Uniform Convexity of
Banach Spaces $\ell(\{p_i\})$

K. Sundaresan

Report 70 - 1

January, 1970

Uniform Convexity of Banach Spaces $\ell(\{p_i\})$

K. Sundaresan

The class of Banach sequence spaces $\ell(\{p_i\})$ studied originally by Nakano [4] has received attention in some of the recent papers. Klee [3] studied bounded summability property in the spaces $\ell(\{p_i\})$ while Waterman et all [6] characterised reflexive $\ell(\{p_i\})$ spaces. In the present note we sharpen the main theorem in [6] by showing that the hypothesis in that theorem provides a characterization of Uniformly Convex $\ell(\{p_i\})$ spaces and that a reflexive $\ell(\{p_i\})$ space is uniformly Convex. We accomplish the proofs of these results without appealing to the theorem in [6].

Let $\{p_i\}$ be a sequence of real numbers $1 \leq p_i < \infty$. Then $\ell(\{p_i\})$ is the set of all real sequences x such that

$$\sum_{i\geq 1} \frac{1}{p_i} |ax_i|^{p_i} < \infty$$

for some $a > 0$ depending on x. It is verified that with the usual definition of sum of two sequences and scalar multiple of a sequence the set $\ell(\{p_i\})$ is a real vector space. Further if for $x \in \ell(\{p_i\})$

$$(*) \quad M(x) = \sum_{i\geq 1} \frac{1}{p_i} |x_i|^{p_i}$$

then M is a modular on $\ell(\{p_i\})$. For a detailed account of modulars on vector spaces we refer to Nakano [4]. If M is a modular on a vector space the norm induced by the modular M is given by the formula

$$\|x\| = \inf \{\tfrac{1}{\xi} \mid \xi > 0,\ M(\xi x) \leq 1\}.$$

The spaces $\ell(p_i\})$ under the norm induced by the modular M defined in (*) is a Banach space.

Before proceeding to the main result of this note we recall some terminology from Nakano [5] concerning modulars and state a theorem useful in the subsequent discussion.

Let M be a modular on a vector space E and let the norm induced by M be denoted by $\|\cdot\|$. A vector $x \in E$ is said to be finite if $M(\lambda x) < \infty$ for all real values of λ. The modular M is said to be finite if every vector $x \in E$ is finite. The modular M is said to be uniformly finite (uniformly simple) if

$$\sup_{M(x)\leq 1} M(\xi x) < \infty \quad \left(\inf_{M(x)\geq 1} M(\xi x) > 0\right) \quad \text{for every real number } \xi.$$

The modular M is said to be uniformly convex if corresponding to any pair of positive real numbers r, ϵ there exists a $\delta > 0$ such that $M(x) \leq r, M(y) \leq r, M(x-y) \geq \epsilon \Rightarrow$

$$M\left(\frac{x+y}{2}\right) \leq \frac{1}{2}\,[M(x) + M(y)] - \delta$$

For a definition of Uniformly Convex Banach spaces, see Day [2]. The theorem which is stated below relates the uniform convexity of the modular M with the uniform Convexity of the norm induced by M.

Theorem [Nakano] If a modular M is uniformly convex, uniformly finite and uniformly simple, then the norm induced by M is uniformly convex. For a proof see Theorem 3 on page 227 in Nakano [5].

We proceed next to the main theorems of this note. Let P be the set of positive integers. If $Q \subset P$ we denote by M_Q the function on $\ell(\{p_i\})$ defined by

$$M_Q(x) = \sum_{n\in Q} \frac{1}{p_n} |x_n|^{p_n} .$$

We note M_Q is a convex function. We further recall the following inequalities. (i_1) If $p \geq 2$ then

$|a+b|^p + |a-b|^p \leq 2^{p-1} [|a|^p+|b|^p]$ for any two real numbers a,b. (i_2) If $1 < p \leq 2$ then

$$\left|\frac{a+b}{2}\right|^p + \frac{p(p-1)}{2} \left|\frac{a-b}{|a|+|b|}\right|^{2-p} \left|\frac{a-b}{2}\right|^p \leq \frac{|a|^p+|b|^p}{2}$$

with a,b as in (i_1). For a proof of (i_1) see Clarkson [1]. (i_2) follows from the Taylor expansion of $(1+t)^p$ for small t.

