d\lambda > \int_0^{y^2(\bar{x}_{k+1})} F(\lambda, x_2(\lambda))\,d\lambda,$$

but in view of (41) and (42) this yields a contradiction, thus $y^2(\bar{x}_{k+1}) \geq y^2(\bar{x}_k)$. Consider now the sequence $\{|y(\bar{x}_k)|\}$, which is non-decreasing and hence must tend to a positive limit, finite or infinite. In any case, there exists a constant $c > 0$, and a sequence of zeros of $y(x)$, say $\{x_k\}$, such that

$$(43) \qquad \liminf_{k\to\infty} y(x_k) = c > 0$$

From Theorem 1 it follows that there exists a nonoscillatory $z(x)$ which tends monotonically to c from below. Hence there must exist for sufficiently large k points s_1 and s_2 in an interval $[x_k, x_{k+1}]$ such that $y(s_i) = z(s_i)$, $i = 1,2, 0 < z(x) < y(x)$ for $x \in (s_1, s_2), y'(s_1) > z'(s_1)$ and $y'(s_2) < z'(s_2)$. Now consider the Wronskian of $y(x)$ and $z(x)$, defined by $W(y,z)(x) = y(x)z'(x) - z(x)y'(x)$. Using (2) and (4), we find, for $s_1 \leq x \leq s_2$,

$$(44) \quad \frac{d}{dx} W(y,z)(x) = y(x)z(x)(F(y^2(x),x) - F(z^2(x),x)) \leq 0.$$

Integrating (44) from s_1 and s_2, we obtain

$$W(y,z)(s_2) - W(y,z)(s_1) \leq 0,$$

or

$$(45) \quad y(s_2)(z'(s_2) - y'(s_2)) - y(s_1)(z'(s_1) - y'(s_1)) \leq 0.$$

However, the left hand side of (45) is positive, which is impossible. This contradiction proves that all solutions of (1) are nonoscillatory.

Remark 1. It is clear that Theorem D follows from the result above. In fact, we have relaxed the positiveness assumption on $q(x)$ to that of non-negativeness, a hypothesis crucial in Heidel's proof.

Remark 2. The technique used in connection with the derivation of inequality (42) has been introduced in Moroney [7] in treating a similar problem arisen from a different context.

3. In this section, we first prove another generalization of Theorem D for the general equation (2), and then in accordance with the duality described in section 1 we obtain a dual result in the superlinear case

THEOREM 4. Assume that $F(t,x)$ satisfies (4) and

(i) $F(t,x) > 0$ for all $t,x > 0$,

(ii) For each $c > 0$, (10) holds,

(iii) There exists a nondecreasing and bounded continuous function $h(x)$ such that for every $t > 0$,

$$\log F(t,x_2) - \log F(t,x_1) \leq h(x_2) - h(x_1), \tag{46}$$

for $x_2 \geq x_1$,

(iv) There exists a strictly increasing differentiable function $\varphi(t)$ with $\varphi(0) = 0$ such that $\varphi'(t)F(\varphi(t),x)$ is nondecreasing in t for each $x > 0$.

Then all solutions of (2) are nonoscillatory.

PROOF. We assume first that $F(t,x)$ is of class C^1 and then proceed in much the same manner as in Theorem 3. Let $y(x)$ be an oscillatory solution of (2) with $\{x_k : k = 1,2,3,...\}$ denoting the sequence of consecutive zeros of $y(x)$. In view of the discussion at the beginning of the proof of

Theorem 3, we have that (40) holds. Introduce a function $G(t,x)$ defined by

$$G(t,x) = \int_0^t F(u,x)\,du = 2\int_0^t vF(v^2,x)\,dv, \tag{47}$$

and an energy function $\Phi(x)$ defined by

$$\Phi(x) = y'^2(x) + G(y^2(x),x). \tag{48}$$

Because of assumption (i), we may, for fixed $x > 0$, define $\Gamma(w,x)$ implicitly by

$$\tilde{G}(t,x) = w, \qquad \Gamma(w,x) = t, \tag{49}$$

where

$$\tilde{G}(t,x) = G(\varphi(t),x), \tag{50}$$

with φ as given in assumption (iv), so that $\tilde{G}(t,x)$ is a convex function of t.

