

NOTICE WARNING CONCERNING COPYRIGHT RESTRICTIONS:
The copyright law of the United States (title 17, U.S. Code) governs the making of photocopies or other reproductions of copyrighted material. Any copying of this document without permission of its author may be prohibited by law.

APPROACH OF ELECTROMAGNETIC FIELDS
TO STEADY STATE

R. C. MacCamy

Report 69-37

December 1969

FEB 1 '71

University Libraries
Carnegie Mellon University
Pittsburgh PA 15213-3890

Approach of Electromagnetic Fields to Steady State

R. C. MacCamy *

* This work was supported by the National Science Foundation under Grant GP 11945.

Abstract

A special class of electromagnetic fields in the interior of a cylinder is studied, assuming finite conductivity. Two problems are considered. In one the tangential magnetic field is given on the boundary and in the other the tangential electric field is given. In each case the boundary values tend to limits as time tends to infinity. It is shown that the solution of the magnetic problem tends to the corresponding steady state field while the solution of the electric problem can grow linearly with time.

1. Introduction.

A basic problem in the study of dynamical systems is that of approach to steady state. The problem can be phrased as follows: Consider an evolution equation,

$$(1.1) \quad \dot{u}(t) + A(t)u(t) = f(t)$$

on some Banach space B . Suppose $A(t) \rightarrow A_0$ (in some sense) and $f(t) \rightarrow f_0$ as $t \rightarrow \infty$. Is it true that $u(t) \rightarrow u_0$? Roughly speaking it will be true, with $u_0 = A_0^{-1}f_0$, if A_0 has a bounded inverse (see for example [1]).

The present paper considers a special case of this problem arising in the study of Maxwell's equations. The goal is to illustrate what can happen if A_0^{-1} fails to exist as a bounded operator. We show, in the special case, that u can grow linearly with time.

In order to keep the computations simple we have restricted ourselves to a special geometry. The techniques are those of [2]. They relate large time behavior to the study of monochromatic fields, as the frequency tends to zero. For our problem the necessary results are contained in [3]. One can obtain similar results for more general electromagnetic fields by using the work of [4].

2. Statement of Results.

We consider Maxwell's equations in a bounded region Ω , under the assumption of finite conductivity σ . The equations are,

$$\begin{aligned}
 (2.1) \quad & \nabla \times \underline{E} + \mu \underline{H}_t = \underline{Q}, & \mu > 0, \\
 & \nabla \times \underline{H} - \epsilon \underline{E}_t = \sigma \underline{E}, & \epsilon > 0, \\
 & \nabla \cdot \underline{E} = 0, \quad \nabla \cdot \underline{H} = 0.
 \end{aligned}$$

We study two problems for these equations. These are denoted by (P.1) and (P.2). In each we require the initial condition,

$$\begin{aligned}
 (c) \quad & \underline{E}(\underline{x}, 0) = \underline{f}(\underline{x}) & \nabla \cdot \underline{f} = 0 \\
 & \underline{H}(\underline{x}, 0) = \underline{g}(\underline{x}) & \nabla \cdot \underline{g} = 0
 \end{aligned}
 \quad \text{in } \Omega.$$

For (P.1) we have the boundary condition,

$$(B.1) \quad \underline{H} \times \underline{n} = \underline{\varrho} \quad \text{on } \partial\Omega.$$

and for (P.2),

$$(B.2) \quad \underline{E} \times \underline{n} = \underline{\varrho} \quad \text{on } \partial\Omega.$$

Here \underline{n} is the exterior normal to $\partial\Omega$.

It is well known that (P.1) and (P.2) have unique solutions which we denote by $\underline{E}^k(\underline{x}; \underline{\varrho})$ and $\underline{H}^k(\underline{x}; \underline{\varrho})$ respectively. We are going to study problems in which Ω is not bounded but consists of a cylinder of cross section D and axis along the z -axis. We seek solutions \underline{E} and \underline{H} which depend only on x and y and accordingly we take the boundary functions $\underline{\varrho}$ to have the form,

$$(2.2) \quad \varrho = \varphi_1(x, y, t) \underline{\tau} + \varphi_2(x, y, t) \underline{k}$$

where $\underline{\tau}$ is the unit tangent to ∂D .