Theorem 1. The Banach space $\ell(\{p_i\})$ is uniformly convex if and only if

$$(*) \quad 1 < \inf_{i\geq 1} p_i \leq \sup_{i\geq 1} p_i < \infty .$$

Proof. Let the sequence $\{p_i\}_{i\geq 1}$ satisfy the inequality stated in $(*)$. Thus there exist real numbers A and B such that $1 < A \leq p_i \leq B < \infty$. We proceed to verify that the modular M is uniformly convex, uniformly finite and uniformly simple. Let r, ϵ be two positive numbers and $x, y \in \ell(\{p_i\})$ such that

$$M(x) \leq r, M(y) \leq r \text{ and } M(x-y) \geq \epsilon.$$

Let us partition the set of positive integers into sets E, F defined by $n \in E$ if $p_n \geq 2$ and $n \in F$ if $p_n < 2$. We note that $M(x) = M_E(x) + M_F(x)$ for all $x \in \ell(\{p_i\})$. Thus $M(x-y) \geq \epsilon$ implies either $M_E(x-y) \geq \epsilon/2$ or $M_F(x-y) \geq \epsilon/2$.

Case 1. Let $M_E(x-y) \geq \epsilon/2$. Since $p_n \leq B$

$$M_E\left(\frac{x-y}{2}\right) \geq \frac{1}{2^B} M_E(x-y) \geq \frac{\epsilon}{2^{B+1}} .$$

Further since for $n \in E$ $p_n \geq 2$ it follows from the inequality (i_1) that

$$M_E\left(\frac{x+y}{2}\right) + M_E\left(\frac{x-y}{2}\right) \leq \frac{1}{2}[M_E(x) + M_E(y)]$$

Now noting that M_F is a convex function it is verified using the above inequalities that

$$\frac{1}{2}[M(x)+M(y)] \geq M_E\left(\frac{x+y}{2}\right) + M_E\left(\frac{x-y}{2}\right) + M_F\left(\frac{x+y}{2}\right)$$

$$\geq M\left(\frac{x+y}{2}\right) + \frac{\epsilon}{2^{B+1}} .$$

Case 2. Let $M_F(x-y) \geq \epsilon/2$. Let G be the subset of F consisting of the $n \in G$ such that

$$|x_n - y_n| \geq C(|x_n| + |y_n|)$$

where $C = \text{Min}(\frac{1}{2}, \frac{\epsilon}{8r})$. With $G_o = F \sim G$ it is verified that,

$$\sum_{n\in G_o} \frac{1}{p_n} |x_n - y_n|^{p_n} \leq \sum_{n\in G_o} \frac{1}{p_n} \{c^{p_n}(|x_n|+|y_n|)^{p_n}\}$$

$$\leq \sum_{n\in G_o} \frac{2^{p_n}}{p_n} [\frac{c(|x_n|+|y_n|)}{2}]^{p_n}$$

$$\leq \sum_{n\in G_o} \frac{2^{p_n}}{2p_n} [|cx_n|^{p_n} + |cy_n|^{p_n}]$$

$$\leq \frac{1}{2}[M(2cx) + M(2cy)]$$

Since $0 \leq 2c \leq 1$ and $M(x)$, $M(y) \leq r$

$$M(2cx) + M(2cy) \leq 4cr.$$

Thus it is verified that

$$\sum_{n\in G_o} \frac{1}{p_n} |x_n - y_n|^{p_n} \leq 2cr \leq \frac{\epsilon}{4} \quad \text{since} \quad c \leq \frac{\epsilon}{8r} .$$

Since $M_F(x-y) \geq \epsilon/2$ it follows from the definition of G_o that

(**) $$\sum_{n\in G} \frac{1}{p_n} |x_n - y_n|^{p_n} > \epsilon/4.$$

Then from inequality (i_2) it follows that

(***) $$\frac{1}{2}[M_G(x) + M_G(y)] \geq M_G(\frac{x+y}{2}) + M_G(\frac{x-y}{2}) \frac{(A-1)c}{2}$$