The function $\Gamma(w,x)$ obeys the following rules of differentiation

$$\Gamma_w(w,x) = (\varphi'(\Gamma(w,x))F(\varphi(\Gamma(w,x),x)),^{-1} \tag{51}$$

$$\Gamma_x(w,x) = -\frac{G_x(\Gamma(w,x),x)}{\varphi'(\Gamma(w,x))F(\varphi(\Gamma(w,x)),x)}. \tag{52}$$

If $y(x)$ is a solution of (2), then differentiating (48) we find

$$\Phi'(x) = \tilde{G}_x(\varphi^{-1}(y^2(x)),x). \tag{53}$$

Denote $\Psi(x) = \Gamma(\Phi(x),x)$. Using (51),(52) and (53), we obtain

$$\Psi'(x) = \frac{-(G_x(\Psi(x),x) - G_x(\varphi^{-1}(y^2(x)),x))}{\varphi'(\Psi(x))F(\varphi(\Psi(x)),x)} \tag{54}$$

Note that $\tilde{G}(t,x)$, as defined by (50), may be rewritten as

$$(55) \qquad \tilde{G}(t,x) = \int_0^{\varphi(t)} F(s,x)ds = \int_0^t F(\varphi(s),x)\varphi'(s)ds.$$

Differentiating $\tilde{G}(t,x)$, as given by (55), with respect to x, we have

$$(56) \qquad G_x(t,x) = \int_0^t F_x(\varphi(s),x)\varphi'(s)ds.$$

Using (56), we may rewrite (54) as

$$(57) \quad \Psi'(x) = \frac{-1}{\varphi'(\Psi(x))F(\varphi(\Psi(x)),x)} \int_{\varphi^{-1}(y^2(x))}^{\Psi(x)} F_x(\varphi(s),x)\varphi'(s)ds$$

Since $F(t,x)$ is positive for all $t,x > 0$, $\tilde{G}(t,x)$ is strictly increasing in t for every x, therefore $\Phi(x) \geq G(\varphi^{-1}(y^2(x)),x)$ and (48) implies

$$(58) \qquad \Psi(x) = \Gamma(\Phi(x),x) \geq \varphi^{-1}(y^2(x)).$$

For fixed x, the assumption (iv) implies for $0 \leq s < \Psi(x)$,

$$(59) \qquad \varphi'(\Psi(x))F(\varphi(\Psi(x)),x) \geq \varphi'(s)F(\varphi(s),x).$$

Thus for those x such that $y^2(x) > 0$, we have, upon substituting (58),(59) into (57), that

$$\Psi'(x) \geq -\int_{\varphi^{-1}(y^2(x))}^{\Psi(x)} \frac{[F_x(\varphi(s),x)]_+\varphi'(s)ds}{F(\varphi(s),x)\varphi'(s)}$$

Here for any function $m(x)$, $m_+(x) = \mathrm{Max}(m(x),0)$. Using assumption (46) in the above estimate, we obtain

$$(60) \qquad \Psi'(x) \geq -\int_{\varphi^{-1}(y^2(x))}^{\Psi(x)} h'(x)ds \geq -h'(x)\Psi(x),$$

and since $\Psi'(x)$ is continuous, (60) holds also for $y^2(x) = 0$. Thus,

$$\Psi'(x) \geq -h'(x)\Psi(x)$$

or

$$\Psi(x) \geq \Psi(x_o)\exp(-\int_{x_o}^{x} h'(x)dx)$$

$$= \Psi(x_o)\exp(h(x_o) - h(x)) \tag{61}$$

$$\geq \lambda\Psi(x_o).$$

where $\lambda = e^{-M}$ and $M = \sup_{x>0} |h(x)|$. The computation given in (60) and (61) is carried out under the assumption that $h(x)$ is differentiable. However, due to the non-decreasing character of $h(x)$, we know that $h'(x)$ exists almost everywhere and clearly (61) remains valid in this slightly more general case. The assumption that $F(t,x)$ is of class C' can also be dropped; see the discussion of similar problems in [4]. At $x = \bar{x}_k$, where $y'(\bar{x}_k) = 0$, we have

$$\Psi(\bar{x}_k) = \varphi^{-1}(y^2(\bar{x}_k)),$$

from which it follows that

$$\liminf_{k \to \infty} y^2(\bar{x}_k) \geq c = \varphi(\lambda\Psi(x_o)) > 0. \tag{62}$$

The conclusion in (62) allows us to repeat the argument given in the proof of Theorem 3 following statement (43).

<u>Remark 3</u>. If we assume in Theorem 3 certain monotonicity conditions on $F(t,x)$ with respect to t, e.g. the condition that for each $x > 0$,

$$y_2F(y_2^2,x) \geq y_1F(y_1^2,x) \qquad y_2 \geq y_1, \tag{63}$$

then we can show in these cases that Theorem 4 is stronger than Theorem 3. In fact condition (63) implies assumption (iv) in Theorem 4, with the function $\varphi(t) = t^2$ (cf. (47)). Also if assumption (i) of Theorem 4 fails, i.e. if $F(t_o,x_o) = 0$ for some $t_o > 0$, then (4), (14) and (63) imply that $F(t,x) \equiv 0$ for $x \geq x_o$ and all $t > 0$, in which case (2) is obviously nonoscillatory.