We make three further assumptions which will simplify the calculations. First we assume D to be simply connected. Second we take \underline{f} and \underline{g} in (c) to be zero. Finally we assume that,

$$(2.3) \quad \varrho = \varrho^0(x, y) + \psi(x, y, t), \quad \varrho^0 = \varphi_1^0 \underline{\tau} + \varphi_2^0 \underline{k},$$

where $\psi_m(x, y, t) = O(e^{-at})$ uniformly in (x, y) as $t \rightarrow \infty$.

We consider also the steady state problems corresponding to (P.1) and (P.2). We denote these by $(P.1)_0$ and $(P.2)_0$. They involve finding a solution of the equations,

$$(2.4) \quad \begin{aligned} \nabla \times \underline{E} &= \underline{Q} \\ \nabla \times \underline{H} &= \sigma \underline{E} \\ \nabla \cdot \underline{E} &= 0, \quad \nabla \cdot \underline{H} = 0, \end{aligned}$$

subject to (B.1) or (B.2) respectively, when $\varrho \equiv \varrho^0, \varrho^0$ as in (2.3). We can now state our results. In all of them the assumptions listed above are to be understood.

Theorem (1) (i) Problem $(P.1)_0$ has a unique solution

$$E_0^1(\underline{x}; \varrho^0), H_0^1(\underline{x}; \varrho^0).$$

(ii) There is a $\beta > 0$ such that,

$$(2.5) \quad \begin{aligned} E^1(\underline{x}; \varrho) - E_0^1(\underline{x}; \varrho^0) &= O(e^{-\beta t}) \\ H^1(\underline{x}; \varrho) - H_0^1(\underline{x}; \varrho^0) &= O(e^{-\beta t}) \end{aligned} \quad \underline{\text{as}} \quad t \rightarrow \infty.$$

Our results for (P.2) require that the following additional condition be satisfied:

$$(2.6) \quad \varphi_1^0(x,y) = N, \quad N \text{ a constant.}$$

Theorem (2) Let A denote the area of D , L the length of ∂D and set,

$$\alpha = \int_{\partial D} \varphi_2^0 ds.$$

(i) If $\alpha = 0$ there exist solutions of (P.2)₀. The solution is not unique but there exists one solution $E_0^2(x;\varphi^0)$, $H_0^2(x;\varphi^0)$ such that (2.5) holds with one replaced by two.

(ii) If $\alpha \neq 0$ then there exists no solution of (P.2)₀. There does, however, exist a solution of (P.2)₀ with (B.2)₀ replaced by,

$$(B.2)_0' \quad \underline{E} \times \underline{n} = \varphi^0 - \frac{\alpha}{L} \underline{k}.$$

Again this solution is not unique but there exists one solution $\hat{E}_0^2(x;\varphi^0)$, $\hat{H}_0^2(x;\varphi^0)$ such that,

$$(2.7) \quad \begin{aligned} E^2(x;\varphi) - \hat{E}_0^2(x;\varphi^0) &= o(e^{-\beta t}) \\ H^2(x;\varphi) - \frac{\alpha}{A} t \underline{k} - \hat{H}_0^2(x;\varphi^0) &= o(e^{-\beta t}) \end{aligned} \quad \text{as } t \rightarrow \infty.$$

3. Steady State.

The special solutions arising in this section and the next derive from the general idea of "Hertz Potentials" [5]. We seek first to find all solutions of (2.4) which are functions only of x and y . For $\underline{A} = A^1 \underline{i} + A^2 \underline{j}$ we introduce the notation

$\underline{A}^\perp = A^2 \underline{i} - A^1 \underline{j}$. We note the following formulas:

$$(3.1) \quad \nabla \times (\nabla \varphi(x,y))^\perp = -\Delta \varphi \underline{k}$$

$$(3.2) \quad \nabla \times (\psi(x,y) \underline{k}) = (\nabla \psi)^\perp$$

Observe also that a harmonic function $\varphi(x,y)$ and its conjugate ψ satisfy,

$$(3.3) \quad (\nabla \varphi)^\perp = \nabla \psi$$

It is not difficult to check, using (3.1) - (3.3), that the expressions,

$$(3.4) \quad \underline{E} = \nabla \psi + N \underline{k} ; \quad \underline{H} = (\nabla \gamma)^\perp + \sigma \chi \underline{k}$$

will be solutions of (2.4) if N is a constant and ψ , γ and χ are functions of (x,y) satisfying,

$$(3.5) \quad \Delta \psi = 0, \quad \nabla \chi = (\nabla \psi)^\perp, \quad \Delta \gamma = -\sigma N.$$