Since for $n\in G$, $p_n < 2$ it is verified

$$M_G\left(\frac{x-y}{2}\right) \geq \frac{1}{4} M_G(x-y).$$

But from (**) it follows that $M_G(\frac{x-y}{2}) \geq \frac{\epsilon}{16}$

Thus the inequality (***) yields

$$\frac{1}{2}[M_G(x) + M_G(y)] \geq M_G\left(\frac{x+y}{2}\right) + \frac{(A-1)c\epsilon}{32}$$

Noting that the function M_Q is convex it is deduced from the above inequality that

$$\frac{1}{2}[M(x) + M(y)] \geq M_G\left(\frac{x+y}{2}\right) + M_{P\sim G}\left(\frac{x+y}{2}\right) + \frac{(A-1)c\epsilon}{32}$$

$$= M\left(\frac{x+y}{2}\right) + \frac{(A-1)c\epsilon}{32}$$, where P is the set of positive integers.

Thus choosing $\delta = \mathrm{Min}\left(\frac{\epsilon}{2^{B+1}}, \frac{(A-1)c\epsilon}{32}\right)$ it is verified that the modular M is uniformly convex.

The modular M is uniformly finite for if S is the function defined on the real line by setting $S(\xi) = |\xi|^B$ if $|\xi| \geq 1$ and $S(\xi) = |\xi|^A$ if $|\xi| < 1$ it is verified that $M(\xi x) \leq S(\xi)M(x)$. Thus $\sup_{M(x)\leq 1} M(\xi x) \leq S(\xi)$.

Next we proceed to verify that M is uniformly simple. Let L be the function defined on the real line by setting $L(\xi) = |\xi|^A$ if $|\xi| \geq 1$ and $L(\xi) = |\xi|^B$ if $|\xi| < 1$. Then it follows that $M(\xi x) \geq L(\xi)M(x)$. Hence M is uniformly simple. Thus it follows from Nakano's theorem the norm induced by M is uniformly convex.

We next proceed to the Converse of the above theorem.

Theorem 2. If $\ell(\{p_i\})$ is uniformly convex then $1 < \liminf p_i \leq \limsup p_i < \infty$.

Proof. If possible let $\ell(\{p_i\})$ be uniformly convex and $\liminf p_i = 1$. Thus there exists an infinite subsequence

$\{p_{i_j}\}$ of $\{p_i\}$ such that $p_{i_j} \to 1$. By considering the vectors $x \in \ell(\{p_i\})$ such that $x_n = 0$ if $n \neq i_j$ for some j it is seen that the Banach space $\ell(\{p_{i_j}\})$ is isometrically isomorphic with a subspace of $\ell(\{p_i\})$. Thus $\ell(\{p_{i_j}\})$ is uniformly convex. Hence it is a reflexive Banach space. However since $p_{i_j} \to 1$ by the Theorem 2 in Nakano [4] the weak sequential convergence and norm convergence coincide in $\ell(\{p_{i_j}\})$. Since $\ell(\{p_{i_j}\})$ is reflexive the unit cell in $\ell(\{p_{i_j}\})$ is a weakly compact. Thus it follows readily from Eberlein theorem (see page 51 [2] that the unit cell in $\ell(\{p_{i_j}\})$ is compact in the norm topology. Hence $\ell(\{p_{i_j}\})$ is finite dimensional contradicting that $\{p_{i_j}\}$ is an infinite sequence. Hence $1 < \inf p_i$. If $\limsup p_i = \infty$ it is verified as in Lemma 1 in [6] that $\ell(\{p_i\})$ contains a subspace isomorphic to ℓ^{∞} contradicting that the space $\ell(\{p_i\})$ is reflexive. The proof of Theorem 2 is complete.

In conclusion we note that from the theorem 1 and proof of theorem 2 in this note it is readily inferred that the Banach space $\ell(\{p_i\})$ is uniformly convex if and only if it is reflexive.