The superlinear analogue is the following.

THEOREM 5. *Assume that* $F(t,x)$ *satisfies* (3) *and for each* $c > 0$, (11) *holds. If in addition, we assume that there exists a non-decreasing bounded continuous function* $h(x)$ *such that for each fixed* $t > 0$,

$$\log G(t,x_2) - \log G(t,x_1) \leq h(x_2) - h(x_1), \tag{64}$$

for all $0 \leq x_1 \leq x_2$. *Then all solutions of* (2) *are non-oscillatory*.

PROOF. We may assume that $F(t,x)$ is of class C^1. The general case is again treated by a standard approximation argument, see e.g. [4]. Suppose that $y(x)$ is any solution of (2) and consider the energy function $\Phi(x)$ defined by (48). Differentiating $\Phi(x)$ we obtain from (64),

$$\Phi'(x) = G_x(y^2,x) \leq h'(x)G(y^2,x)$$

$$\leq h'(x)\Phi(x),$$

and consequently, for $x \geq x_o$,

$$\Phi(x) \leq \Phi(x_o)\exp\int_{x_o}^{x} h'(x)dx. \tag{65}$$

It follows from (65) that there exists a constant $K > 0$ such that

(66) $$|y'(x)| \leq K, \qquad x \geq x_o.$$

It follows from a result of Nehari [9,Theorem II], mentioned in section 1, that (2) has a solution $u(x)$ satisfying

(67) $$\lim_{x\to\infty} \frac{u(x)}{x} > c$$

for some $c > K$. Thus, for x_o sufficiently large, we have

(68) $$|y(x)| \leq Kx < u(x) \qquad x \geq x_o .$$

Condition (4) and (68) together imply

(69) $$F(y^2(x),x) \leq F(u^2(x),x) \qquad x \geq x_o .$$

It then follows from (69) and the Sturm comparison theorem that $y(x)$ is nonoscillatory. This completes the proof.

Remark 3. The above theorem improves another result of Nehari [9;Theorem III], mentioned earlier in section 1, in that condition (14) is weakened to that of (64). We note also that condition (46) always implies (64). In case of equation (1), these two conditions are equivalent.

We now apply Theorems 4 and 5 to the following equation

(70) $$y'' + q(x)f(y) = 0$$

where $f(y)$ is continuous and satisfies

(71) $$yf(y) > 0, \qquad y \neq 0.$$

Corollary 2. Assume that $q(x)$ is positive and differentiable and satisfies (15). If $y^{-1}f(y)$ is nonincreasing *in* y *for* $y \neq 0$, *and* $q(x)$ *satisfies* (7), *or if* $y^{-1}f(y)$ *is nondecreasing in* y *for* $y \neq 0$, *and* $q(x)$ *satisfies* (8), *then all solutions of* (70) *are nonoscillatory*.

PROOF. Condition (71) and $q(x) > 0$ imply hypothesis (i) of Theorem (4). Condition (7) implies (10) holds for every $c > 0$. The assumption on $q(x)$ that (15) holds guaranties the existence of a nondecreasing bounded function $h(x)$ defined by

$$h(x) = \int^x \frac{q_+'(x)}{q(x)}\, dx$$

for which condition (64) is satisfied. In this case $G(t,x)$ as defined by (47) is given by

$$G(t,x) = 2\int_0^t vF(v^2,x)\,dv = 2q(x)\int_0^t f(u)\,du,$$

which in view of (71) is strictly increasing in t for all x. Define $\Sigma(t) = \int_0^t f(u)\,du$ and $\varphi(t) = \Sigma^{-1}(t)$. We find $\tilde{G}(t,x) = G(\varphi(t),x) = 2q(x)t$ which is clearly convex in t. $y^{-1}f(y)$ nonincreasing is the required condition of sub-linearity. Thus, the first part of the Corollary follows from Theorem 4. The second part follows from Theorem 5 in a similar way.

Finally, to see that condition (15) in fact improves upon the stronger hypothesis that $q'(x) \leq 0$ and is yet compatible with the remaining assumptions, we consider

$$q(x) = \frac{1}{x^3}\left(1 + 3\,\frac{\sin x}{x} + \frac{\sin \pi x}{x}\right)$$

In this case $q(x) > 0$ and satisfies (3) but $q'((2n)^2\pi) > 0$ for all $n = 1,2,\ldots.$ On the other hand, for large x,

$$q_+'(x) \leq \frac{1}{2x^4} \frac{\pi}{2} \frac{\cos \pi x}{x^{1/2}}$$

which gives

$$\frac{q_+'(x)}{q(x)} \leq x^{-\frac{3}{2}}$$

satisfying (15).

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