When D is simply connected, as we assume, then it can be verified that (3.4), (3.5) in fact yield the only solutions of (2.4) which depend only on x and y . We note that if \underline{E} and \underline{H} are as in (3.4) then,

$$(3.6) \quad \begin{aligned} \underline{E} \times \underline{n} &= N \underline{\tau} + \nabla \chi \cdot \underline{n} \underline{k}; \\ \underline{H} \times \underline{n} &= \sigma \chi \underline{\tau} + (\nabla \gamma \cdot \underline{n}) \underline{k}, \end{aligned}$$

where $\underline{\tau}$ is the unit tangent to ∂D .

We seek solutions of (P.1)₀ and (P.2)₀ in the form (3.4).

For (P.1)₀ we must have, by (B.1), (3.5) and (3.6),

$$(3.7) \quad \Delta\chi = 0 \text{ in } D, \quad \chi = \frac{\varphi_1^0}{\sigma} \text{ on } \partial D,$$

$$(3.8) \quad \Delta\gamma = -\sigma N \text{ in } D, \quad \nabla\gamma \cdot \underline{n} = \varphi_2^0 \text{ on } \partial D.$$

Equations (3.7) uniquely determine χ . If we choose N by the condition,

$$-\int_D \sigma N dA = -\sigma N A = \int_{\partial D} \varphi_2^0$$

then (3.8) has a solution, unique up to a constant. Let χ be the conjugate of ψ . Again ψ is unique up to a constant. Then (3.4) yields a solution of (P.1)₀.

For (P.2)₀, (B.2), (3.5) and (3.6) yield, (note (2.6)),

$$(3.9) \quad \Delta\chi = 0 \text{ in } D, \quad \nabla\chi \cdot \underline{n} = \varphi_2^0 \text{ on } \partial D.$$

The problem (3.9) has no solutions unless

$$(3.10) \quad \alpha = \int_s \varphi_2^0 ds = 0.$$

If (3.10) is satisfied then (3.9) does have a solution, unique up to a constant. Note also that if the second condition in (3.9) is replaced by,

$$(3.11) \quad \nabla\chi \cdot \underline{n} = \varphi_2^0 - \frac{\alpha}{L}$$

then the resulting problem has a solution unique up to a constant.

We conclude that if (3.10) is violated (P.2)₀ has no solution of the form (3.4). If (3.10) holds then (P.2)₀ does have a solution of the form (3.4). However this solution is not unique since χ is undetermined except for the condition $\Delta\chi = -\sigma N$. Note that the non-uniqueness is represented by a magnetostatic field.

Remark: We remark that for multiply connected domains the degree of non-uniqueness is greater. We hope to return to this question at a later time.

4. Time-dependent Problems.

The special solutions of the last section are capable of generalization to problems (P.1) and (P.2). Suppose that $u(x,y)$ and $v(x,y)$ satisfy the equations,

$$(4.1) \quad \begin{aligned} \Delta u &= Tu_t & Tw &= \mu\epsilon w_t + \mu\sigma w. \\ \Delta v &= Tv_t \end{aligned}$$

Then one can verify that the expressions,

$$(4.2) \quad \begin{aligned} \underline{E} &= -Tu_t \underline{k} - \mu(\nabla v_t)^\perp \\ \underline{H} &= (Tu)^\perp - Tv_t \underline{k} \end{aligned}$$

will yield solutions of (2.1). Accordingly we seek solutions of (P.1) and (P.2) in the form (4.2).