References

1. J. A. Clarkson, Uniformly Convex Spaces, Trans. Amer. Math. Soc., 40(1936), 396-414.

2. M. M. Day, Normed Linear Spaces. Berlin: Spr inger, 1958.

3. V. Klee, Summability in $\ell(p_{11} \; p_{21},\ldots)$ Spaces, Studia Math. (1965) p. 277-280.

4. H. Nakano, Modulared Sequence Spaces, Proc. Jap. Acad. 27 (1951), p. 508-512.

5. H. Nakano, Topology and Linear Topological Spaces: Tokyo Maruzen and Co., 1951.

6. D. Waterman, T. Ito, F. Barber and J. Ratti, Reflexivity and Summability: The Nakano $\ell(p_i)$ spaces, Studia Math. 33^1(1969) p. 141-146.

FIXED POINT THEOREMS FOR SUMS OF OPERATORS

James S. W. Wong

Report 69-40

December, 1969

ERRATA on ' Fixed Point Theorems for Sums of operators ', James S. W. Wong.

p. 1 line 11 $Ax < Ay$ should read $Ax - Ay$.

p. 3 line 2 completely should read strongly.

p. 6 line 6 Insert after Note that the words ' for large i '.

line 8 $Bx_i\|$ should read $\|Bx_i\|$.

p. 7 line 7 The whole line should read ' Let $t > 0$, we obtain from above with $x = x_0 + th$, '

line 2 from bottom. The whole line should read

$$' \frac{1}{1-q} \limsup_{\|x\|\to\infty} \frac{\|Bx\|}{\|x\|} + \limsup_{\|x\|\to\infty} \frac{\|(I-A)^{-1}B0\|}{\|x\|} \text{ '}$$

FIXED POINT THEOREMS FOR SUMS OF OPERATORS

James S. W. Wong

Let T be a mapping from a Banach space X into itself and K be a closed bounded convex subset. The celebrated Schauder fixed point theorem states that if $T(K) \subseteq K$ and T is completely continuous then T has a fixed point in K. We are here concerned with extensions of Schauder's theorem to sums of operators. An operator A defined on X is called a contraction if there exists some constant $0 < q < 1$, such that $\|Ax < Ay\| \leq q\|x-y\|$ for all $x,y \in X$. The following is a generalization of Schauder's theorem due to Krasnoselskii for sum of operators.

THEOREM I. (Krasnoselskii [8], Sadovskii [11]) *Let* $T = A + B$, *where* A *is a contraction and* B *completely continuous, and* $T(K) \subseteq K$. *Then* T *has a fixed point in* K.

The operator A is called non-expansive if $\|Ax - Ay\| \leq \|x - y\|$ for all $x,y \in X$. We call the operator B strongly continuous if for every weakly convergent sequence $\{x_k\}$, with limit x, there exists a subsequence $\{Bx_{k_i}\}$ such that $Bx_{k_i} \longrightarrow x$ strongly. Recent interests in the theory of non-expansive mappings led to the following analogue of Theorem I:

THEOREM II. (Zabreiko, Kachurovskii and Krasnoselskii [12]). *Let* X *be a real Hilbert space*, $T = A + B$, *where* A *is nonexpansive and* B *is strongly continuous, and* $T(K) \subseteq K$.

Then T *has a fixed point in* K.

When the operator $B \equiv 0$, Theorem II reduces to the recent well known result of Browder [1], Kirk [7] and Gohde [4], establishing the existence of fixed points of nonexpansive mappings on Hilbert spaces.

In a number of applications of Schauder's theorem, it is sometimes difficult to find a desired bounded convex set K which is mapped into itself by T. One is thus led to impose other conditions directly on the operator T which ascertains the existence of some large closed ball being mapped into itself by T. For this purpose, the notion of *quasi-norm* of T is introduced which is defined by

$$\|T\| = \limsup_{\|x\|\to\infty} \frac{\|Tx\|}{\|x\|} . \tag{1}$$

By requiring that $\|T\|$ is small, we have the following analogue of Schauder's theorem concerning the solvability of functional equations.

THEOREM III. (Dubrovskii [2], Granas [5]). *If* T *is completely continuous, and* $\|T\| < 1$, *then* $\Re(I-T) = X$, *where* $\Re(T)$ *denotes the range of* T.

The purpose of this note is to prove analogues of Theorems I and II by imposing quasinorm conditions on A and B in place of the condition $T(K) \subseteq K$.

THEOREM 1. *If* $T = A + B$, *where* A *is a contraction and* B *is completely continuous, and* $\|A\| + \|B\| < 1$, *then* $\Re(I-T) = X$.

THEOREM 2. *Let X be a real Hilbert space. If $T = A + B$, where A is nonexpansive and B is completely continuous, and $\|A\| + \|B\| < 1$, then $\mathcal{R}(I-T) = X$.*

Roughly speaking, Theorems I and II remain valid when the condition that $T(K) \subseteq K$ is being replaced by the quasinorm condition that $\|A\| + \|B\| < 1$. Of course, in Theorems I and II the operator T need only to be defined on K rather than the entire space X. However, the conclusions of Theorems 1 and 2 are also stronger.

As an immediate consequence of Theorem 1, we obtain the main theorem of Nashed and Wong [10] as a corollary:

COROLLARY 1. *If $T = A + B$, where A is a contraction and B is completely continuous, and $\|B\| < 1 - q$, then $\mathcal{R}(I-T) = X$.*

Note that if A is a contraction with contractive constant q, then we have

$$\begin{aligned}\|A\| &= \limsup_{\|x\|\to\infty} \frac{\|Ax\|}{\|x\|}\\ &\leq \limsup_{\|x\|\to\infty} \frac{\|Ax - A0\| + \|A0\|}{\|x\|}\\ &\leq \limsup_{\|x\|\to\infty} \frac{q\|x\| + \|A0\|}{\|x\|} = q.\end{aligned}$$

Thus if $\|B\| < 1 - q$ then $\|A\| + \|B\| < 1$, and the result follows from Theorem 1.

PROOF OF THEOREM 1. For each $y \in X$, define $A_y x = Ax + y$ for every $x \in X$ and $T_y = A_y + B$. It is easy to see that A_y is a contraction with the same contractive constant q, and the operator T_y satisfies the same hypothesis as that of T. Moreover, $0 \in \mathcal{R}(I-T_y)$ if and only if $y \in \mathcal{R}(I-T)$. Thus, it suffices to show that $0 \in \mathcal{R}(I-T)$. For any fixed element $z \in X$, let Lz denotes the unique solution of

$$(2) \qquad Lz = ALz + Bz,$$

which is possible because A is a contraction. For any pair of elements $u, v \in X$, we deduce from (2) the following inequality

$$\|Lu - Lv\| \leq \frac{1}{1-q} \|Bu - Bv\|,$$

from which and the complete continuity of B it follows that L is also completely continuous. For each positive integer n, denote $B_n = \{x : \|x\| \leq n\}$. We wish to show that there exists a positive integer N such that $L(B_N) \subseteq B_N$. Suppose not, there must exist a sequence $\{u_n\} \in B_n$ such that $\|Lu_n\| \geq n$ for all n. Since L is completely continuous, so $\|u_n\| \to \infty$ as $n \to \infty$. Note that from (2), we have

$$(3) \quad \|u_n\| \leq n \leq \|Lu_n\| \leq \|ALu_n\| + \|Bu_n\|.$$

For each $\epsilon > 0$, we may choose n_o such that for all $n \geq n_o$, $\|ALu_n\| \leq (\|A\| + \frac{\epsilon}{2})\|Lu_n\|$, and also $\|Bu_n\| \leq (\|B\| + \frac{\epsilon}{2})\|u_n\|$. Using these estimates, we can obtain from (3)

$$\|Lu_n\|(1-\|A\| - \frac{\epsilon}{2}) \leq \|Bu_n\|$$

which implies

$$(1-\|A\| - \frac{\epsilon}{2}) \leq \frac{\|Bu_n\|}{\|Lu_n\|} \leq \frac{\|Bu_n\|}{\|u_n\|} \leq \|B\| + \frac{\epsilon}{2},$$

from which it follows $1 \leq \|A\| + \|B\| + \epsilon$. Since $\epsilon > 0$ is arbitrary, this provides the desired contradiction and proves the theorem.