For (P.1), (B.1) (4.1) and (4.2) yield,

$$(4.3) \quad \begin{aligned} \Delta u &= Tu \text{ in } D, & Tu &= -\varphi_1 \text{ on } \partial D \\ \Delta v &= Tv \text{ in } D, & \mu \nabla v_t \cdot \underline{n} &= -\varphi_2 \text{ on } \partial D. \end{aligned}$$

For (P.2) we obtain,

$$(4.4) \quad \begin{aligned} \Delta u &= Tu \quad \text{in } D, & Tv &= -\varphi_1 \quad \text{on } \partial D \\ \Delta v &= Tv \quad \text{in } D, & (T\nabla u) \cdot \underline{n} &= -\varphi_2 \quad \text{on } \partial D. \end{aligned}$$

We observe that (c) (with $\underline{f} = \underline{g} = 0$) is satisfied if,

$$(4.5) \quad u, u_t, v, v_t = 0 \quad \text{at } t = 0.$$

The problems (4.5) and (4.3) or (4.4), which we denote by (P.1)', (P.2)' have unique solutions.* When substituted into (4.2) these yield solutions $E^k(x, y; \varrho)$, $H^k(x, y; \varrho)$ of (P.1) and (P.2). We are going to discuss these problems by means of the Laplace transform techniques of [2]. We outline the ideas here and present more detail in the next section.

We form the Laplace transform of (P.1)' and (P.2)'. This yields boundary-value problems for the transforms \mathcal{U} and \mathcal{V} of u and v . The methods of [2] show that these are meromorphic functions of the transform variable s in $\Re s > -\beta'$, where $\beta' > 0$ is some number less than the α in (2.3). The results of [3] can be invoked to show that \mathcal{U} and \mathcal{V} have poles of order at most three at $s = 0$ and no other poles in $\Re s > -\beta'$. From these results and (4.2) it will follow that the transforms e^k and h^k of E^k and H^k satisfy the relations,

* Here certain compatibility conditions on $\varrho(x, y, 0)$ are needed.

$$\begin{aligned}
 \tilde{\mathcal{C}}^k(x, y, s) &= \frac{\mathcal{A}^k(x, y)}{s^2} + \frac{\mathcal{B}^k(x, y)}{s} + \mathcal{R}^k(x, y, s) \\
 \tilde{\mathcal{H}}^k(x, y, s) &= \frac{\mathcal{C}^k(x, y)}{s^2} + \frac{\mathcal{D}^k(x, y)}{s} + \mathcal{S}^k(x, y, s)
 \end{aligned}
 \tag{4.6}$$

where \mathcal{R}^k and \mathcal{S}^k are regular in $\operatorname{Re} s > -\beta'$.

The equations (4.6), together with the inversion formula for the Laplace transform yield the estimates,

$$\begin{aligned}
 \mathcal{E}^k(x, y, t) &= \mathcal{A}^k(x, y)t + \mathcal{B}^k(x, y) + O(e^{-\beta t}) \\
 \mathcal{H}^k(x, y, t) &= \mathcal{C}^k(x, y)t + \mathcal{D}^k(x, y) + O(e^{-\beta t}),
 \end{aligned}
 \tag{4.7}$$

for any $\beta < \beta'$.

The next section is devoted to establishing the formulas (4.6) and identifying the various coefficients with the corresponding steady-state solutions in order to complete the proofs of theorems (1) and (2). We refer to [2] for the steps necessary to verify that (4.6) implies (4.7). The only problem is to estimate the quantities \mathcal{u} and \mathcal{v} , and hence \mathcal{C} and \mathcal{H} , for large values of s and this question is discussed in [2].

5. Laplace Transforms.

Let \mathcal{u} and \mathcal{v} denote the Laplace transforms of the solutions (u, v) of (P.1)' or (P.2)' and let Φ_k denote the transforms of φ_k in (2.2). Let

$$k^2 = -\mu s(\sigma + \epsilon s).
 \tag{5.1}$$

Then (P.1)' and (P.2)' are transformed into the following two problems:

$$(P.1)'' \quad \begin{array}{ll} \Delta u + k^2 u = 0 & \text{in } D \\ \Delta v + k^2 v = 0 & \text{in } D \end{array} \quad \begin{array}{l} (\sigma + \epsilon s) \nabla u \cdot \underline{n} = -\Phi_1 \text{ on } \partial D \\ -k^2 v = -\Phi_2 \text{ on } \partial D \end{array}$$

$$(P.2)'' \quad \begin{array}{ll} \Delta u + k^2 u = 0 & \text{in } D \\ \Delta v + k^2 v = 0 & \text{in } D \end{array} \quad \begin{array}{l} k^2 u = -\Phi_1 \text{ on } \partial D \\ \mu s \nabla v \cdot \underline{n} = -\Phi_2 \text{ on } \partial D. \end{array}$$