PROOF OF THEOREM 2. As in the proof of Theorem 1, it suffices to show that $0 \in \mathcal{R}(I-T)$. To this end, we define for $0 < \lambda < 1, A_\lambda = \lambda A, B_\lambda = \lambda B$ and $T_\lambda = \lambda T$. First note that A nonexpansive implies A_λ is a contraction with contractive constant λ. Next, since B is strongly continuous and X is reflexive, B is also completely continuous so does B_λ for every λ. Thus, an application of Theorem 1 to the operator T_λ shows that there exists $x_\lambda \in X$ satisfying

$$x_\lambda - (A_\lambda + B_\lambda)x_\lambda = 0, \tag{4}$$

for each $\lambda, 0 < \lambda < 1$. We claim that the set $\{x_\lambda : 0 < \lambda < 1\}$ is bounded. For otherwise, there exists a sequence $\{\lambda_i\}$ such that $\|x_{\lambda_i}\| \to +\infty$ as $i \to \infty$. Using (4), we observe that

$$1 = \frac{1}{\|x_{\lambda_i}\|} \|(A_{\lambda_i} + B_{\lambda_i})x_{\lambda_i}\|$$

$$\leq \frac{\lambda_i}{\|x_{\lambda_i}\|} \|Ax_{\lambda_i} + Bx_{\lambda_i}\|$$

$$\leq \frac{1}{\|x_{\lambda_i}\|} (\|Ax_{\lambda_i}\| + \|Bx_{\lambda_i}\|),$$

which upon letting $i \to \infty$, gives a contradiction. Now since the set $\{x_\lambda\}$ is bounded, there exists a subsequence $\{x_{\lambda_i}\}$, $\lambda_i \uparrow 1$, which converges to an element $x_o \in X$. Since B is strongly continuous, there exists a subsequence $\{\lambda_{n_i}\}$ such that $\lambda_{n_i} \uparrow 1$ and $Bx_{\lambda_{n_i}}$ converges strongly to Bx_o.

We write $x_i = x_{\lambda_{n_i}}$ for short. Note that

$$\begin{aligned} \|Tx_i\| &= \|Ax_i + Bx_i\| \\ &\leq \|Ax_i - A0\| + \|A0\| + Bx_i\| \\ &\leq \|x_i\| + 2\|A0\| + (\|Bx_o\| + 1), \end{aligned}$$

from which it follows that the sequence $\{Tx_i\}$ is also bounded independent of i, say $\|Tx_i\| \leq M$. Now, we observe that by (4)

$$\begin{aligned} \|Tx_i - x_i\| &= \|Tx_i - \lambda_{n_i} Tx_i\| \\ &\leq (1 - \lambda_{n_i})\|Tx_i\| \leq (1 - \lambda_{n_i})M, \end{aligned}$$

hence $\lim_{i\to\infty} \|Tx_i - x_i\| = 0$. Also the strong convergence of Bx_i to Bx_o may be used to prove that the sequence $\{x_i - Ax_i\}$ converges strongly to Bx_o, since

$$\|x_i - Ax_i - Bx_o\| \leq \|x_i - Tx_i\| + \|Bx_i - Bx_o\|.$$

Finally, let $x \in X$; and obtain from the nonexpansiveness of A the following inequality

(5) $$(x - Ax - x_i + Ax_i,\ x - x_i) \geq 0.$$

Note that

$$|(x - Ax - x_i + Ax_i, x - x_i) - (x - Ax - Bx_o, x - x_o)|$$

$$\leq |(-x_i + Ax_i + Bx_o, x - x_i)| + |(x - Ax - Bx_o, x_o - x_i)|,$$

which tends to zero as $i \to \infty$. Thus passing the limit in (5), we obtain

$$(x - Ax - Bx_o, x - x_o) \geq 0.$$

Since $t > 0$, we obtain from above

$$(x_o - A(x_o + th) - Bx_o, h) \geq 0.$$

Letting $t \to 0$ in the above inequality, we find $(x_o - Ax_o - Bx_o, h) \geq 0$. The fact that h is arbitrary yields $x_o = Tx_o$. This completes the proof of the theorem.

Remark 1. We note that the original proof of Corollary 1 is similar to that of Theorem 1. However, by a direct application of Theorem III, we can now provide a shorter proof. It is well known that if A is contraction then $(I - A)^{-1}$ exists and is Lipschitzian with Lipschitzian constant $(1 - q)^{-1}$. Since B is completely continuous and $(I - A)^{-1}$ Lipschitzian, it follows that $(I - A)^{-1}B$ is completely continuous. Now we observe that

$$\|(I-A)^{-1}B\| \leq \limsup_{\|x\|\to\infty} \frac{\|(I-A)^{-1}(Bx-BO)\| + \|(I-A)^{-1}BO\|}{\|x\|}$$

$$\leq \frac{1}{1-q} \limsup_{\|x\|\to\infty} \frac{\|Bx\| + \|BO\| + \|(I-A)^{-1}BO\|}{\|x\|}$$

$$\leq \frac{1}{1-q} \|B\| < 1.$$

Applying Theorem III to the operator $(I-A)^{-1}B$, we obtain $\mathcal{R}(I-(I-A)^{-1}B) = X$. Thus,

$$R(I-T) = R((I-A)(I-(I-A)^{-1}B)) = (I-A)X = R(I-A).$$

Again since A is a contraction, we have $\mathcal{R}(I-A) = X$.

Remark 2. As a historical remark, we wish to point out that in [8], Krasnoselskii assumed the stronger condition that $Ax + By \in K$ for every pair $x, y \in K$. The stronger result was first given in Sadovskii [11]. An alternative proof of Theorem I in case X is a Hilbert space was also given in Zabreiko, Kachurovskii and Krasnoselskii [12]. We remark also that under the above stipulated stronger condition, Theorem II was first proved by Kachurovskii [6]. Dubrovskii [2] originally proved Theorem III under the stronger hypothesis that $\|T\| = 0$, such an operator is also called asymptotically zero. The introduction of quasinorm and the present improvement was due to Granas [5]. Extensions of Theorems I and II in a different direction may also be found in Fučik [3]. For applications of fixed point theorems for sums of operators to the study of nonlinear integral equations we refer to Krasnoselskii [9] and Nashed and Wong [10].

Remark 3. Although Theorem 2 is stated and proved for a real Hilbert space it obviously remains valid for Hilbert spaces over complex numbers. In particular, inequality (5) and the following arguments remain valid if one simply replaces the inner product by its real part.

REFERENCES

[1] F. E. Browder, "Fixed point theorems for noncompact mappings in Hilbert space", Proc. Natl. Acad. Sci. U. S. A., 53 (1965), 1272-1276.

[2] W. Dubrovskii, "Sur certaines equations integrales nonlineares", Uchen. Zap. Moskov. Gos. Univ., 30(1939), 49-60.

[3] S. Fučik, "Fixed point theorems for sum of nonlinear mappings", Comment. Math. Univ. Carolinae, 3 (1967), 503-514.

[4] D. Göhde, "Zum Prinzip der kontraktiven Abbildung", Math. Nachr., 30 (1966), 251-258.

[5] A. Granas, "The theory of compact vector fields and some of its applications to topology of functional spaces I", Rozpr. Matematyczne, XXX (1962), 1-93.

[6] R. I. Kachurovskii, "Nonlinear monotone operators in Banach spaces", Russian Math. Surveys, 23 (1968), 117-165.

[7] W. A. Kirk, "A fixed point theorem for mappings which do not increase distance", Amer. Math. Monthly, 72 (1965), 1004-1006.

[8] M. A. Krasnoselskii, "Two remarks about the methods of successive approximation", Uspekhi Mat. Nauk, 19 (1955), 123-127.

[9] M. A. Krasnoselskii, Topological methods in the theory of nonlinear integral equations, MacMillan Co., New York, 1964.

[10] M. Z. Nashed and J. S. W. Wong, "Some variants of fixed point theorems of Krasnoselskii and applications to nonlinear integral equations", J. Math. Mech., 18 (1969), 767-778.

[11] B. N. Sadovskii, "A fixed point principle", Functional Analysis, 1 (1967), p. 74-76. English Translation 1(1967), p. 151-153.

[12] P. P. Zabreiko, R. I. Kachurovskii and M. A. Krasnoselskii, "Fixed points of a sum of two operators", Functional Analysis, 1 (1967), 93-94. English Translation, 1 (1967), 168-169.

Carnegie-Mellon University
Pittsburgh, Pennsylvania 15213