Consider the two problems:

$$(I) \quad \Delta W + \lambda W = 0 \text{ in } D \quad W = \psi \text{ on } \partial D$$

$$(II) \quad \Delta W + \lambda W = 0 \text{ in } D \quad \nabla W \cdot \underline{n} = \psi \text{ on } \partial D.$$

(I) has a unique solution $D(\underline{x}; \psi; \lambda)$ for any $\lambda \neq \lambda_1, \lambda_2, \dots$, where $0 < \lambda_1 < \lambda_2 < \dots$. (II) has a unique solution $N(\underline{x}; \psi; \lambda)$ for $\lambda \neq \tilde{\lambda}_0, \tilde{\lambda}_1, \tilde{\lambda}_2, \dots$, where $0 = \tilde{\lambda}_0 < \tilde{\lambda}_1 < \tilde{\lambda}_2, \dots$. It follows easily then from (5.1) that (P.1)'' and (P.2)'' have solutions for all $s \neq 0$ in $\Re s > -\beta'$ for some $\beta' > 0$. Moreover it is easy to see that these solutions are analytic functions of s in $\Re s > -\beta'$.

We observe that in general the solutions of (P.1)'' and (P.2)'' will be singular at $s = 0$. These singularities reflect those of D and N at $s = 0$. The singularities of D and N were the subject of [3] and we summarize the results here.

Lemma. Suppose $\Psi(\underline{x}, k) = \sum_{m=0}^{\infty} f_m(\underline{x}) k^{2m}$. Then we have

$$(5.2) \quad D(\underline{x}; \Psi; k^2) = \sum_{m=0}^{\infty} d_m(\underline{x}) k^{2m},$$

where

$$(5.3) \quad \Delta d_0 = 0, \quad \Delta d_m = -d_{m-1}, \quad m \geq 1, \quad \text{in } D,$$

$$d_m = f_m \quad \text{on } \partial D;$$

$$(5.4) \quad N(\underline{x}; \Psi; k^2) = \sum_{m=-1}^{\infty} n_m(\underline{x}) k^{2m},$$

where

$$(5.5) \quad n_{-1} = \frac{\alpha}{A} \quad \Delta n_m = -n_{m-1}, \quad m \geq 0 \quad \text{in } D: \quad \alpha = \int_{\partial D} f_0 ds$$

$$\nabla n_m \cdot n = f_m \quad \text{on } \partial D.$$

The series are all to converge uniformly (except for the singular terms) in $0 \leq |k|^2 \leq r$ for some $r > 0$.

We observe that (2.3) shows that,

$$(5.6) \quad \Phi_k(\underline{x}) = \frac{\varphi_k^0(\underline{x})}{s} + \Psi_k(\underline{x}, s)$$

where Ψ_k is analytic near $s = 0$. We note next that from (5.1) we can define s as an analytic function $s(k^2)$ of k^2 such that $s(0) = 0$. It is clear then that each of the functions u and v in (P.1)" or (P.2)", when multiplied by either s or s^2 , will be a solution of (I) or (II), with a ψ which is a power series in k^2 . Thus our lemma will show that each of these has a pole of order at most three at $s = 0$.

Now let us take the Laplace transform of (4.2). If $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{H}}$ denote the transforms of \mathcal{E} and \mathcal{H} we have,

$$(5.7) \quad \begin{aligned} \tilde{\mathcal{E}} &= k^2 \mathcal{U}_k - \mu_s (\nabla \mathcal{V})^\perp \\ \tilde{\mathcal{H}} &= \mu (\sigma + \epsilon s) \mathcal{U}^\perp + k^2 \mathcal{U}_k. \end{aligned}$$

From the remarks of the preceding paragraphs and (5.7) one sees that \mathcal{E} and \mathcal{H} can have at most poles of order three at $s = 0$ (actually they are of order two). The calculations are tedious, but straightforward, and we omit them. It turns out that equations (4.6) are indeed satisfied and we merely list the results here.

Define d_0, n_0 by,

$$(5.8) \quad \begin{aligned} \Delta d_0 &= 0 \quad \text{in } D & d_0 &= -\varphi_1^0 \quad \text{on } \partial D \\ \Delta n_0 &= -\frac{\alpha}{A} \quad \text{in } D & \nabla n_0 \cdot \tilde{n} &= -\varphi_2^0 \quad \text{in } \partial D. \end{aligned}$$

Then we have:

$$(5.9) \quad \begin{aligned} \mathcal{A}^1(x, y) &= 0, \quad \mathcal{B}^1(x, y) = -\frac{1}{\sigma} (\nabla d_0)^\perp + \frac{\alpha}{A\sigma} \mathcal{K} \\ \mathcal{C}^1(x, y) &= 0, \quad \mathcal{D}^1(x, y) = (\nabla n_0)^\perp - d_0 \mathcal{K}, \end{aligned}$$

and,

$$(5.10) \quad \begin{aligned} \mathcal{A}^2(x, y) &= 0; \quad \mathcal{B}^2(x, y) = -(\nabla n_0)^\perp - d_0 \mathcal{K} \\ \mathcal{C}^2(x, y) &= \frac{1}{\mu} (\nabla d_0)^\perp + \frac{\alpha}{A\mu} \mathcal{K} \\ \mathcal{D}^2(x, y) &= -\sigma (\nabla d_1)^\perp - \sigma n_0 \mathcal{K} \end{aligned}$$

In (5.10) d_1 is a solution of $\Delta d_1 = -d_0$. Its boundary values on ∂D can be computed but they do not really concern us.

Equations (5.9) and (5.10) together with (4.7) yield the asymptotic behavior of \mathbb{E} and \mathbb{H} for large t . In order to complete the proofs of theorems (1) and (2) we have only to identify the terms in (5.9) and (5.10) with the steady state solutions.

Consider first (5.9), that is (P.1). Let $\chi = -\frac{1}{\sigma} d_0$. Then,

$$\Delta \chi = 0 \text{ in } D, \quad \chi = \frac{\varphi_1^0}{\sigma} \text{ on } \partial D.$$

Let χ be the harmonic conjugate of ψ . Set $N = \frac{\alpha}{A\sigma}$ and $\gamma = n_0$.

Then $\Delta \gamma = -\frac{\alpha}{A} = -\sigma N$ in D , $\nabla \gamma \cdot \underline{n} = \varphi_2^0$ on ∂D . Thus we have,

$$\mathbb{E}^1(x, y) = \nabla \psi + N \underline{k}, \quad \mathbb{C}^1(x, y) = (\nabla \gamma)^\perp + \sigma \chi \underline{k}.$$

and (4.7) then yields the formula (2.5) of theorem (1).

Next consider (5.10), that is (P.2). In this case recall we were to take $\varphi_1^0 \equiv N$, a constant. It follows that $d_0 \equiv -N$ and $\nabla d_0 = 0$. Now let $\chi = -n_0$ and let χ be the conjugate of ψ . Then (5.10) yields,

$$\mathbb{B}^2(x, y) = \nabla \psi + N \underline{k}.$$

Next let $\gamma = -\sigma d_1$. Then $\Delta \gamma = -\sigma \Delta d_1 = \sigma d_0 = -\sigma N$ in D and thus, by (5.10),

$$\mathbb{D}^2(x, y) = (\nabla \gamma)^\perp + \sigma \chi \underline{k}.$$

These results, together with (4.7), yield the estimates stated in theorem (2).

References

- [1] Friedman, A. "Partial Differential Equations", Holt, Rinehart and Winston, 1969.
- [2] MacCamy, R. C. "The Laplace Transform and Initial-Boundary Value Problems", Journ. of Math. Anal. and Appl. Vol. 16, No. 1, October, 1966.
- [3] MacCamy, R. C. "Low Frequency Acoustic Oscillations", Quart. of App. Math. Vol. 28, No. 3, October, 1965.
- [4] Werner, P. "On the Behavior of Stationary Electromagnetic Wave Fields for Small Frequencies", Report 605, Math. Research Center, Univ. of Wisconsin, February, 1966.
- [5] Stratton, J. A. "Electromagnetic Theory", McGraw-Hill.